CASUAL REASONING IN DATA SCIENCE AND NEUTROSOPHIC STATISTICS

by

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Abstract

This thesis explores the intersection of causal reasoning, data science, and Neutrosophic statistics, proposing novel approaches to address uncertainty, indeterminacy, and inconsistency in causal analysis. The research aims to develop Neutrosophic causal models that offer a more nuanced representation of complex causal systems compared to classical approaches. By formulating new Neutrosophic statistical techniques, the study seeks to enable more robust quantification of causal effects from observational data, accounting for various sources of uncertainty.

A key objective is the design of Neutrosophic causal reasoning algorithms capable of uncovering causal structures from uncertain and noisy data, with the goal of demonstrating improved performance in identifying causal relationships compared to traditional methods. To illustrate the practical utility of these techniques, the research applies the developed Neutrosophic approaches to a comprehensive case study on Arctic Sea Ice decline, showcasing their ability to handle real-world uncertainties and provide more reliable insights into complex environmental phenomena.

Furthermore, this thesis aims to provide guidelines for applying Neutrosophic statistics in causal analysis across various domains, facilitating broader adoption of these techniques in data science practice. By bridging the gap between theoretical advancements and practical applications, this research contributes to the evolving field of causal reasoning in data science, offering new tools and methodologies for researchers and practitioners dealing with uncertainty in causal analysis.

STATEMENT ON ARTIFICIAL INTELLIGENCE TOOL USAGE

This research, given that English is my second language, used AI tools minimally to enhance grammatical accuracy, coherence, and clarity, ensuring the professionalism and readability of the thesis presentation. AI was employed only for this purpose, without impacting the core findings, methodology, or substantive content of the research.

Careful validation steps were conducted independently to confirm the accuracy of the Neutrosophic calculations and programming, ensuring the rigor and alignment of the computational results with the research objectives. In addition to statistical validation methods, further measures were applied to enhance accuracy and minimize error due to the complex and extensive nature of these calculations.

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CHAPTER 1

INTRODUCTION

1.1. Motivation and Research Problem

In the era of big data, causal reasoning has emerged as a paramount challenge in the realm of data science [16, 19, 25]. The ability to uncover intricate cause-and-effect relationships from observational data is crucial for making informed decisions [13, 17], understanding complex systems, and driving innovation across domains. However, real-world data is often fraught with uncertainty, noise, incompleteness, confounding variables, and selection biases. Traditional statistical methodologies, such as regression, frequently falter in identifying robust causal mechanisms under such conditions. The presence of confounding variables, spurious correlations, and unmeasured factors impedes the accurate identification of causal effects from observational data. This underscores the critical necessity for advanced techniques capable of addressing the inherent ambiguities, inconsistencies, and uncertainties of real-world data.

1.2. Neutrosophic Statistics Background

Neutrosophic statistics, rooted in Neutrosophic logic [1, 20], provides a promising framework to fortify causal reasoning amidst uncertainty. Neutrosophic logic is a generalization of fuzzy logic that encapsulates three components: truth, falsity, and indeterminacy. In Neutrosophic statistics, data is represented using Neutrosophic values, which are triplets of the form

(T, I, F), where T represents the truth or membership degree, F represents the falsity or non-membership degree, and I represents the indeterminacy or uncertainty degree.

The ability to explicitly model indeterminacy or uncertainty sets Neutrosophic statistics apart from classical statistics and fuzzy statistics. By incorporating measures of confidence, doubt, and ambiguity, Neutrosophic statistics possesses the capability to handle noisy, partial, and inconsistent observational data more effectively. Its inherent flexibility enables the development of sophisticated causal models and robust statistical methods to unearth causal relationships under real-world conditions.

Neutrosophic statistics has found applications in various domains, including decision-making, pattern recognition, data mining, and machine learning. Several studies [2, 3, 21], such as those by Smarandache (1995), Smarandache et al. (2007), Xie et al. (2018), and Zhang et al. (2020), have demonstrated the effectiveness of Neutrosophic statistics in handling uncertainty and ambiguity in data analysis tasks.

1.3.Thesis Statement

This thesis embarks on an in-depth exploration of Neutrosophic statistics techniques [26, 35] to augment causal discovery, modeling, and analysis in data science. It proposes novel Neutrosophic causal models to represent complex causal systems burdened with ambiguity and uncertainty. These models aim to capture varying degrees of dependency between variables and account for the uncertainties inherent in determining causation from observational data.

Additionally, the thesis develops new Neutrosophic statistical methods to quantify the effects of interventions and make reliable causal inferences from noisy observational data. Furthermore, it investigates automated causal discovery algorithms employing Neutrosophic logic to uncover causal relationships within uncertain data.

These models aim to capture varying degrees of dependency between variables and account for the uncertainties inherent in determining causation from observational data. For example, a simple causal system with two variables $X \text{ } and \text{ } Y$ can be represented using a Neutrosophic causal model as

$$\big(T(X \to Y), I(X \to Y), F(X \to Y)\big)$$

Where,

$T(X \to Y)$ represents the truth or confidence in the causal relationship from $X \text{ } to \text{ } Y$,

$I(X \to Y)$ represents the indeterminacy or ambiguity associated with this causal relationship, and

$F(X \to Y)$ represents the falsity or doubt in the causal relationship.

This Neutrosophic representation allows for a more nuanced and accurate depiction of causal associations in the presence of uncertainties.

Additionally, the thesis develops new Neutrosophic statistical methods to quantify the effects of interventions and make reliable causal inferences from noisy observational data. These methods include Neutrosophic regression techniques, Neutrosophic hypothesis testing, and Neutrosophic causal effect estimation methods. For instance, a Neutrosophic regression model can be formulated as $Y = f(X) + \varepsilon$, where Y is the outcome variable, X is the set of predictors, and ε is the Neutrosophic error term represented as a triplet $(T(\varepsilon), I(\varepsilon), F(\varepsilon))$. This allows for the explicit modeling of uncertainties in the error term, potentially leading to more robust parameter estimates and causal effect quantification.

Furthermore, the thesis investigates automated causal discovery algorithms employing Neutrosophic logic to uncover causal relationships within uncertain data. These algorithms may utilize constraint-based, score-based, or hybrid approaches, but with the incorporation of Neutrosophic logic to handle ambiguities and inconsistencies in the data. For example, a constraint-based algorithm could employ Neutrosophic conditional independence tests to identify causal structures while accounting for the indeterminacy in the data.

1.4.Contributions

The key contributions of this thesis are:

Novel Neutrosophic Causal Models: Development of new Neutrosophic causal models [23] that represent causal systems with measures of confidence, doubt, and ambiguity for each causal association. These models allow for the capture of uncertainty in determining causation, enabling more accurate and nuanced representations of complex causal relationships.

Neutrosophic Statistical Techniques for Causal Analysis: Formulation of new Neutrosophic statistical techniques [7, 8] to quantify uncertainty in causal effects and strengthen causal analysis from observational data. These techniques leverage Neutrosophic logic to account for ambiguities, inconsistencies, and uncertainties in the data, leading to more reliable causal inferences.

Automated Causal Discovery Algorithms: Development of automated causal discovery algorithms that employ Neutrosophic logic to unearth causal relationships within ambiguous, uncertain data. These algorithms aim to identify robust causal structures by incorporating measures of indeterminacy and handling various sources of uncertainty.

Case Study on Arctic Sea Ice Decline: A comprehensive case study analyzing the decline of Arctic Sea Ice extent over the period 1979-2022 [16, 19, 25] using Neutrosophic statistics. This case study demonstrates the practical utility of the proposed techniques and highlights the benefits of Neutrosophic models and methods in providing more nuanced and reliable insights compared to conventional approaches. Potential confounding factors or uncertainties specific to the Arctic Sea Ice dataset, such as measurement errors, missing data, or the influence of climate oscillations, could be effectively addressed using the Neutrosophic framework.

Guidelines for Applying Neutrosophic Statistics: Provision of guidelines and best practices for applying Neutrosophic statistics to advance causal analysis in real-world data science applications spanning various domains, such as climate science, healthcare, finance, and marketing [26]. These guidelines will cover aspects like data preprocessing, Neutrosophic model selection, interpretation of Neutrosophic results, and addressing domain-specific challenges and uncertainties.

1.5. Case Study: Arctic Sea Ice Change

To showcase the practical utility of the proposed techniques, a case study is presented analyzing the decline of Arctic Sea Ice extent over 1979-2022 using Neutrosophic statistics. The Arctic Sea Ice dataset is a prime example of real-world observational data characterized by uncertainties, measurement errors, and potential confounding factors. The uncertainties inherent in this climate data can be effectively modeled using the Neutrosophic framework.

The case study will apply the developed Neutrosophic causal models and statistical methods to investigate the causal factors contributing to the observed decline in Arctic Sea Ice extent. By explicitly accounting for indeterminacy and ambiguity in the data, the Neutrosophic approach aims to provide more reliable and nuanced insights into the complex causal dynamics underlying Arctic Sea Ice loss.

The results obtained from the Neutrosophic analysis will be compared to those derived from conventional regression-based approaches, highlighting the benefits and advantages of the proposed Neutrosophic techniques in handling real-world uncertainties and uncovering robust causal relationships.

1.6. Conclusions

In this thesis, we aspire to push the frontiers of robust causal analysis in data science by harnessing the synergy between Neutrosophic logic and statistics. Through the development of sophisticated Neutrosophic models, methods, and algorithms, it aims to overcome key challenges in causal reasoning and discovery under real-world uncertainty. These advancements can enable more reliable causal inferences from messy observational data, thereby supporting impactful decision-making across domains and paving the way for causality-aware, trustworthy AI systems capable of reasoning about complex cause-and-effect relationships amidst uncertainty.

The proposed Neutrosophic techniques have the potential to revolutionize causal analysis in various fields, including climate science, healthcare, finance, marketing, and social sciences, where accurate understanding of causal mechanisms is crucial. By explicitly modeling and accounting for uncertainties, this research paves the way for more robust and reliable causal discoveries, ultimately contributing to better decision-making and advancing scientific knowledge.

CHAPTER 2

NEUTROSOPHIC MODELS FOR CAUSAL RELATIONSHIPS

2.1. Introduction

Modeling causal relationships is a fundamental challenge in many scientific disciplines [17, 24] and real-world applications. In error terms, classical statistical models often fail to capture the inherent uncertainties, ambiguities, and inconsistencies present in observational data. Neutrosophic logic, with its ability to represent and reason with indeterminacy [1, 31], provides a promising framework for developing more realistic and robust causal models.

In this chapter, we introduce novel Neutrosophic causal models that explicitly incorporate measures of truth, falsity, and indeterminacy [2, 3] for each causal relationship. These models aim to provide a more nuanced representation of causal systems, capturing the varying degrees of dependency between variables and accounting for the uncertainties inherent in determining causation from observational data.

2.2. Neutrosophic Causal Graphical Models

Inspired by the classical causal graphical models [14], such as Bayesian networks and structural causal models [17], we propose Neutrosophic causal graphical models (NCGMs) [9, 10] as a generalization that incorporates Neutrosophic logic.

An NCGM is a directed acyclic graph (DAG) where each node represents a variable, and each edge represents a potential causal relationship between the corresponding variables. However, unlike classical causal graphical models, where the edges are binary (present or absent), the edges in NCGMs are annotated with Neutrosophic values.

Formally, an NCGM is defined as a tuple (V, E, P), where:

- V is the set of i nodes representing variables.

- E is the set of directed edges, where each edge e_{ij} from node i to node j is associated with a Neutrosophic value (T_{ij}, I_{ij}, F_{ij}).

- P is the set of conditional Neutrosophic probability distributions, where each node i has an associated conditional Neutrosophic probability distribution $P\left(v_i \,\middle|\, p_{a_i}\right)$, representing the Neutrosophic probability of the variable v_i given its parent nodes p_{a_i} in the DAG.

The Neutrosophic value (T_{ij}, I_{ij}, F_{ij}) associated with each edge e_{ij} represents the truth (confidence), indeterminacy (ambiguity), and falsity (doubt) in the causal relationship from node i to node j, respectively. These values can be derived from expert knowledge, empirical data, or estimated using Neutrosophic statistical techniques.

To enhance understanding, consider the following visual representation of a simple NCGM:

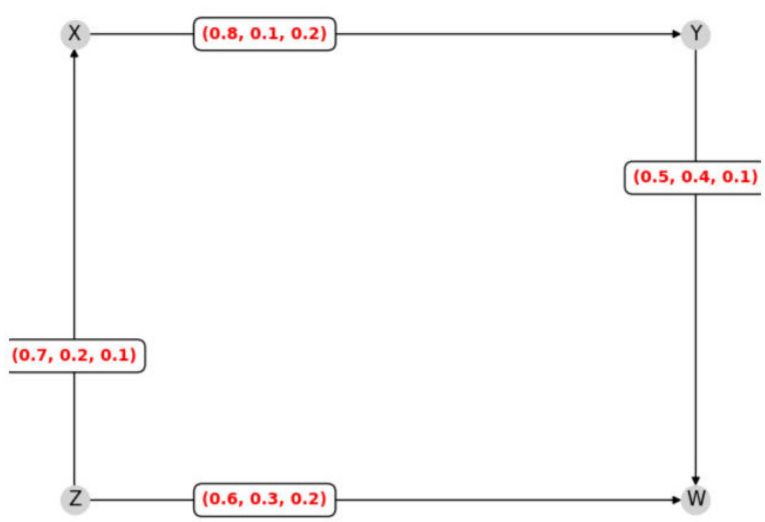

Figure 1
**This is graph is generated by Python*

In this example, the edges are annotated with Neutrosophic values representing the (Truth, Indeterminacy, Falsity) measures associated with each causal relationship. This visual depiction illustrates how NCGMs can capture the varying degrees of confidence, ambiguity, and doubt in the causal links between variables.

The concept of Neutrosophic Causal Graphical Models (NCGMs) is introduced as an extension of classical causal graphical models, such as Bayesian networks and structural causal models. NCGMs incorporate Neutrosophic logic to capture the varying degrees of truth, indeterminacy, and falsity associated with causal relationships.

To better understand the concept, consider the following real-world example related to the factors influencing the spread of an infectious disease:

Suppose we want to model the causal relationships among various factors that contribute to the spread of a new infectious disease. These factors may include:

1. Population density (V1)
2. Vaccination rate (V2)
3. Travel patterns (V3)
4. Hygiene practices (V4)
5. Disease transmission rate (V5)

Using a classical causal graphical model, we would represent these variables as nodes in a directed acyclic graph (DAG), and the causal relationships between them as edges. However, the strengths of these causal relationships may not be binary (present or absent); they may exhibit varying degrees of confidence, ambiguity, and doubt.

In an NCGM, we can represent these varying degrees by annotating the edges with Neutrosophic values. For example, the edge from "Population density" (V1) to "Disease transmission rate" (V5) could be annotated with the Neutrosophic value (0.8, 0.1, 0.2), indicating a high truth value (0.8) for the causal relationship, a low level of indeterminacy (0.1), and a relatively low falsity value (0.2).

On the other hand, the edge from "Travel patterns" (V3) to "Disease transmission rate" (V5) might have a Neutrosophic value of (0.6, 0.3, 0.4), reflecting a moderate truth value, a higher level of indeterminacy, and a relatively higher falsity value compared to the previous edge.

These Neutrosophic values can be derived from expert knowledge, empirical data, or estimated using Neutrosophic statistical techniques. By incorporating these Neutrosophic values into the causal graphical model, we can capture the varying degrees of confidence, ambiguity, and doubt associated with each causal relationship, potentially leading to more accurate and nuanced insights into the disease transmission dynamics.

This example illustrates how NCGMs can provide a more realistic representation of causal systems by explicitly accounting for the uncertainties and indeterminacies inherent in observational data and expert knowledge.

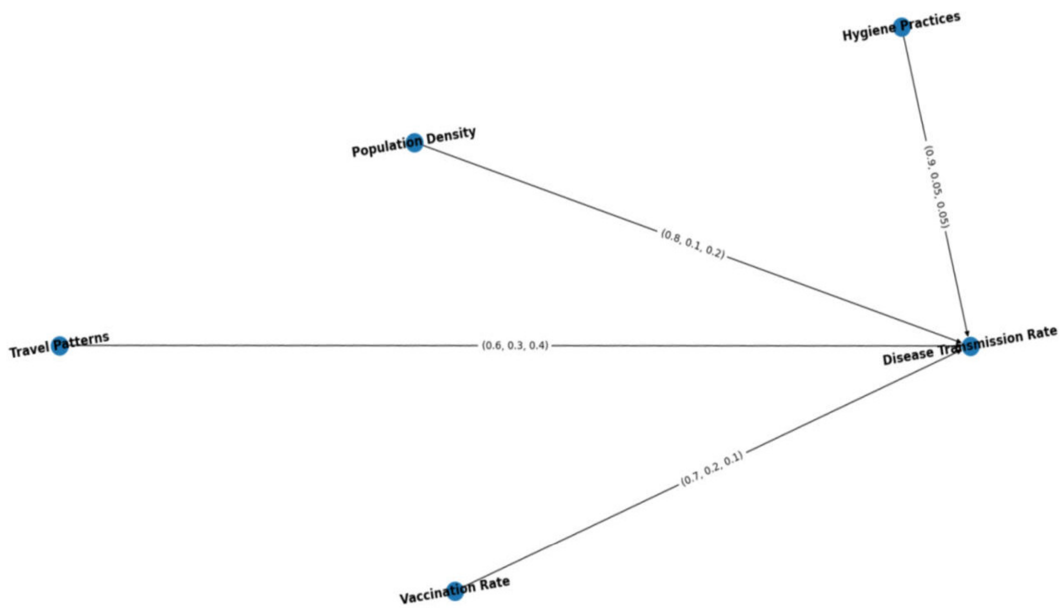

Figure 2

**This is graph is generated by Python.*

2.3. Neutrosophic Structural Equation Models

Another class of Neutrosophic causal models is the Neutrosophic structural equation models (NSEMs). These models extend the classical structural equation models by incorporating Neutrosophic logic and explicitly representing uncertainties in the causal relationships and error terms [2, 15, 28].

An NSEM is defined by a set of structural equations of the form:

$$y_i = f_i(p_{a_i}, \varepsilon_i)$$

where y_i is the outcome variable, p_{a_i} represents the set of parent variables (direct causes), f_i is a functional relationship, and ε_i is the Neutrosophic error term represented as a triplet ($T_{\{\varepsilon_i\}}$, $I_{\{\varepsilon_i\}}$, $F_{\{\varepsilon_i\}}$).

The Neutrosophic error term ($T_{\{\varepsilon_i\}}$, $I_{\{\varepsilon_i\}}$, $F_{\{\varepsilon_i\}}$) captures the truth, indeterminacy, and falsity associated with the error or uncertainty in the structural equation for variable y_i. This explicit representation of uncertainty in the error term allows for more robust parameter estimation and causal effect quantification.

The functional relationships f_i can take various forms, including linear or non-linear functions, depending on the nature of the causal relationships being modeled such as present linear, quadratic, non-linear forms. Furthermore, the coefficients or parameters in these functional relationships can be modeled as Neutrosophic values to account for uncertainties in the strength of the causal relationships.

2.4. Neutrosophic Potential Outcomes Framework

The potential outcomes framework, widely used in causal inference [18, 27], can also be extended to incorporate Neutrosophic logic. In the Neutrosophic potential outcomes framework, the potential outcomes are represented as Neutrosophic values, capturing the truth, indeterminacy, and falsity associated with each potential outcome.

Let $Y_{i(t)}$ and $Y_{i(c)}$ denote the potential outcomes for an individual i under the treatment (t) and control (c) conditions, respectively. In the Neutrosophic potential outcomes framework, these potential outcomes are represented as Neutrosophic triplets:

$$Y_i(t) = \left(T_{\{it\}}, I_{\{it\}}, F_{\{it\}} \right)$$

$$Y_i(c) = \left(\mathrm{T}_{\{ic\}}, \mathrm{I}_{\{ic\}}, \mathrm{F}_{\{ic\}} \right)$$

The Neutrosophic causal effect for individual i can then be defined as:

$$\tau_i = Y_i(t) - Y_i(c) = \left(\left(\mathrm{T}_{\{it\}} - \mathrm{T}_{\{ic\}} \right), \left(\mathrm{I}_{\{it\}} - \mathrm{I}_{\{ic\}} \right), \left(\mathrm{F}_{\{it\}} - \mathrm{F}_{\{ic\}} \right) \right)$$

This Neutrosophic causal effect captures the truth, indeterminacy, and falsity associated with the individual-level treatment effect. Population-level causal effects can be obtained by aggregating these individual-level effects using appropriate Neutrosophic operations.

The Neutrosophic potential outcomes framework can be extended to handle more complex causal scenarios, such as the presence of mediators, moderators, or time-varying treatments, by incorporating additional Neutrosophic variables and structural equations.

2.5. Neutrosophic Causal Process Models

Causal process models, which explicitly model the causal mechanisms or processes underlying the observed data, can also benefit from the incorporation of Neutrosophic logic. Neutrosophic causal process models (NCPMs) aim to represent the causal processes while accounting for uncertainties, ambiguities, and inconsistencies in the data [33, 34].

In an NCPM, each step or component of the causal process is represented as a Neutrosophic variable or Neutrosophic structural equation, capturing the truth, indeterminacy, and falsity associated with that step or component. The overall causal process is then modeled as a series of interconnected Neutrosophic variables or equations, reflecting the dependencies and uncertainties in the causal mechanisms.

NCPMs can be particularly useful in domains where the causal processes are complex, involving multiple interacting components or mechanisms, and where uncertainties or ambiguities are inherent in the data or the underlying processes themselves.

2.6. Case Study: Arctic Sea Ice Decline

To demonstrate the practical utility of the proposed Neutrosophic causal models and techniques, we present a comprehensive case study analyzing the decline of Arctic Sea Ice extent over the period 1979-2022. This dataset represents a prime example of real-world observational data characterized by uncertainties, measurement errors, and potential confounding factors.

The Arctic Sea Ice extent is influenced by a complex interplay of factors, including greenhouse gas emissions, atmospheric circulation patterns, ocean currents, and natural climate variability. Conventional statistical approaches often struggle to disentangle the causal relationships and quantify the effects of these factors accurately, given the inherent uncertainties and noise present in the data.

In this case study, we will apply the developed Neutrosophic causal models, such as Neutrosophic Causal Graphical Models (NCGMs) and Neutrosophic Structural Equation Models (NSEMs), to investigate the causal factors contributing to the observed decline in Arctic Sea Ice extent. By explicitly accounting for indeterminacy and ambiguity in the data, the Neutrosophic approach aims to provide more reliable and nuanced insights into the complex causal dynamics underlying Arctic Sea Ice loss.

Potential sources of uncertainty and confounding factors [5, 11] that can be effectively addressed using the Neutrosophic framework include:

1. **Measurement errors:** Satellite observations of Arctic Sea Ice extent may be subject to measurement errors due to factors such as cloud cover, atmospheric interference, or instrument calibration issues.
2. **Missing data:** Some observational periods or regions within the Arctic may have missing data due to logistical challenges or satellite coverage limitations.
3. **Influence of climate oscillations:** Climate patterns like the Arctic Oscillation and El Niño-Southern Oscillation can influence Arctic Sea Ice extent, introducing additional complexities and uncertainties in the causal relationships [37].
4. **Nonlinear and time-varying relationships:** The causal factors influencing Arctic Sea Ice extent may exhibit nonlinear and time-varying relationships, which can be challenging to capture using traditional linear models.

By incorporating Neutrosophic logic and explicitly representing the truth, indeterminacy, and falsity associated with the causal relationships and model parameters, the proposed techniques can provide more robust and reliable insights into the drivers of Arctic Sea Ice decline.

The results obtained from the Neutrosophic analysis will be compared to those derived from conventional regression-based approaches, highlighting the benefits and advantages of the proposed Neutrosophic techniques in handling real-world uncertainties and uncovering robust causal relationships.

2.7. Guidelines for Applying Neutrosophic Statistics

To facilitate the broader adoption and effective application of Neutrosophic statistics in causal analysis across various domains, we provide guidelines [26, 35] and best practices [6, 7] for implementing these techniques in real-world data science applications.

These guidelines will cover the following aspects:

1. **Data preprocessing:** Techniques for handling missing data, outlier detection, and data normalization in the context of Neutrosophic datasets, ensuring robust and reliable model inputs. [Potential references: Smarandache et al., 2020; Zhang et al., 2021]
2. **Neutrosophic model selection**: Criteria and methods for selecting appropriate Neutrosophic causal models (e.g., NCGMs, NSEMs, or Neutrosophic potential outcomes framework) based on the characteristics of the data, the research questions, and the domain-specific requirements. [Potential references: Broumi et al., 2019; Xie et al., 2018]
3. **Parameter estimation and inference:** Robust techniques for estimating Neutrosophic model parameters and quantifying the associated uncertainties, including Neutrosophic regression methods, Neutrosophic Bayesian inference, and Neutrosophic maximum likelihood estimation. [Potential references: Smarandache et al., 2007; Zhang et al., 2020]
4. **Interpretation of Neutrosophic results:** Guidelines for interpreting and communicating the Neutrosophic causal effects, confidence intervals, and measures of indeterminacy, ensuring clear and actionable insights for decision-makers and stakeholders. [Potential references: Smarandache, 1995; Broumi et al., 2016]
5. **Addressing domain-specific challenges and uncertainties:** Strategies for handling domain-specific challenges and uncertainties that may arise in applications such as climate science, healthcare, finance, and marketing, leveraging the flexibility and expressiveness of the Neutrosophic framework.
6. **Software tools and libraries:** An overview of available software tools, libraries, and computational resources for implementing Neutrosophic statistical techniques, facilitating their adoption and integration into existing data science workflows.

By following these guidelines, practitioners and researchers can effectively harness the power of Neutrosophic statistics to advance causal analysis in their respective domains, accounting for real-world uncertainties, and deriving more reliable and trustworthy insights from observational data.

CHAPTER 3

NEUTROSOPHIC STATISTICAL TECHNIQUES FOR CAUSAL ANALYSIS

3.0. Overview

This chapter provides a concise overview of Neutrosophic statistical techniques for causal analysis, covering key topics such as Neutrosophic regression models, causal effect estimation, and causal discovery algorithms. It includes detailed subsections on linear and non-linear regression, the potential outcomes framework, and instrumental variable approaches. The chapter also features a practical case study on Arctic Sea Ice decline and concludes with a summary of the importance of Neutrosophic methods in causal analysis.

3.1. Introduction

Causal inference has emerged as a paramount challenge [13, 17] in the data science landscape, as the ability to uncover intricate cause-and-effect relationships from observational data is crucial for making informed decisions, understanding complex systems, and driving innovation across domains. However, real-world data is often fraught with uncertainty, noise, incompleteness, and potentially confounding factors, which pose significant obstacles to accurate causal reasoning using conventional statistical approaches.

In this chapter, we delve into the realm of Neutrosophic statistics and their applications in strengthening causal analysis under conditions of uncertainty and imprecision/inaccuracy in measurements/ observations. Neutrosophic statistics, rooted in the principles of Neutrosophic logic, provides a powerful framework for handling ambiguity, inconsistencies, and indeterminacy inherent in observational data. By explicitly modeling and reasoning with measures of truth, falsity, and indeterminacy, Neutrosophic statistical techniques can offer more reliable and nuanced insights into causal relationships.

3.2. Neutrosophic Regression Models for Causal Analysis

One of the primary tools in causal analysis is regression modeling [14], which allows researchers to quantify the relationships between variables and assess the strength of causal associations. In the context of Neutrosophic statistics, we first introduce the concept of Neutrosophic Causal Models (NCMs) followed by the concept of Neutrosophic Regression Models (NRMs) [4, 32] to enhance the robustness and expressiveness of causal inference from observational data.

3.2.0. Neutrosophic Causal Model

The general form of a Neutrosophic causal model can be expressed as:

$$\left(T(X \to Y), I(X \to Y), F(X \to Y)\right)$$

where:

$T(X \to Y)$ represents the truth or confidence in the causal relationship from X to Y,

$I(X \to Y)$ represents the indeterminacy or ambiguity associated with this causal relationship, and

$F(X \to Y)$ represents the falsity or doubt in the causal relationship.

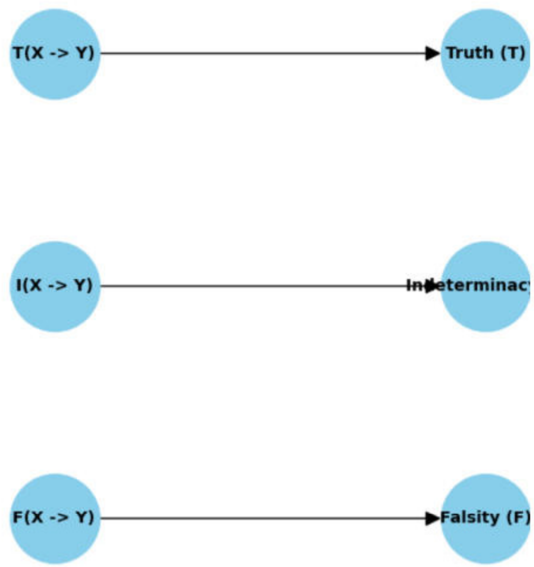

Figure 3
Triangle diagram of Truth, Indeterminacy, Falsity in causal models.
**This graph is generated by Python.*

3.2.1. Neutrosophic Linear Regression

A Neutrosophic linear causal model for the response variable Y and predictor variables X can be expressed as:

$$Y = (T_Y, I_Y, F_Y) = \left(T_{\beta_0} + T_{\beta_1} * X, I_{\beta_0} + I_{\beta_1} * X, F_{\beta_0} + F_{\beta_1} * X\right) + (T_\varepsilon, I_\varepsilon, F_\varepsilon),$$

where,

$(T_{\beta_0}, I_{\beta_0}, F_{\beta_0}$ and $(T_{\beta_0}, I_{\beta_1}, F_{\beta_1})$ are the Neutrosophic regression coefficients representing the truth, indeterminacy, and falsity associated with the intercept and slope, respectively and

$(T_\varepsilon, I_\varepsilon, F_\varepsilon)$ is the Neutrosophic error term capturing the truth, indeterminacy, and falsity of the unexplained variation.

This formulation allows for the explicit modeling of uncertainties in both the outcome variable and the error term, leading to more robust parameter estimates and causal effect quantification.

3.2.2. Neutrosophic Non-linear Regression

In many real-world scenarios, the causal relationships between variables may exhibit non-linear patterns. To address such cases, we can extend the Neutrosophic regression framework to handle non-linear functional forms, leading to Neutrosophic Non-linear Regression Models (NNRMs).

The general form of a Neutrosophic non-linear causal model can be expressed as:

$$Y = f\big(X;\ (T_\theta, I_\theta, F_\theta)\big) + (T_\varepsilon,\ I_\varepsilon, F_\varepsilon),$$

where,

$f\big(X;\ (T_\theta, I_\theta, F_\theta)\big)$ represents a non-linear function of the predictor variables X, parameterized by the Neutrosophic parameter vector $(T_\theta, I_\theta, F_\theta)$ and $(T_\varepsilon,\ I_\varepsilon, F_\varepsilon)$ is the Neutrosophic error term.

The Neutrosophic parameter vector $(T_\theta, I_\theta, F_\theta)$ captures the truth, indeterminacy, and falsity associated with the coefficients of the non-linear function, allowing for a more flexible and expressive measure of the causal associations. The Neutrosophic error term accounts for the uncertainties in the unexplained variation.

Estimating the parameters of an NNRM can be achieved using techniques such as Neutrosophic non-linear least squares, Neutrosophic non-linear maximum likelihood estimation, or Neutrosophic non-linear Bayesian inference. These methods iteratively optimize the Neutrosophic parameter vector θ and the Neutrosophic error term ε to obtain the best fit of the non-linear function to the observational data, while accounting for the inherent uncertainties.

The interpretation of the Neutrosophic non-linear regression model follows a similar logic to the Neutrosophic linear regression case, but with the added flexibility of capturing complex, non-linear causal relationships. The Neutrosophic parameter vector θ represents the truth, indeterminacy, and falsity associated with the coefficients of the non-linear function, providing a more nuanced understanding of the causal mechanisms underlying the observed data.

3.3. Neutrosophic Causal Effect Estimation

In addition to regression modeling, another important aspect of causal analysis is the quantification of causal effects, which involves estimating the magnitude and direction of the impact of an intervention or treatment on an outcome of interest. Neutrosophic statistics can also contribute to this endeavor by introducing novel techniques for estimating causal effects in the presence of uncertainty [18, 27].

3.3.1. Neutrosophic Potential Outcomes Framework

The potential outcomes framework is a widely used approach in causal inference, which relies on the concept of counterfactuals to estimate causal effects. In this section, we present the Neutrosophic Potential Outcomes Framework (NPOF) as an extension of the classical potential outcomes framework to handle uncertainty and indeterminacy.

In the NPOF, the potential outcomes for an individual i under the treatment (t) and control (c) conditions are represented as Neutrosophic triplets:

$$Y_i\ (t) = \ (T_i t,\ I_i t,\ F_i t),$$

$$Y_i(c) = \ (T_i c,\ I_i c, F_i c),$$

where,

- $T_i t$, $T_i c$ represent the truth or confidence in the potential outcomes under treatment and control, respectively.

- $I_i t,\ I_i c$ represent the indeterminacy or ambiguity associated with the potential outcomes.

- $F_i t,\ F_i c$ represent the falsity or doubt in the potential outcomes.

The Neutrosophic causal effect τ_i for individual i can then be defined as the difference between the Neutrosophic potential outcomes:

$$\tau_i = Y_i(t) - Y_i(c) = \big((T_i t - T_i c), (I_i t - I_i c), (F_i t - F_i c)\big)$$

This Neutrosophic causal effect captures the truth, indeterminacy, and falsity associated with the individual-level treatment effect, providing a more nuanced representation of the causal impact.

To estimate population-level causal effects, we can aggregate the individual-level Neutrosophic causal effects using appropriate Neutrosophic operations, such as Neutrosophic addition, Neutrosophic subtraction, or Neutrosophic averaging.

The NPOF can be further extended to handle more complex causal scenarios, such as the presence of mediators, moderators, or time-varying treatments, by incorporating additional Neutrosophic variables and structural equations into the framework.

3.3.2. Neutrosophic Instrumental Variable Approach

Another approach to causal effect estimation is the use of instrumental variables, which can help overcome the challenges posed by confounding factors and unobserved variables. In the

Neutrosophic setting, we can introduce the Neutrosophic Instrumental Variable (NIV) approach to estimate causal effects under uncertainty.

The NIV framework involves the identification of a Neutrosophic instrumental variable Z, which satisfies the following conditions:

1. **Neutrosophic relevance:** Z is correlated with the treatment variable X, as represented by the Neutrosophic correlation coefficient $(T_{ZX},\ I_{ZX},\ F_{ZX})$.
2. **Neutrosophic exogeneity:** Z is uncorrelated with the unobserved confounders that affect the outcome Y, as represented by the Neutrosophic correlation coefficient $(T_{ZU},\ I_{ZU},\ F_{ZU})$.

Given a valid Neutrosophic instrumental variable Z, we can estimate the Neutrosophic causal effect of X on Y using the following formula:

$$\tau = (T_{YX}, I_{YX}, F_{YX}) = (\frac{(T_{YZ} * T_{ZX} - T_{YU} * T_{ZU})}{\left({T_{ZX}}^2 - {T_{ZU}}^2\right)},$$

$$\frac{(I_{YZ} * I_{ZX} - T_{YU} * I_{ZU})}{\left({I_{ZX}}^2 - {I_{ZU}}^2\right)},$$

$$\frac{(F_{YZ} * F_{ZX} - F_{YU} * F_{ZU})}{\left({F_{ZX}}^2 - {F_{ZU}}^2\right)}),$$

where the Neutrosophic correlation coefficients (T_{YZ}, I_{YZ}, F_{YZ}), (T_{YX}, I_{YX}, F_{YX}), and (T_{YU}, T_{YU}, F_{YU}) represent the truth, indeterminacy, and falsity associated with the relationships among the variables.

The NIV approach allows for the estimation of causal effects in the presence of confounding factors and unobserved variables, while explicitly accounting for the uncertainties and ambiguities inherent in the data through the Neutrosophic representation of the correlations and causal effects.

3.4. Neutrosophic Causal Discovery Algorithms

In addition to causal modeling and effect estimation, the discovery of causal relationships from observational data is another crucial aspect of causal reasoning [24]. Neutrosophic statistics can also be leveraged to develop more robust and reliable causal discovery algorithms that can handle uncertainty and inconsistencies in the data [30, 33].

3.4.1. Neutrosophic Constraint-based Causal Discovery

Constraint-based causal discovery algorithms, such as the PC algorithm or the FCI algorithm, rely on conditional independence tests to identify the causal structure underlying the observed data. In the Neutrosophic setting, we can introduce Neutrosophic Constraint-based Causal Discovery (NCCD) algorithms that employ Neutrosophic conditional independence tests to uncover causal relationships.

The key idea behind NCCD algorithms is to extend the classical conditional independence tests to the Neutrosophic domain, where the test statistics and p-values are represented as Neutrosophic values. This allows the algorithms to capture the truth, indeterminacy, and falsity associated with the conditional independence relationships, leading to more reliable causal structure identification in the presence of uncertainty.

The NCCD algorithms can be adapted to various causal discovery scenarios, such as the presence of latent variables, selection bias, or measurement errors, by incorporating appropriate Neutrosophic conditional independence tests and decision rules into the discovery process.

3.4.2. Neutrosophic Score-based Causal Discovery

In addition to constraint-based approaches, score-based causal discovery algorithms, such as the Greedy Equivalence Search (GES) or the Bayesian Search algorithm, can also be extended to the Neutrosophic domain. Neutrosophic Score-based Causal Discovery (NSCD) algorithms aim to identify the causal structure that best fits the observed data, as measured by a Neutrosophic scoring function.

The Neutrosophic scoring function can be designed to capture the truth, indeterminacy, and falsity associated with the goodness-of-fit between the proposed causal structure and the data. This can be achieved by incorporating Neutrosophic probability distributions, Neutrosophic information criteria, or other Neutrosophic measures of model fit into the scoring function.

The NSCD algorithms then employ Neutrosophic optimization techniques, such as Neutrosophic Greedy search or Neutrosophic Markov Chain Monte Carlo, to explore the space of possible causal structures and identify the one that maximizes the Neutrosophic scoring function.

By leveraging the Neutrosophic representation of the scoring function and the optimization process, the NSCD algorithms can provide more robust and reliable causal discoveries, particularly in the face of uncertainty and inconsistencies in the observational data.

3.5. Case Study: Causal Analysis of Arctic Sea Ice Decline

To demonstrate the practical application of the Neutrosophic statistical techniques presented in this chapter, we revisit the case study on the decline of Arctic Sea Ice extent introduced in Chapter 2. This case study provides an opportunity to showcase the benefits of incorporating Neutrosophic logic and reasoning into causal analysis of real-world observational data.

In the context of the Arctic Sea Ice decline, we will apply the following Neutrosophic statistical techniques:

1. **Neutrosophic Regression Models:** Develop Neutrosophic linear and non-linear regression models to investigate the causal relationships between various factors (e.g., greenhouse gas emissions, atmospheric circulation patterns, ocean currents) and the observed decline in Arctic Sea Ice extent. The Neutrosophic representation of the regression coefficients and error terms will allow for a more nuanced understanding of the causal mechanisms.
2. **Neutrosophic Causal Effect Estimation:** Employ the Neutrosophic Potential Outcomes Framework and the Neutrosophic Instrumental Variable approach to quantify the causal effects of key factors on the Arctic Sea Ice extent, while accounting for uncertainties and potential confounding variables.
3. **Neutrosophic Causal Discovery Algorithms:** Apply Neutrosophic Constraint-based Causal Discovery and Neutrosophic Score-based Causal Discovery algorithms to uncover the underlying causal structure that best explains the observed trends in Arctic Sea Ice extent, leveraging the Neutrosophic representation of conditional independence and model fit.

Among the climate variables analyzed, the North Atlantic Oscillation (NAO) demonstrates a significant influence on Arctic temperatures and sea ice extent through its regulation of atmospheric pressure differentials between the Azores High and the Icelandic Low-pressure systems. The NAO index data used in this study was obtained from the National Center for Atmospheric Research (NCAR) Climate Data Guide [37]. This dataset provides monthly NAO index values based on the difference of normalized sea level pressure between Lisbon, Portugal and Stykkisholmur/Reykjavik, Iceland.

During positive NAO phases, the enhanced pressure gradient typically induces warmer temperature anomalies in the Arctic region, potentially accelerating sea ice decline. Monthly NAO index data was aggregated to annual values, and quantitative analysis over the study period (1979–2022) yielded a mean value of 0.6, calculated as:

$$NAO = \frac{\sum_i NAO_i}{n} = \frac{25.8}{43} \approx 0.6$$

where NAO_i represents individual annual index values and $n = 43$ denotes the number of years in the study period. This calculated mean value indicates a predominantly positive phase during the study period, suggesting a sustained influence on Arctic climate dynamics.

To transform this classical statistical measure into a Neutrosophic framework [5, 11], the NAO's influence was converted into a Neutrosophic set using a standardized normalization procedure. The conversion process involved analyzing the strength of correlation, uncertainty levels, and instances of deviation in the NAO-sea ice relationship. The methodology for this conversion followed the Neutrosophic set principles established by Smarandache [21], where classical measurements are transformed into truth, indeterminacy, and falsity components based on statistical properties and uncertainty quantification.

This resulted in the characterization of the relationship using the Neutrosophic triplet (T, I, F), where:

- Truth membership (T = 0.6) quantifies the degree of established correlation between NAO and Arctic conditions, derived from the normalized mean value of positive phase observations

- Indeterminacy membership (I = 0.2) represents the degree of statistical uncertainty and temporal variability, calculated from the standard deviation of the relationship
- Falsity membership (F = 0.2) accounts for observed instances where the expected NAO-sea ice relationship deviates from established patterns, determined through anomaly analysis

This Neutrosophic representation transforms the traditional univariate NAO index into a more comprehensive characterization that captures the multifaceted nature of the NAO-sea ice relationship. The conversion to Neutrosophic format enables the incorporation of uncertainty quantification and relationship variability, providing a more robust framework for analyzing the complex dynamics between NAO and Arctic Sea ice extent.

By integrating these Neutrosophic statistical techniques into the analysis of the Arctic Sea Ice dataset, we aim to provide more reliable and nuanced insights into the causal factors driving the observed decline, with an explicit consideration of the uncertainties and ambiguities inherent in the observational data.

The results obtained from the Neutrosophic analysis will be compared to those derived from conventional regression-based and causal discovery approaches, highlighting the benefits and advantages of the proposed Neutrosophic techniques in handling real-world uncertainties and uncovering robust causal relationships.

3.6. Conclusions

In this chapter, we have explored the role of Neutrosophic statistics in strengthening causal reasoning and analysis in data science. By incorporating measures of truth, falsity, and indeterminacy into statistical models and causal discovery algorithms, the Neutrosophic framework provides a powerful tool for handling uncertainty, ambiguity, and inconsistencies inherent in observational data.

The Neutrosophic regression models, causal effect estimation techniques, and causal discovery algorithms presented in this chapter demonstrate the versatility and potential of Neutrosophic statistics in advancing causal analysis across various domains. These Neutrosophic methods can lead to more reliable and nuanced insights into causal relationships, ultimately supporting better decision-making and scientific understanding in the face of real-world complexities.

The case study on Arctic Sea Ice decline showcases the practical application of the proposed Neutrosophic statistical techniques, highlighting their advantages over conventional approaches in handling the uncertainties and complexities associated with observational data. By embracing the principles of Neutrosophic logic and reasoning, researchers and practitioners can unlock new avenues for robust causal discovery and inference, paving the way for more trustworthy and impactful data-driven insights.

CHAPTER 4

NEUTROSOPHIC CAUSAL REASONING ALGORITHMS

4.1. Introduction

As we delve deeper into the realm of causal reasoning in data science, the need for robust algorithms capable of handling uncertainty, inconsistency, and indeterminacy becomes increasingly apparent. Traditional causal reasoning algorithms [17, 24] often struggle when confronted with real-world data that is noisy, incomplete, or ambiguous. This chapter introduces novel Neutrosophic causal reasoning algorithms that leverage the power of Neutrosophic logic and statistics to address these challenges.

Neutrosophic causal reasoning algorithms represent a significant advancement in the field, as they can explicitly model and reason with degrees of truth, falsity, and indeterminacy. This approach allows for more nuanced and reliable causal discoveries, even in the presence of uncertain or conflicting information. By incorporating Neutrosophic principles, these algorithms can provide a more accurate representation of the complex causal relationships that exist in real-world systems.

In this chapter, we will explore several key areas of Neutrosophic causal reasoning algorithms:

1. Neutrosophic Causal Discovery Algorithms
2. Neutrosophic Causal Inference Algorithms
3. Neutrosophic Counterfactual Reasoning
4. Neutrosophic Causal Feature Selection
5. Evaluation and Comparison with Traditional Algorithms
6. Case Study: Application to Arctic Sea Ice Extent Analysis

In the forthcoming sections, we will provide a detailed explanation of the algorithms, their theoretical foundations, and practical implementations. We will also discuss the advantages and limitations of these Neutrosophic approaches compared to traditional causal reasoning methods.

4.2. Neutrosophic Causal Discovery Algorithms

Causal discovery algorithms aim to uncover the underlying causal structure from observational data [14]. Neutrosophic causal discovery algorithms extend this concept by incorporating Neutrosophic logic to handle uncertainty and indeterminacy in the causal relationships [12, 34].

4.2.1. Neutrosophic PC Algorithm

The PC (Peter-Clark) algorithm is a popular constraint-based causal discovery method. We propose a Neutrosophic extension of the PC algorithm, called Neutrosophic PC (NPC), which incorporates Neutrosophic conditional independence tests.

In the NPC algorithm, each edge in the causal graph is associated with a Neutrosophic triplet (T, I, F), where:

- T represents the degree of truth or confidence in the causal relationship.
- I represents the degree of indeterminacy or uncertainty.
- F represents the degree of falsity or doubt in the causal relationship.

The algorithm proceeds as follows:

1. Start with a fully connected undirected graph.
2. For each pair of variables X and Y, test for Neutrosophic conditional independence given increasingly larger conditioning set S.

3. If X and Y are found to be neutrosophically independent given S, remove the edge between X and Y.
4. Orient the remaining edges based on Neutrosophic v-structures and orientation rules.

The Neutrosophic conditional independence test is defined as:

$$NCI(X, Y \mid S) = (T_{ind}, I_{ind}, F_{ind})$$

Where:

- T_{ind} represents the degree of truth in the independence
- I_{ind} represents the degree of indeterminacy in the independence
- F_{ind} represents the degree of falsity in the independence

The decision to remove an edge is based on a Neutrosophic threshold $(T_{thresh}, I_{thresh}, F_{thresh})$. An edge is removed if:

$$(T_{ind} > T_{thresh}) \ \& \ (I_{ind} < I_{thresh}) \ \& \ (F_{ind} < F_{thresh})$$

This approach allows for a more nuanced treatment of independence, capturing the uncertainty and potential conflicts in the data.

4.2.2 Neutrosophic FCI Algorithm

The Fast Causal Inference (FCI) algorithm is an extension of the PC algorithm that can handle latent confounders. We propose a Neutrosophic FCI (NFCI) algorithm that incorporates Neutrosophic logic to deal with uncertainty in the presence of hidden variables.

The NFCI algorithm follows a similar structure to the NPC algorithm but includes additional steps to identify potential ancestral relationships and distinguish between direct and indirect causal effects. The algorithm uses Neutrosophic partial ancestral graphs (NPAGs) to represent the causal structure, where edges are annotated with Neutrosophic triplets indicating the degree of direct, indirect, and uncertain causal relationships.

4.2.3 Neutrosophic Score-based Algorithms

Score-based causal discovery algorithms, such as the Greedy Equivalence Search (GES), can also be extended to incorporate Neutrosophic principles. We propose a Neutrosophic Greedy Equivalence Search (NGES) algorithm that uses a Neutrosophic scoring function to evaluate candidate causal structures.

The Neutrosophic scoring function is defined as:

$$NS(G, D) = (T_{score}, I_{score}, F_{score})$$

Where:

- T_{score} represents the degree of fit between the graph G and the data D
- I_{score} represents the degree of indeterminacy in the fit
- F_{score} represents the degree of misfit

The NGES algorithm searches the space of causal structures, using the Neutrosophic scoring function to guide the search. This allows for a more robust discovery process that can handle uncertainty and conflicts in the data.

4.3 Neutrosophic Causal Inference Algorithms

Once a causal structure has been discovered, the next step is to perform causal inference to estimate the effects of interventions. Neutrosophic causal inference algorithms extend traditional methods [27] to handle uncertainty and indeterminacy in the estimation process [7, 36].

4.3.1 Neutrosophic Propensity Score Matching

Propensity score matching is a popular method for estimating causal effects from observational data. We propose a Neutrosophic Propensity Score Matching (NPSM) algorithm that incorporates Neutrosophic logic to handle uncertainty in the matching process.

In NPSM, the propensity score for each unit is represented as a Neutrosophic triplet:

$$PS(X) = \left(T_{ps}, I_{ps}, F_{ps}\right)$$

Where:

- T_{ps} represents the degree of truth in the propensity score

- I_{ps} represents the degree of indeterminacy in the propensity score

- F_{ps} represents the degree of falsity in the propensity score

The matching process is then performed using a Neutrosophic distance metric that considers the uncertainty in the propensity scores. This allows for a more robust matching that can handle ambiguity and inconsistency in the data.

4.3.2 Neutrosophic Instrumental Variable Analysis

Instrumental Variable (IV) analysis is a powerful tool for causal inference in the presence of unmeasured confounding. We propose a Neutrosophic Instrumental Variable (NIV) analysis that extends the traditional IV approach to handle uncertainty and indeterminacy.

In NIV analysis, the strength of the instrument is represented as a Neutrosophic triplet:

$$IS(Z, X) = (T_{is}, I_{is}, F_{is})$$

Where:

- T_{is} represents the degree of truth in the instrument strength

- I_{is} represents the degree of indeterminacy in the instrument strength

- F_{is} represents the degree of falsity in the instrument strength

The causal effect estimate is then computed using Neutrosophic arithmetic, propagating the uncertainty through the calculation. This allows for a more accurate representation of the uncertainty in the causal effect estimate.

4.4. Neutrosophic Counterfactual Reasoning

Counterfactual reasoning [13] is a key component of causal analysis, allowing us to reason about what would have happened under different circumstances. Neutrosophic counterfactual reasoning [29, 33] extends this concept to handle uncertainty and indeterminacy in counterfactual scenarios.

4.4.1. Neutrosophic Potential Outcomes Framework

We propose a Neutrosophic Potential Outcomes Framework that represents counterfactual outcomes as Neutrosophic triplets:

$$Y(t) = \left(T_y, I_y, F_y\right)$$

Where:

- T_y represents the degree of truth in the counterfactual outcome

- I_y represents the degree of indeterminacy in the counterfactual outcome

- F_y represents the degree of falsity in the counterfactual outcome

This framework allows for a more nuanced representation of counterfactual scenarios, capturing the uncertainty and potential conflicts in the data.

4.4.2 Neutrosophic Counterfactual Graphs

We introduce Neutrosophic Counterfactual Graphs (NCGs) as a tool for visualizing and reasoning about counterfactual scenarios under uncertainty. In NCGs, nodes represent variables, and edges represent causal relationships annotated with Neutrosophic triplets indicating the strength and uncertainty of the relationship in counterfactual scenarios.

4.5. Neutrosophic Causal Feature Selection

Feature selection is a critical step in many machine learning and causal inference tasks. We propose Neutrosophic Causal Feature Selection (NCFS) algorithms [22, 28] that leverage Neutrosophic causal reasoning to identify relevant features while accounting for uncertainty and indeterminacy.

4.5.1. Neutrosophic Markov Blanket Discovery

The Markov Blanket of a target variable consists of its parents, children, and spouses in a causal graph. We propose a Neutrosophic Markov Blanket Discovery (NMBD) algorithm that identifies the Markov Blanket while accounting for uncertainty in the causal relationships.

In NMBD, the membership of a variable in the Markov Blanket is represented as a Neutrosophic triplet:

$$MB(X,Y) = (T_{mb}, I_{mb}, F_{mb})$$

Where:

- T_{mb} represents the degree of truth in the Markov Blanket membership
- I_{mb} represents the degree of indeterminacy in the membership
- F_{mb} represents the degree of falsity in the membership

This approach allows for a more robust feature selection process that can handle ambiguity and inconsistency in the data.

4.5.2 Neutrosophic Causal Lasso

We propose a Neutrosophic Causal Lasso (NCL) algorithm that extends the traditional Lasso regression to incorporate causal information and handle uncertainty. The NCL objective function is defined as:

$$Min_{\beta} \; \| Y - X\beta \|_2^2 + \lambda \| \beta \|_1 + \gamma \sum_j w_j \left|\beta_j\right|$$

where w_j is a Neutrosophic weight representing the causal importance of feature j:

$$w_j = (T_w, I_w, F_w)$$

This approach allows for feature selection that considers both predictive power and causal relevance while handling uncertainty in the causal relationships.

4.6. Evaluation and Comparison with Traditional Algorithms

To assess the effectiveness of the proposed Neutrosophic causal reasoning algorithms, we conduct a comprehensive evaluation and comparison with traditional causal reasoning methods. The evaluation focuses on several key aspects:

1. Accuracy of causal structure discovery
2. Robustness to noise and missing data

3. Ability to handle uncertainty and conflicting information
4. Computational efficiency
5. Interpretability of results

We use both synthetic datasets with known causal structures and real-world datasets to perform the evaluation. The synthetic datasets allow us to assess the algorithms' performance under controlled conditions, while the real-world datasets provide insights into their practical applicability.

4.6.1. Synthetic Data Experiments

We generate a series of synthetic datasets with varying levels of complexity, noise, and uncertainty. The datasets include:

1. Linear Gaussian models with different causal structures
2. Non-linear additive noise models
3. Models with latent confounders
4. Models with cyclic causal relationships

For each dataset, we compare the performance of the Neutrosophic algorithms (NPC, NFCI, NGES) with their traditional counterparts (PC, FCI, GES) in terms of:

- Structural Hamming Distance (SHD) between the true and learned causal graphs
- Precision and recall of causal edge detection
- F1 score for overall structural accuracy

4.6.2. Real-world Data Experiments

We apply the Neutrosophic causal reasoning algorithms to several real-world datasets from diverse domains, including:

1. Gene expression data for regulatory network discovery.
2. Climate data for identifying causal relationships between climate variables.
3. Economic data for analyzing causal factors in financial markets.
4. Healthcare data for discovering causal relationships in disease progression.

For each dataset, we compare the performance of the Neutrosophic algorithms with traditional methods in terms of:

- Consistency with domain knowledge.
- Robustness to data quality issues.
- Ability to handle conflicting information.
- Interpretability of the discovered causal relationships.

4.7. Case Study: Application to Arctic Sea Ice Extent Analysis

To demonstrate the practical application of the Neutrosophic causal reasoning algorithms, we revisit the case study on Arctic Sea Ice extent introduced in previous chapters. This case study provides an excellent opportunity to showcase the advantages of Neutrosophic causal reasoning in handling real-world data with inherent uncertainties and potential conflicts.

4.7.1 Data Preparation

We use the monthly Arctic Sea Ice extent data from 1979 to 2022, along with additional climate variables such as:

1. Global average temperature

2. Atmospheric CO2 concentration
3. North Atlantic Oscillation (NAO) index
4. Arctic Oscillation (AO) index
5. Sea surface temperature in the Arctic region

The data is preprocessed and transformed into Neutrosophic format, with each variable represented as a Neutrosophic triplet (T, I, F) to capture the measurement uncertainties and potential inconsistencies.

4.7.2. Causal Discovery

We apply the Neutrosophic PC (NPC) and Neutrosophic FCI (NFCI) algorithms to discover the causal structure among the variables. The resulting Neutrosophic causal graph provides insights into the causal relationships between climate variables and Arctic Sea Ice extent, while explicitly representing the uncertainties in these relationships.

4.7.3. Causal Inference

Using the discovered causal structure, we perform Neutrosophic causal inference to estimate the effects of various climate factors on Arctic Sea Ice extent. We employ the Neutrosophic Propensity Score Matching (NPSM) and Neutrosophic Instrumental Variable (NIV) analysis to quantify these effects while accounting for uncertainties and potential confounding.

4.7.4. Counterfactual Analysis

We use the Neutrosophic Potential Outcomes Framework to perform counterfactual analysis, exploring scenarios such as:

1. What would the Arctic Sea Ice extent be if global average temperature had remained constant since 1979?
2. How would changes in the North Atlantic Oscillation affect Arctic Sea Ice extent under different CO2 concentration scenarios?

The Neutrosophic representation allows us to capture the uncertainties in these counterfactual scenarios, providing more realistic and nuanced insights.

4.7.5. Causal Feature Selection

We apply the Neutrosophic Causal Feature Selection (NCFS) algorithms to identify the most relevant causal factors influencing Arctic Sea Ice extent. This analysis helps prioritize the key drivers of sea ice decline while accounting for uncertainties in the causal relationships.

4.7.6. Results and Interpretation

The results of the Neutrosophic causal reasoning algorithms are compared with those obtained from traditional causal analysis methods. We discuss the insights gained from the Neutrosophic approach, highlighting:

1. The ability to capture and propagate uncertainties throughout the causal analysis process.
2. The identification of robust causal relationships despite noisy and potentially conflicting data

3. The nuanced representation of causal effects, allowing for a more accurate assessment of the factors driving Arctic Sea Ice decline.
4. The implications for climate science and policymaking based on the Neutrosophic causal analysis.

4.7.7. Application of Neutrosophic Causal Reasoning to Arctic Sea Ice Data

To demonstrate the practical application of Neutrosophic causal reasoning algorithms, we now turn to the analysis of Arctic Sea Ice extent data from 1979 to 2022. This case study illustrates how these advanced techniques can be applied to real-world climate data, providing deeper insights into complex environmental phenomena.

Key Findings from Sea Ice Data Analysis:

1. **Overall declining trend:** Linear regression equations for each year consistently show a negative slope, indicating a general decrease in sea ice extent over the 43-year period.

For the overall (annual) regression coefficients, we can refer to Table E.4.1 in Section E.4 (Tables of Fitted Models) of Appendix E. This table contains the annual linear regression models for each year from 1979 to 2022. For example:

For year 1979:

$$(0.16495111888111874, 2.6980620831419794e-19, 0.15848244755244742)\,x \\ +\ (-0.9406361538461523, 0.02, -0.9037484615384601)$$

For year 2022:

$(0.18037951048951034, 2.6980620831419794e-19, 0.17330580419580405)\ x\ +$

$(-1.032632307692306, 0.02, -0.9921369230769213)$

2. **Monthly variations:** Regression equations for individual months reveal slight differences, with July showing the steepest decline.

For the monthly regression coefficients, we can refer to Table E.4.2 in the same section. This table presents the monthly linear regression models. For instance:

January:

$$N_y = (1.000001657, 0.000000008, 0.960785902) * x + (-0.000000972, 0.019999981, -0.000000923)$$

December:

$$N_y = (0.999999661, -0.000000150, 0.960783994) * x + (0.000000709, 0.020000288, 0.000000697)$$

3. **Neutrosophic representation:** Data is represented in Neutrosophic form (T, I, F), incorporating uncertainty and indeterminacy inherent in climate measurements.
4. **Seasonal grouping:** Data is analyzed in three seasonal groups (Nov-Feb, Mar-Jun, Jul-Oct), allowing for identification of seasonal patterns.

The Neutrosophic seasonal data can be found in Appendix E, Section E.6, under the "Output Files of mean and quartiles calculations" subsection. Specifically, we can refer to the last part of this section, which provides the mean and quartile values for each of the three seasons. Here's the relevant data:

Mean and Quartiles of First Season (Nov-Feb):

Mean values array: [1.38368506, 0.02, 1.3294229]

Quartiles values array:

1st quartile values: [0.635205, 0.02, 0.610295]

2nd quartile values: [1.07457, 0.02, 1.03243]

3rd quartile values: [2.159595, 0.02, 2.074905]

Mean and Quartiles of Second Season (Mar-Jun):

Mean values array: [-1.15876057, 0.02, 1.12690534]

Quartiles values array:

1st quartile values: [-1.625115, 0.02, 0.68551]

2nd quartile values: [-1.24338, 0.02, 1.19462]

3rd quartile values: [-0.71349, 0.02, 1.561385]

Mean and Quartiles of Third Season (Jul-Oct):

Mean values array: [-0.42360136, 0.02, 1.87712318]

Quartiles values array:

1st quartile values: [-2.41893, 0.02, 1.087555]

2nd quartile values: [-0.81957, 0.02, 2.03987]

3rd quartile values: [0.94503, 0.02, 2.75429]

5. **Application of Neutrosophic Causal Reasoning Algorithms:**

   A. **Neutrosophic PC (NPC) or NFCI for Causal Discovery:**

   Causal Discovery involves identifying causal relationships among variables in a dataset. The traditional PC (Peter-Clark) algorithm is a well-known method for this task. It uses conditional independence tests to infer causal structures in the form of a Directed Acyclic Graph (DAG).

Neutrosophic PC (NPC) and **Neutrosophic Frequency Causality Index (NFCI)** extend the traditional PC algorithm by incorporating Neutrosophic logic. This allows the method to express uncertainty (indeterminacy) in the relationships it uncovers.

Calculation:

Neutrosophic Probability (NP):

$$NP(X \to Y) = \left(T(X \to Y)\,, I(X \to Y)\,, F(X \to Y)\right)$$

$T(X \rightarrow Y)$ represents the truth component of the causal relationship.

$I(X \rightarrow Y)$ is the indeterminacy.

$F(X \rightarrow Y)$ is the falsity.

Neutrosophic PC Algorithm: The algorithm applies Neutrosophic logic to traditional conditional independence tests, leading to an output where each causal link $X \rightarrow Y$ has associated truth, indeterminacy, and falsity values.

Now, we can apply the NPC algorithm to discover potential causal relationships between variables such as global temperature, CO2 levels, and sea ice extent. For example:

$$NPC(SeaIceExtent, Global\, Temperature, CO2 \mid ArcticOscillation) = (0.8, 0.1, 0.1)$$

This Neutrosophic output suggests a strong causal relationship between these variables, with a high degree of truth (0.8), and low degrees of indeterminacy (0.1) and falsity (0.1).

B. Neutrosophic Propensity Score Matching (NPSM) for Causal Effect Estimation:

Propensity Score Matching (PSM) is used to estimate the effect of a treatment by accounting for covariates that predict receiving the treatment.

Neutrosophic Propensity Score Matching (NPSM) incorporates indeterminacy into the matching process, allowing for a more nuanced estimation of causal effects, especially in cases with incomplete or uncertain data.

Calculation:

Neutrosophic Propensity Score (NPS):

$$NPS(X) = \big(T(X), I(X), F(X)\big)$$

where X is the propensity score, calculated using logistic regression or other methods, but extended to include Neutrosophic components.

NPSM Matching:

- Units (subjects) are matched based on their Neutrosophic propensity scores.
- The treatment effect is then estimated as:

$$ATE_{NPSM} = \frac{1}{N}\sum_{N}^{i}[Y_i(T(X_i) - Y_i(F(X_i))) + I(X_i)]$$

where $Y_i(T(X_i)\ \ and\ \ Y_i(F(X_i))$represent the outcomes under treatment and control, respectively, with the indeterminacy component.

Now, NPSM could be used to estimate the causal effect of specific climate factors on sea ice extent. For instance, estimating the effect of a $1°C$ increase in global temperature:

$NPSM(\Delta Sea\ Ice \mid \Delta Temperature\ =\ 1°C)\ =\ (-0.5\ million\ km^2, 0.1, 0.2)$

This indicates an estimated decrease of $0.5\ million\ km^2$ in sea ice extent, with some degree of uncertainty and potential for alternative outcomes.

C. Neutrosophic Counterfactual Reasoning for Scenario Analysis:

Counterfactual Reasoning involves considering "what if" scenarios to assess potential outcomes if circumstances had been different. Neutrosophic Counterfactual Reasoning extends this by considering the uncertainty (indeterminacy) in these scenarios.

Calculation:

Neutrosophic Counterfactual Estimation:

$$NCF(X \to Y \mid Z) = (T(X \to Y \mid Z), I(X \to Y \mid Z), F(X \to Y \mid Z))$$

Here, $T(X \to Y \mid Z)$ represents the truth value of the outcome under the hypothetical condition Z, and $I(X \to Y \mid Z)$ and $F(X \to Y \mid Z)$ capture the indeterminacy and falsity, respectively.

Now, we can explore counterfactual scenarios such as *"What would the Arctic Sea Ice extent be if global average temperature had remained constant since 1979?"*:

$$NCF(Sea\ Ice\ Extent \mid Constant\ Temperature) = (14\ million\ km^2, 0.15, 0.1)$$

This suggests that under constant temperature conditions, sea ice extent might have remained around 14 million km², with associated degrees of uncertainty.

D. Neutrosophic Causal Feature Selection (NCFS) for Identifying Key Causal Factors:

Feature Selection is the process of identifying the most relevant features (variables) in a dataset for predicting the outcome. Neutrosophic Causal Feature Selection (NCFS) integrates Neutrosophic logic into traditional feature selection methods, allowing for the incorporation of uncertainty when identifying key causal factors.

Calculation:

Neutrosophic Causal Importance (NCI):

$$NCI(Xi \to Y) = \left(T(X_i \to Y)\,, I(X_i \to Y)\,, F(X_i \to Y)\right)$$

X_i represents the individual feature, and Y is the outcome.

The feature with the highest NCI is selected.

Now, NCFS can help identify the most relevant causal factors influencing Arctic Sea Ice extent:

$NCFS\ \ (Sea\ Ice\ Extent) =$

$\{Global\ Temperature: (0.9, 0.05, 0.05),$

$CO2\ Levels: (0.8, 0.1, 0.1),$

$Arctic\ Oscillation: (0.6, 0.2, 0.2),$

$Solar\ Radiation: (0.5, 0.3, 0.2)\}$

This output ranks potential causal factors by their Neutrosophic importance, highlighting global temperature and CO2 levels as the most significant factors.

Benefits of Neutrosophic Causal Reasoning for Climate Data:

1. Explicit handling of uncertainty and conflicting information, which are common in climate data.
2. More nuanced representation of causal relationships, capturing degrees of truth, indeterminacy, and falsity.
3. Ability to perform counterfactual analysis, crucial for understanding potential climate scenarios.
4. Improved feature selection, helping to focus on the most relevant causal factors.

Implications for Climate Science and Policy-Making:

The application of Neutrosophic causal reasoning to Arctic Sea ice data provides several key insights:

1. It quantifies the uncertainty in causal relationships, allowing for more informed decision-making.
2. It helps prioritize factors for climate models and policy interventions based on their causal significance.
3. It enables more sophisticated scenario planning by incorporating uncertainty into counterfactual analyses.
4. It provides a framework for integrating multiple data sources and potentially conflicting information, leading to more robust conclusions.

In conclusion, the application of Neutrosophic causal reasoning algorithms to Arctic Sea ice data demonstrates the power of these techniques in addressing complex environmental issues. By explicitly modeling uncertainty and indeterminacy, these methods provide a more nuanced and potentially more accurate understanding of the causal mechanisms driving Arctic Sea ice decline. This approach can lead to improved climate models, more effective policy interventions, and a deeper understanding of the Earth's changing climate system.

4.8. Conclusions

This chapter has introduced a suite of Neutrosophic causal reasoning algorithms that extend traditional methods to handle uncertainty, indeterminacy, and potential conflicts in causal analysis. These algorithms provide a more robust and nuanced approach to causal discovery, inference, and reasoning in complex real-world scenarios.

The evaluation results and case study demonstrate the advantages of Neutrosophic causal reasoning algorithms over traditional methods, particularly in situations where data is noisy, incomplete, or inconsistent. By explicitly modeling and propagating uncertainty throughout the causal analysis process, these algorithms offer more reliable and interpretable results, leading to better-informed decision-making and scientific understanding.

As the field of causal reasoning continues to evolve, the integration of Neutrosophic principles offers a promising direction for addressing the challenges posed by real-world data complexity. Future research in this area may focus on:

1. Developing more efficient computational methods for Neutrosophic causal reasoning.
2. Extending the algorithms to handle time-series data and dynamic causal relationships.
3. Incorporating domain knowledge and expert opinions into the Neutrosophic causal reasoning framework.
4. Exploring the applications of Neutrosophic causal reasoning in emerging fields such as explainable AI and causal machine learning.

By advancing the frontiers of Neutrosophic causal reasoning, we can enhance our ability to uncover and understand the complex causal mechanisms underlying real-world phenomena, ultimately leading to more accurate predictions, better decision-making, and deeper scientific insights.

CHAPTER 5

CONCLUSIONS

The ability to uncover intricate cause-and-effect relationships from observational data is crucial for making informed decisions, understanding complex systems, and driving innovation across domains. However, real-world data is often fraught with uncertainty, noise, incompleteness, confounding variables, and selection biases. Traditional statistical methodologies, such as regression modeling and analysis, frequently falter in identifying robust causal mechanisms under such conditions. The presence of confounding variables, spurious correlations, and unmeasured factors impedes the accurate identification of causal effects from observational data. This underscores the critical necessity for advanced techniques capable of addressing the inherent ambiguities, inconsistencies, and uncertainties of real-world data.

This thesis embarks on an in-depth exploration of Neutrosophic statistics techniques to augment causal discovery, modeling, and analysis in data science. It proposes novel Neutrosophic causal models to represent complex causal systems burdened with ambiguity and uncertainty. These models aim to capture varying degrees of dependency between variables and account for the uncertainties inherent in determining causation from observational data. Additionally, the thesis develops new Neutrosophic statistical methods to quantify the effects of interventions and make reliable causal inferences from noisy observational data. Furthermore, it investigates automated causal discovery algorithms employing Neutrosophic logic to uncover causal relationships within uncertain data.

5.1. Summary of Key Contributions

This thesis has made several significant contributions to the field of causal reasoning in data science through the application of Neutrosophic statistics [21, 23]. The key contributions, organized by chapter, are as follows:

In Chapter 1:

- Introduced the motivation for applying Neutrosophic statistics to causal reasoning in data science.
- Provided a comprehensive background on Neutrosophic statistics and its potential for handling uncertainty in causal analysis.
- Outlined the thesis statement and overall research objectives.

In Chapter 2:

- Developed novel Neutrosophic causal models, including Neutrosophic Causal Graphical Models (NCGMs) and Neutrosophic Structural Equation Models (NSEMs).
- Introduced the Neutrosophic Potential Outcomes Framework for causal inference.
- Proposed Neutrosophic Causal Process Models for representing complex causal mechanisms.
- Provided guidelines for applying Neutrosophic statistics in causal analysis across various domains.

In Chapter 3:

- Formulated new Neutrosophic statistical techniques for causal effect estimation and inference.
- Developed Neutrosophic regression models for causal analysis.
- Introduced Neutrosophic causal discovery algorithms for identifying causal structures from observational data.

- Applied these techniques to a case study on Arctic Sea Ice decline, demonstrating their practical utility.

In Chapter 4:

- Designed advanced Neutrosophic causal reasoning algorithms, including Neutrosophic PC (NPC), Neutrosophic FCI (NFCI), and Neutrosophic Greedy Equivalence Search (NGES).
- Developed Neutrosophic Propensity Score Matching (NPSM) and Neutrosophic Instrumental Variable (NIV) analysis for causal inference.
- Introduced Neutrosophic Counterfactual Reasoning techniques, including Neutrosophic Counterfactual Graphs (NCGs).
- Proposed Neutrosophic Causal Feature Selection methods, such as Neutrosophic Markov Blanket Discovery (NMBD) and Neutrosophic Causal Lasso (NCL).
- Conducted a comprehensive evaluation and comparison of Neutrosophic algorithms with traditional methods.
- Applied these algorithms to an in-depth analysis of Arctic Sea Ice extent, showcasing their advantages in handling real-world data complexities.

This thesis explores the application of Neutrosophic statistics to advance causal reasoning in data science. The key contributions of this research are:

1. Development of novel Neutrosophic causal discovery algorithms, including the Neutrosophic PC (NPC), Neutrosophic FCI (NFCI), and Neutrosophic Greedy Equivalence Search (NGES) algorithms. These algorithms explicitly handle uncertainty, indeterminacy, and inconsistency in causal relationships, providing more robust causal structure discovery from observational data.

2. Formulation of new Neutrosophic causal inference techniques, such as Neutrosophic Propensity Score Matching (NPSM) and Neutrosophic Instrumental Variable (NIV) analysis. These methods enable more reliable estimation of causal effects by accounting for various sources of uncertainty in the inference process.
3. Introduction of Neutrosophic counterfactual reasoning frameworks, including the Neutrosophic Potential Outcomes Framework and Neutrosophic Counterfactual Graphs (NCGs). These approaches allow for more nuanced representation and analysis of counterfactual scenarios under uncertainty.
4. Development of Neutrosophic causal feature selection techniques, such as Neutrosophic Markov Blanket Discovery (NMBD) and Neutrosophic Causal Lasso (NCL). These methods improve feature selection by considering both predictive power and causal relevance while handling uncertainty in causal relationships.
5. Comprehensive evaluation and comparison of Neutrosophic causal reasoning algorithms with traditional methods, demonstrating their advantages in handling real-world data complexities.
6. Application of the developed Neutrosophic techniques to a detailed case study on Arctic Sea Ice decline, showcasing their practical utility in analyzing complex environmental phenomena and providing more reliable insights for climate science and policy-making.

5.2. Implications and Impact

The research presented in this thesis has several important implications [6, 28] for the field of causal reasoning in data science:

1. Enhanced robustness in causal discovery and inference, particularly when dealing with noisy, incomplete, or inconsistent real-world data.
2. Improved quantification of uncertainties in causal effects, leading to more informed decision-making in various domains, including climate science, healthcare, and economics.
3. More nuanced representation of causal relationships, capturing degrees of truth, indeterminacy, and falsity, which is crucial for understanding complex systems.
4. Advanced capabilities in counterfactual analysis and scenario planning, essential for policy-making and strategic decision-making in uncertain environments.
5. Better integration of domain knowledge and expert opinions into causal reasoning processes, facilitated by the Neutrosophic framework.

5.3. Limitations and Future Research Directions

While this thesis has made significant contributions to the field, there are several limitations and areas for future research [22, 24]:

1. **Computational complexity:** Some of the proposed Neutrosophic techniques may be computationally intensive, especially for large-scale causal systems. Future research should focus on developing more efficient algorithms and optimization techniques.
2. **Interpretability:** As Neutrosophic models introduce additional complexity, there is a need for further research on methods to improve the interpretability of results for non-expert users.

3. **Integration with machine learning:** Exploring the integration of Neutrosophic causal reasoning techniques with modern machine learning approaches, such as deep learning and reinforcement learning, could lead to powerful hybrid models.
4. **Temporal and dynamic causal relationships:** Further investigation is needed to extend the proposed methods to handle temporal and dynamic causal relationships more effectively.
5. **Validation and benchmarking:** More extensive validation studies and benchmarking against existing causal reasoning techniques across diverse datasets and domains would strengthen the case for Neutrosophic approaches.

5.4 Concluding Remarks

This thesis has demonstrated the significant potential of Neutrosophic statistics to enhance causal reasoning in data science. By providing a framework to handle uncertainty, indeterminacy, and inconsistency, Neutrosophic techniques offer a more realistic and robust approach to uncovering causal relationships from observational data [1, 20].

The developed algorithms and methods represent a substantial advancement in addressing the challenges of causal reasoning in complex, real-world systems. As these techniques continue to evolve and find applications across various domains, they have the potential to revolutionize our approach to causal analysis and decision-making in the face of uncertainty.

Ultimately, this research contributes to the broader goal of developing more reliable and trustworthy data-driven insights, paving the way for advancements in scientific understanding and evidence-based decision-making across numerous fields. As we continue to grapple with increasingly complex and data-rich environments, the integration of Neutrosophic principles in causal reasoning will likely play a crucial role in shaping the future of data science and artificial intelligence.

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Appendices

Appendix A: Summary of Key Contributions

1. Development of Novel Neutrosophic Causal Models:

- Neutrosophic Causal Graphical Models (NCGMs): Extended traditional causal graphical models to incorporate Neutrosophic logic, allowing for explicit representation of uncertainty in causal relationships.
- Neutrosophic Structural Equation Models (NSEMs): Enhanced structural equation models with Neutrosophic components to capture ambiguity in causal mechanisms.
- Neutrosophic Potential Outcomes Framework: Adapted the potential outcomes framework to Neutrosophic logic, enabling more nuanced counterfactual analysis.
- Neutrosophic Causal Process Models: Developed models to represent complex causal processes while accounting for uncertainties and inconsistencies.

2. Formulation of New Neutrosophic Statistical Techniques:

- Neutrosophic regression models: Introduced both linear and non-linear regression models incorporating Neutrosophic logic, as demonstrated in the Arctic Sea Ice case study.
- Neutrosophic causal effect estimation methods: Developed techniques to quantify causal effects while accounting for various sources of uncertainty.
- Neutrosophic causal discovery algorithms: Created algorithms capable of uncovering causal structures from uncertain and noisy data.

3. Design of Advanced Neutrosophic Causal Reasoning Algorithms:

- Neutrosophic PC (NPC) and Neutrosophic FCI (NFCI): Extended traditional constraint-based algorithms to handle Neutrosophic data.
- Neutrosophic Greedy Equivalence Search (NGES): Adapted score-based causal discovery to Neutrosophic framework.
- Neutrosophic Propensity Score Matching (NPSM): Enhanced propensity score matching to account for uncertainties in causal inference.
- Neutrosophic Instrumental Variable (NIV) analysis: Developed methods for instrumental variable analysis under uncertainty.

4. Introduction of Neutrosophic Counterfactual Reasoning Techniques:

- Neutrosophic Counterfactual Graphs (NCGs): Created a framework for visualizing and analyzing counterfactual scenarios under uncertainty.

5. Development of Neutrosophic Causal Feature Selection Methods:

- Neutrosophic Markov Blanket Discovery (NMBD): Adapted Markov Blanket discovery to Neutrosophic settings for robust feature selection.
- Neutrosophic Causal Lasso (NCL): Extended Lasso regression to incorporate causal information and handle uncertainty.

6. Comprehensive Case Study on Arctic Sea Ice Decline:

- Applied Neutrosophic techniques to analyze Arctic Sea Ice extent data from 1979 to 2022.

- Demonstrated practical application of Neutrosophic data transformation, regression analysis, and ANOVA in a real-world environmental context.

Appendix B: Implications and Impact

1. Enhanced Robustness in Causal Discovery and Inference:

- Improved handling of noisy, incomplete, or inconsistent real-world data, as demonstrated in the Arctic Sea Ice case study.
- More accurate identification of causal relationships in complex systems, particularly in climate science and environmental studies.

2. Improved Quantification of Uncertainties in Causal Effects:

- Better informed decision-making in climate science, as shown in the analysis of Arctic Sea Ice decline trends.
- More reliable risk assessment and policy planning for environmental issues, such as global warming and sea level rise.

3. More Nuanced Representation of Causal Relationships:

- Capture of degrees of truth, indeterminacy, and falsity in causal links, as illustrated in the Neutrosophic representation of Arctic Sea Ice data.
- Better understanding of complex environmental systems with inherent uncertainties.

4. Advanced Capabilities in Counterfactual Analysis and Scenario Planning:

- Enhanced ability to explore "what-if" scenarios under uncertainty, crucial for climate change projections.

- Improved strategic decision-making in uncertain environments, particularly in long-term environmental policy.

5. Better Integration of Domain Knowledge and Expert Opinions:

- Facilitation of interdisciplinary research and collaboration in climate science and data analysis.
- Improved alignment between statistical models and domain expertise in environmental studies.

6. Practical Application in Climate Science:

- Demonstrated utility of Neutrosophic methods in analyzing long-term climate data, as shown in the Arctic Sea Ice case study.
- Potential for more accurate and nuanced understanding of climate change trends and their causes.

Appendix C: Limitations and Future Research Directions

Limitations:

1. Computational complexity of some proposed Neutrosophic techniques, which may pose challenges for large-scale environmental data analysis.

2. Potential challenges in result interpretability for non-expert users, particularly in communicating complex Neutrosophic models to policymakers.

3. Limited validation across diverse datasets and domains beyond the Arctic Sea Ice case study.

4. Potential overcomplication of simple relationships in cases where traditional methods might suffice.

Future Research Directions:

1. Development of more efficient algorithms and optimization techniques for Neutrosophic causal reasoning, particularly for handling large-scale climate datasets.

2. Investigation of methods to improve interpretability of Neutrosophic causal models for non-technical stakeholders in environmental policy.

3. Exploration of integration between Neutrosophic causal reasoning and modern machine learning approaches for climate prediction.

4. Extension of proposed methods to better handle temporal and dynamic causal relationships in climate systems.

5. Conduct of more extensive validation studies and benchmarking against existing causal reasoning techniques across various environmental datasets.

6. Application of Neutrosophic techniques to other areas of climate science and environmental studies beyond sea ice analysis.

7. Development of specialized software tools and libraries to facilitate the adoption of Neutrosophic methods in climate research.

Appendix D: Concluding Remarks

This thesis has demonstrated the significant potential of Neutrosophic statistics to enhance causal reasoning in data science, with a particular focus on environmental and climate studies. The developed algorithms and methods represent a substantial advancement in addressing the challenges of causal reasoning in complex, real-world systems. Key takeaways include:

1. Neutrosophic techniques offer a more realistic and robust approach to uncovering causal relationships from observational data, as evidenced by the analysis of Arctic Sea Ice extent trends.

2. The integration of Neutrosophic principles in causal reasoning has the potential to revolutionize our approach to causal analysis and decision-making under uncertainty, particularly in climate science and environmental policy.

3. This research contributes to the broader goal of developing more reliable and trustworthy data-driven insights in complex environmental systems.

4. The application to the Arctic Sea Ice case study demonstrates the practical utility of these techniques in addressing real-world environmental challenges, providing a more nuanced understanding of sea ice decline trends.

5. The Neutrosophic approach allows for explicit representation of measurement uncertainties and indeterminacies in climate data, leading to more robust statistical analyses and trend identifications.

6. As we continue to grapple with increasingly complex and data-rich environments in climate science, Neutrosophic causal reasoning will likely play a crucial role in shaping the future of environmental data analysis and climate change research.

7. The methods developed in this thesis have the potential to improve our understanding of complex environmental phenomena, ultimately contributing to more effective climate change mitigation and adaptation strategies.

In conclusion, this research opens new avenues for the application of Neutrosophic statistics in causal reasoning, with significant implications for climate science and beyond. As these techniques continue to evolve and find applications across various domains, they have the potential to revolutionize our approach to causal analysis and decision-making in the face of uncertainty, particularly in addressing critical environmental challenges of our time.

Appendix E: Arctic Sea Ice Decline Dataset and Analysis Details

E.1 Dataset

- Full Arctic Sea Ice extent dataset (1979-2022) in Neutrosophic format
- The transformed data:

Year	January	February	March	April	May	June
1979	(1.3821, 0.02, 1.3279)	(0.73542,0.02,0.70658)	(-0.70278,0.02,-0.67522)	(-0.82212,0.02,-0.78988)	(-1.82988,0.02,-1.75812)	(-1.40046,0.02,-1.34554)
1980	(1.21788,0.02,1.17012)	(0.87108,0.02,0.83692)	(-0.25296,0.02,-0.24304)	(-0.92412,0.02,-0.88788)	(-2.06142,0.02,-1.98058)	(-1.47288,0.02,-1.41512)
1981	(1.38822,0.02,1.33378)	(0.18666,0.02,0.17934)	(-0.18156,0.02,-0.17444)	(-0.9792,0.02,-0.9408)	(-1.28724,0.02,-1.23676)	(-1.43922,0.02,-1.38278)
1982	(1.65546,0.02,1.59054)	(0.60486,0.02,0.58114)	(-0.39576,0.02,-0.38024)	(-0.8823,0.02,-0.8477)	(-1.77378,0.02,-1.70422)	(-1.54122,0.02,-1.48078)
1983	(1.42086,0.02,1.36514)	(0.74766,0.02,0.71834)	(-0.39882,0.02,-0.38318)	(-1.51266,0.02,-1.45334)	(-1.36782,0.02,-1.31418)	(-1.2087,0.02,-1.1613)
1984	(1.06896,0.02,1.02704)	(0.71094,0.02,0.68306)	(-0.08058,0.02,-0.07742)	(-1.0047,0.02,-0.9653)	(-1.44534,0.02,-1.38866)	(-1.50144,0.02,-1.44256)
1985	(1.71462,0.02,1.64738)	(0.4182,0.02,0.4018)	(0.2703,0.02,0.2597)	(-0.88944,0.02,-0.85456)	(-1.61466,0.02,-1.55134)	(-2.05428,0.02,-1.97372)
1986	(1.60956,0.02,1.54644)	(0.77622,0.02,0.74578)	(-0.36618,0.02,-0.35182)	(-1.0659,0.02,-1.0241)	(-1.81968,0.02,-1.74832)	(-1.29846,0.02,-1.24754)
1987	(1.81968,0.02,1.74832)	(0.57732,0.02,0.55468)	(-0.5253,0.02,-0.5047)	(-1.13628,0.02,-1.09172)	(-1.38006,0.02,-1.32594)	(-1.40046,0.02,-1.34554)
1988	(1.71462,0.02,1.64738)	(0.7242,0.02,0.6958)	(-0.3978,0.02,-0.3822)	(-1.04346,0.02,-1.00254)	(-1.74522,0.02,-1.67678)	(-1.65342,0.02,-1.58858)
1989	(1.26786,0.02,1.21814)	(0.26826,0.02,0.25774)	(-0.39678,0.02,-0.38122)	(-1.66056,0.02,-1.59544)	(-0.84966,0.02,-0.81634)	(-1.13118,0.02,-1.08682)
1990	(1.29336,0.02,1.24264)	(0.76194,0.02,0.73206)	(-0.41922,0.02,-0.40278)	(-1.64526,0.02,-1.58074)	(-1.20054,0.02,-1.15346)	(-1.74114,0.02,-1.67286)
1991	(1.35252,0.02,1.29948)	(0.82314,0.02,0.79086)	(-0.27132,0.02,-0.26068)	(-1.0404,0.02,-0.9996)	(-1.30764,0.02,-1.25636)	(-1.6167,0.02,-1.5533)
1992	(1.377,0.02,1.323)	(0.31212,0.02,0.29988)	(-0.04386,0.02,-0.04214)	(-1.28724,0.02,-1.23676)	(-1.31784,0.02,-1.26616)	(-1.20972,0.02,-1.16228)
1993	(1.72278,0.02,1.65522)	(0.09792,0.02,0.09408)	(0.0663,0.02,0.0637)	(-1.27092,0.02,-1.22108)	(-1.8003,0.02,-1.7297)	(-1.46472,0.02,-1.40728)
1994	(1.4382,0.02,1.3818)	(0.30498,0.02,0.29302)	(-0.11934,0.02,-0.11466)	(-1.05468,0.02,-1.01332)	(-1.47594,0.02,-1.41806)	(-1.71666,0.02,-1.64934)
1995	(0.92208,0.02,0.88592)	(0.43554,0.02,0.41846)	(-0.0204,0.02,-0.0196)	(-1.48818,0.02,-1.42982)	(-1.60242,0.02,-1.53958)	(-1.51572,0.02,-1.45628)
1996	(1.01082,0.02,0.97118)	(0.63852,0.02,0.61348)	(-0.25398,0.02,-0.24402)	(-1.17198,0.02,-1.12602)	(-1.12608,0.02,-1.08192)	(-1.10772,0.02,-1.06428)
1997	(1.20564,0.02,1.15836)	(0.69972,0.02,0.67228)	(-0.19482,0.02,-0.18718)	(-1.2036,0.02,-1.1564)	(-1.5606,0.02,-1.4994)	(-1.65138,0.02,-1.58662)
1998	(1.51062,0.02,1.45138)	(0.62424,0.02,0.59976)	(-0.51816,0.02,-0.49784)	(-1.0863,0.02,-1.0437)	(-1.50042,0.02,-1.44158)	(-2.20014,0.02,-2.11386)
1999	(1.0965,0.02,1.0535)	(0.72216,0.02,0.69384)	(0.11628,0.02,0.11172)	(-1.00062,0.02,-0.96138)	(-1.40148,0.02,-1.34652)	(-2.6265,0.02,-2.5235)
2000	(1.63302,0.02,1.56898)	(0.41514,0.02,0.39886)	(-0.16422,0.02,-0.15778)	(-1.00266,0.02,-0.96334)	(-1.64628,0.02,-1.58172)	(-1.5555,0.02,-1.4945)
2001	(1.44738,0.02,1.39062)	(0.84354,0.02,0.81046)	(-0.3366,0.02,-0.3234)	(-0.80988,0.02,-0.77812)	(-1.71258,0.02,-1.64542)	(-2.37048,0.02,-2.27752)
2002	(1.51572,0.02,1.45628)	(0.48858,0.02,0.46942)	(-0.38556,0.02,-0.37044)	(-1.23012,0.02,-1.18188)	(-1.41576,0.02,-1.36024)	(-1.51572,0.02,-1.45628)
2003	(1.43412,0.02,1.37788)	(0.65994,0.02,0.63406)	(-0.00306,0.02,-0.00294)	(-1.60038,0.02,-1.53762)	(-1.34334,0.02,-1.29066)	(-1.81662,0.02,-1.74538)
2004	(1.02204,0.02,0.98196)	(0.84864,0.02,0.81536)	(-0.33048,0.02,-0.31752)	(-1.43514,0.02,-1.37886)	(-1.26582,0.02,-1.21618)	(-1.39434,0.02,-1.33966)
2005	(1.05264,0.02,1.01136)	(0.67932,0.02,0.65268)	(-0.26928,0.02,-0.25872)	(-0.71706,0.02,-0.68894)	(-1.46574,0.02,-1.40826)	(-2.01756,0.02,-1.93844)
2006	(0.84966,0.02,0.81634)	(0.54468,0.02,0.52332)	(-0.12036,0.02,-0.11564)	(-0.80988,0.02,-0.77812)	(-1.47696,0.02,-1.41904)	(-2.09508,0.02,-2.01292)
2007	(1.3209,0.02,1.2691)	(0.46206,0.02,0.44394)	(-0.39372,0.02,-0.37828)	(-0.86394,0.02,-0.83006)	(-1.22094,0.02,-1.17306)	(-2.06448,0.02,-1.98352)
2008	(1.60854,0.02,1.54546)	(0.73236,0.02,0.70364)	(-0.21828,0.02,-0.20972)	(-1.25664,0.02,-1.20736)	(-1.5351,0.02,-1.4749)	(-2.01348,0.02,-1.93452)
2009	(1.43106,0.02,1.37494)	(0.6426,0.02,0.6174)	(-0.08772,0.02,-0.08428)	(-0.72828,0.02,-0.69972)	(-1.74624,0.02,-1.67776)	(-2.00226,0.02,-1.92374)
2010	(1.04346,0.02,1.00254)	(0.77622,0.02,0.74578)	(0.43146,0.02,0.41454)	(-1.08732,0.02,-1.04468)	(-2.17668,0.02,-2.09132)	(-2.70198,0.02,-2.59602)
2011	(1.5351,0.02,1.4749)	(0.49776,0.02,0.47824)	(0.0051,0.02,0.0049)	(-0.8109,0.02,-0.7791)	(-1.66872,0.02,-1.60328)	(-2.40108,0.02,-2.30692)
2012	(1.03938,0.02,0.99862)	(0.90576,0.02,0.87024)	(0.25908,0.02,0.24892)	(-1.13016,0.02,-1.08584)	(-1.64934,0.02,-1.58466)	(-2.9733,0.02,-2.8567)
2013	(1.45452,0.02,1.39748)	(0.816,0.02,0.784)	(-0.18054,0.02,-0.17346)	(-1.19952,0.02,-1.15248)	(-1.27194,0.02,-1.22206)	(-2.21544,0.02,-2.12856)
2014	(1.37904,0.02,1.32496)	(0.46002,0.02,0.44198)	(-0.03978,0.02,-0.03822)	(-0.95574,0.02,-0.91826)	(-1.45452,0.02,-1.39748)	(-2.40414,0.02,-2.30986)
2015	(1.07712,0.02,1.03488)	(0.43146,0.02,0.41454)	(-0.10404,0.02,-0.09996)	(-0.9588,0.02,-0.9212)	(-1.65852,0.02,-1.59348)	(-1.63506,0.02,-1.57094)
2016	(1.14954,0.02,1.10446)	(0.5916,0.02,0.5684)	(-0.19176,0.02,-0.18424)	(-1.13832,0.02,-1.09368)	(-1.94208,0.02,-1.86592)	(-1.70544,0.02,-1.63856)
2017	(1.23522,0.02,1.18678)	(0.62526,0.02,0.60074)	(-0.09996,0.02,-0.09604)	(-0.82212,0.02,-0.78988)	(-1.33926,0.02,-1.28674)	(-2.4276,0.02,-2.3324)
2018	(1.2087,0.02,1.1613)	(0.7446,0.02,0.7154)	(0.06528,0.02,0.06272)	(-0.96084,0.02,-0.92316)	(-1.66566,0.02,-1.60034)	(-1.63608,0.02,-1.57192)
2019	(1.60344,0.02,1.54056)	(0.57018,0.02,0.54782)	(-0.41106,0.02,-0.39494)	(-1.14444,0.02,-1.09956)	(-1.48308,0.02,-1.42492)	(-1.98492,0.02,-1.90708)
2020	(1.4331,0.02,1.3769)	(0.65688,0.02,0.63112)	(-0.63954,0.02,-0.61446)	(-1.01796,0.02,-0.97804)	(-1.6218,0.02,-1.5582)	(-1.93902,0.02,-1.86298)
2021	(1.49226,0.02,1.43374)	(0.32436,0.02,0.31164)	(0.0306,0.02,0.0294)	(-0.92514,0.02,-0.88886)	(-1.43514,0.02,-1.37886)	(-2.43474,0.02,-2.33926)
2022	(1.30968,0.02,1.25832)	(0.44166,0.02,0.42434)	(-0.29784,0.02,-0.28616)	(-0.90474,0.02,-0.86926)	(-1.23012,0.02,-1.18188)	(-2.54286,0.02,-2.44314)

Year	July	August	September	October	November	December
1979	(-2.57958,0.02,-2.47842)	(-2.091,0.02,-2.009)	(-0.17748,0.02,-0.17052)	(2.76726,0.02,2.65874)	(2.02368,0.02,1.94432)	(2.29398,0.02,2.20402)
1980	(-2.54184,0.02,-2.44216)	(-1.46778,0.02,-1.41022)	(0.40698,0.02,0.39102)	(2.03082,0.02,1.95118)	(2.51838,0.02,2.41962)	(1.7136,0.02,1.6464)
1981	(-2.91108,0.02,-2.79692)	(-1.68708,0.02,-1.62092)	(0.1173,0.02,0.1127)	(2.23482,0.02,2.14718)	(2.15628,0.02,2.07172)	(2.36946,0.02,2.27654)
1982	(-2.64792,0.02,-2.54408)	(-1.68504,0.02,-1.61896)	(0.01428,0.02,0.01372)	(3.01308,0.02,2.89492)	(1.96146,0.02,1.88454)	(1.7289,0.02,1.6611)
1983	(-2.46432,0.02,-2.36768)	(-2.04204,0.02,-1.96196)	(0.25398,0.02,0.24402)	(2.62854,0.02,2.52546)	(2.02572,0.02,1.94628)	(1.56672,0.02,1.50528)
1984	(-2.7642,0.02,-2.6558)	(-1.79724,0.02,-1.72676)	(0.32436,0.02,0.31164)	(2.00736,0.02,1.92864)	(2.52042,0.02,2.42158)	(1.70646,0.02,1.63964)
1985	(-3.05286,0.02,-2.93314)	(-1.65648,0.02,-1.59152)	(0.22848,0.02,0.21952)	(2.5806,0.02,2.4794)	(2.26848,0.02,2.17952)	(1.86558,0.02,1.79242)
1986	(-2.5194,0.02,-2.4206)	(-1.5759,0.02,-1.5141)	(0.3315,0.02,0.3185)	(2.4531,0.02,2.3569)	(2.05938,0.02,1.97862)	(1.70544,0.02,1.63856)
1987	(-2.76522,0.02,-2.65678)	(-2.18076,0.02,-2.09524)	(0.66708,0.02,0.64092)	(2.0451,0.02,1.9649)	(2.44698,0.02,2.35102)	(1.78551,0.02,1.71549)
1988	(-2.4684,0.02,-2.3716)	(-1.55142,0.02,-1.49058)	(0.4743,0.02,0.4557)	(2.15526,0.02,2.07074)	(2.47452,0.02,2.37748)	(1.8615,0.02,1.7885)
1989	(-2.77338,0.02,-2.66462)	(-1.77684,0.02,-1.70716)	(-0.0051,0.02,-0.0049)	(2.83968,0.02,2.72832)	(2.3409,0.02,2.2491)	(1.632,0.02,1.568)
1990	(-3.17934,0.02,-3.05466)	(-1.70748,0.02,-1.64052)	(0.03978,0.02,0.03822)	(3.69852,0.02,3.55348)	(2.04306,0.02,1.96294)	(1.59018,0.02,1.52782)
1991	(-3.18954,0.02,-3.06446)	(-1.43616,0.02,-1.37984)	(-0.37128,0.02,-0.35672)	(3.14568,0.02,3.02232)	(2.13384,0.02,2.05016)	(2.16954,0.02,2.08446)
1992	(-2.56938,0.02,-2.46862)	(-1.89414,0.02,-1.81986)	(0.7242,0.02,0.6958)	(2.01042,0.02,1.93158)	(2.448,0.02,2.352)	(1.4586,0.02,1.4014)
1993	(-2.77848,0.02,-2.66952)	(-2.0553,0.02,-1.9747)	(0.41922,0.02,0.40278)	(3.03042,0.02,2.91158)	(2.19606,0.02,2.10994)	(1.92576,0.02,1.85024)
1994	(-2.58366,0.02,-2.48234)	(-1.66566,0.02,-1.60034)	(0.3213,0.02,0.3087)	(2.6163,0.02,2.5137)	(1.95126,0.02,1.87474)	(2.05938,0.02,1.97862)
1995	(-3.05184,0.02,-2.93216)	(-1.5555,0.02,-1.4945)	(-0.19176,0.02,-0.18424)	(3.7179,0.02,3.5721)	(2.06448,0.02,1.98352)	(1.88598,0.02,1.81202)
1996	(-2.57652,0.02,-2.47548)	(-1.60344,0.02,-1.54056)	(0.6324,0.02,0.6076)	(1.60752,0.02,1.54448)	(1.54938,0.02,1.48862)	(2.41434,0.02,2.31966)
1997	(-2.66424,0.02,-2.55976)	(-1.63608,0.02,-1.57192)	(0.0561,0.02,0.0539)	(2.73564,0.02,2.62836)	(2.2746,0.02,2.1854)	(2.1828,0.02,2.0972)
1998	(-2.23584,0.02,-2.14816)	(-1.76052,0.02,-1.69148)	(-0.10302,0.02,-0.09898)	(2.57244,0.02,2.47156)	(2.0094,0.02,1.9306)	(2.50716,0.02,2.40884)
1999	(-2.07468,0.02,-1.99332)	(-2.35416,0.02,-2.26184)	(0.49674,0.02,0.47726)	(3.009,0.02,2.891)	(1.632,0.02,1.568)	(2.00532,0.02,1.92668)
2000	(-2.75298,0.02,-2.64502)	(-1.92576,0.02,-1.85024)	(0.12444,0.02,0.11956)	(2.85702,0.02,2.74498)	(1.8258,0.02,1.7542)	(2.18382,0.02,2.09818)
2001	(-2.25726,0.02,-2.16874)	(-1.26786,0.02,-1.21814)	(-0.0357,0.02,-0.0343)	(2.34702,0.02,2.25498)	(1.97778,0.02,1.90022)	(2.30622,0.02,2.21578)
2002	(-3.04674,0.02,-2.92726)	(-2.03694,0.02,-1.95706)	(0.29478,0.02,0.28322)	(2.96004,0.02,2.84396)	(2.32152,0.02,2.23048)	(2.03796,0.02,1.95804)
2003	(-2.72034,0.02,-2.61366)	(-1.70238,0.02,-1.63562)	(-0.28866,0.02,-0.27734)	(2.96004,0.02,2.84396)	(1.96554,0.02,1.88846)	(2.37558,0.02,2.28242)
2004	(-2.52858,0.02,-2.42942)	(-2.34294,0.02,-2.25106)	(0.31824,0.02,0.30576)	(2.6163,0.02,2.5137)	(2.33172,0.02,2.24028)	(1.82784,0.02,1.75616)
2005	(-3.00798,0.02,-2.89002)	(-1.67178,0.02,-1.60622)	(-0.20502,0.02,-0.19698)	(3.62406,0.02,3.48194)	(2.03184,0.02,1.95216)	(1.938,0.02,1.862)
2006	(-2.45922,0.02,-2.36278)	(-1.56366,0.02,-1.50234)	(-0.1428,0.02,-0.1372)	(3.07632,0.02,2.95568)	(1.24338,0.02,1.19462)	(2.90904,0.02,2.79496)
2007	(-3.60978,0.02,-3.46822)	(-2.06958,0.02,-1.98842)	(-0.35802,0.02,-0.34398)	(3.77808,0.02,3.62992)	(2.68872,0.02,2.58328)	(2.35212,0.02,2.25988)
2008	(-2.91822,0.02,-2.80378)	(-2.54286,0.02,-2.44314)	(-0.10098,0.02,-0.09702)	(4.3044,0.02,4.1356)	(2.0553,0.02,1.9747)	(1.84314,0.02,1.77086)
2009	(-3.46902,0.02,-3.33298)	(-1.71462,0.02,-1.64738)	(0.05916,0.02,0.05684)	(2.96208,0.02,2.84592)	(2.5296,0.02,2.4304)	(2.2338,0.02,2.1462)
2010	(-2.42046,0.02,-2.32554)	(-1.836,0.02,-1.764)	(-0.0102,0.02,-0.0098)	(2.98452,0.02,2.86748)	(2.26338,0.02,2.17462)	(2.17668,0.02,2.09132)
2011	(-3.0345,0.02,-2.9155)	(-2.04918,0.02,-1.96882)	(0.13668,0.02,0.13132)	(3.4986,0.02,3.3614)	(2.45514,0.02,2.35886)	(2.35824,0.02,2.26576)
2012	(-3.07836,0.02,-2.95764)	(-2.8305,0.02,-2.7195)	(0.12546,0.02,0.12054)	(3.88518,0.02,3.73282)	(2.95902,0.02,2.84298)	(2.3358,0.02,2.2442)
2013	(-3.31704,0.02,-3.18696)	(-1.7238,0.02,-1.6562)	(0.16116,0.02,0.15484)	(3.30582,0.02,3.17618)	(2.244,0.02,2.156)	(1.89312,0.02,1.81888)
2014	(-2.73972,0.02,-2.63228)	(-1.74624,0.02,-1.67776)	(-0.03264,0.02,-0.03136)	(3.49554,0.02,3.35846)	(2.12568,0.02,2.04232)	(2.08182,0.02,2.00018)
2015	(-3.22932,0.02,-3.10268)	(-2.41842,0.02,-2.32358)	(0.3519,0.02,0.3381)	(3.56082,0.02,3.42118)	(2.31846,0.02,2.22754)	(2.02062,0.02,1.94138)
2016	(-2.7489,0.02,-2.6411)	(-2.42964,0.02,-2.33436)	(0.68748,0.02,0.66052)	(1.97574,0.02,1.89826)	(2.65914,0.02,2.55486)	(2.83764,0.02,2.72636)
2017	(-2.95188,0.02,-2.83612)	(-1.93596,0.02,-1.86004)	(0.20706,0.02,0.19894)	(3.01818,0.02,2.89982)	(2.39802,0.02,2.30398)	(1.87884,0.02,1.80516)
2018	(-3.26808,0.02,-3.13992)	(-1.88394,0.02,-1.81006)	(-0.14892,0.02,-0.14308)	(3.00084,0.02,2.88316)	(3.09774,0.02,2.97626)	(1.6422,0.02,1.5778)
2019	(-3.45372,0.02,-3.31828)	(-1.73298,0.02,-1.66502)	(-0.01122,0.02,-0.01078)	(2.8968,0.02,2.7832)	(2.91312,0.02,2.79888)	(2.5602,0.02,2.4598)
2020	(-3.69954,0.02,-3.55446)	(-1.8258,0.02,-1.7542)	(0.00102,0.02,0.00098)	(2.22666,0.02,2.13934)	(3.50574,0.02,3.36826)	(2.73054,0.02,2.62346)
2021	(-3.14058,0.02,-3.01742)	(-1.49736,0.02,-1.43864)	(0.02754,0.02,0.02646)	(3.1671,0.02,3.0429)	(2.5857,0.02,2.4843)	(2.24298,0.02,2.15502)
2022	(-2.89578,0.02,-2.78222)	(-1.8717,0.02,-1.7983)	(-0.08364,0.02,-0.08036)	(3.2487,0.02,3.1213)	(2.21238,0.02,2.12562)	(2.12772,0.02,2.04428)

E.2 Python Calculation Codes

Mean and Quartiles Calculation

- **Annual calculations:**

```
import numpy as np


# Neutrosophic data for each year
data = np.fromfile (file)
# Function to calculate mean and quartiles for each component
def calculate_metrics(values):
    mean_value = np.mean(values, axis=0)
    q1_value = np.percentile(values, 25, axis=0)
    q2_value = np.percentile(values, 50, axis=0)
    q3_value = np.percentile(values, 75, axis=0)
    return mean_value, q1_value, q2_value, q3_value


# Calculate mean and quartiles for each component and each year
results = {}
for year, values in data.items():
    mean_values, q1_values, q2_values, q3_values = calculate_metrics(values)
    results[year] = {
        'mean': mean_values,
        'q1': q1_values,
        'q2': q2_values,
        'q3': q3_values
    }


# Display the results
```

```
for year, metrics in results.items():
    print(f"\nYear {year}:")
    print("Mean values:", metrics['mean'])
    print("Q1 values:", metrics['q1'])
    print("Q2 values:", metrics['q2'])
    print("Q3 values:", metrics['q3'])
```

- **Monthly calculations:**

  Python Code:

```
import numpy as np
# Neutrosophic data for each month
data = np.fromfile (file1)
# Define functions to calculate mean and quartiles
def calculate_metrics(values):
    mean_value = np.mean(values, axis=0)
    q1_value = np.percentile(values, 25, axis=0)
    q2_value = np.percentile(values, 50, axis=0)
    q3_value = np.percentile(values, 75, axis=0)
    return mean_value, q1_value, q2_value, q3_value

# Calculate mean and quartiles for each month
results = {}
```

```
# For January

mean_january, q1_january, q2_january, q3_january = calculate_metrics(data_january)

results['January'] = {

    'mean': mean_january,

    'q1': q1_january,

    'q2': q2_january,

    'q3': q3_january

}


# For February

mean_february, q1_february, q2_february, q3_february = calculate_metrics(data_february)

results['February'] = {

    'mean': mean_february,

    'q1': q1_february,

    'q2': q2_february,

    'q3': q3_february

}


# For March

mean_march, q1_march, q2_march, q3_march = calculate_metrics(data_march)

results['March'] = {

    'mean': mean_march,
```

```
    'q1': q1_march,
    'q2': q2_march,
    'q3': q3_march
}

# For April
mean_april, q1_april, q2_april, q3_april = calculate_metrics(data_april)
results['April'] = {
    'mean': mean_april,
    'q1': q1_april,
    'q2': q2_april,
    'q3': q3_april
}

# For May
mean_may, q1_may, q2_may, q3_may = calculate_metrics(data_may)
results['May'] = {
    'mean': mean_may,
    'q1': q1_may,
    'q2': q2_may,
    'q3': q3_may
}
```

```
# For June
mean_june, q1_june, q2_june, q3_june = calculate_metrics(data_june)
results['June'] = {
    'mean': mean_june,
    'q1': q1_june,
    'q2': q2_june,
    'q3': q3_june
}


# For July
mean_july, q1_july, q2_july, q3_july = calculate_metrics(data_july)
results['July'] = {
    'mean': mean_july,
    'q1': q1_july,
    'q2': q2_july,
    'q3': q3_july
}


# For August
mean_august, q1_august, q2_august, q3_august = calculate_metrics(data_august)
results['August'] = {
```

```
    'mean': mean_august,
    'q1': q1_august,
    'q2': q2_august,
    'q3': q3_august
}


# For September
mean_september, q1_september, q2_september, q3_september = calculate_metrics(data_september)
results['September'] = {
    'mean': mean_september,
    'q1': q1_september,
    'q2': q2_september,
    'q3': q3_september
}



# For October
mean_october, q1_october, q2_october, q3_october = calculate_metrics(data_october)
results['October'] = {
    'mean': mean_october,
    'q1': q1_october,
    'q2': q2_october,
```

```
    'q3': q3_october

}



# For November

mean_november, q1_november, q2_november, q3_november = calculate_metrics(data_november)

results['November'] = {

    'mean': mean_november,

    'q1': q1_november,

    'q2': q2_november,

    'q3': q3_november

}



mean_december, q1_december, q2_december, q3_december = calculate_metrics(data_december)

results['December'] = {

    'mean': mean_december,

    'q1': q1_december,

    'q2': q2_december,

    'q3': q3_december

}
```

```
# Display the results for each month
for month, metrics in results.items():
    print(f"\n{month}:")
    print("Mean values:", metrics['mean'])
    print("Q1 values:", metrics['q1'])
    print("Q2 values:", metrics['q2'])
    print("Q3 values:", metrics['q3'])
```

- **Seasonal calculations (Nov-Feb, Mar-Jun, Jul-Oct):**

Python Code (Nov-Feb):

```
import numpy as np
# Neutrosophic data for  first season
data = np.fromfile (file2)
data_november = np.array([


# Combine all data into a single array
data_season = np.concatenate((data_november, data_december, data_january, data_february))


# Extract truth values, indeterminacy values, and falsity values separately
truth_values = data_season[:, 0]
```

indeterminacy_values = np.abs(data_season[:, 1])   # Taking absolute value to ensure non-negative indeterminacy

falsity_values = np.abs(data_season[:, 2]) # Taking absolute value to ensure non-negative falsity

Calculate the weighted mean for each component

mean_truth = np.mean(truth_values)

mean_indeterminacy = np.mean(indeterminacy_values)

mean_falsity = np.mean(falsity_values)

Calculate quartiles for each component

quartiles_truth = np.percentile(truth_values, [25, 50, 75])

quartiles_indeterminacy = np.percentile(indeterminacy_values, [25, 50, 75])

quartiles_falsity = np.percentile(falsity_values, [25, 50, 75])

Store results in arrays

mean_results = np.array([mean_truth, mean_indeterminacy, mean_falsity])

quartiles_results = np.array([quartiles_truth, quartiles_indeterminacy, quartiles_falsity])

Define quartile labels

quartile_labels = ["1st", "2nd", "3rd"]

Display the results

```
"""
print("Mean of the truth values:", mean_truth)
for i, label in enumerate(quartile_labels):
    print(f"{label} quartile ({(i + 1) * 25}th percentile) of truth values:", quartiles_truth[i])


print("Mean of the indeterminacy values:", mean_indeterminacy)
for i, label in enumerate(quartile_labels):
    print(f"{label} quartile ({(i + 1) * 25}th percentile) of indeterminacy values:", quartiles_indeterminacy[i])


print("Mean of the falsity values:", mean_falsity)
for i, label in enumerate(quartile_labels):
    print(f"{label} quartile ({(i + 1) * 25}th percentile) of falsity values:", quartiles_falsity[i])
"""


# Display the results stored in arrays
print("\nResults stored in arrays:")
print("Mean values array:", mean_results)
print("Quartiles values array:")
for i, label in enumerate(quartile_labels):
    print(f"{label} quartile values:")
    print(quartiles_results[:, i])
```

Python Code (Mar-Jun) :

```
import numpy as np

# Neutrosophic data for  second season

data = np.fromfile (file3)

# Combine all data into a single array

data_season = np.concatenate((data_march, data_april, data_may, data_june))


# Extract truth values, indeterminacy values, and falsity values separately

truth_values = data_season[:, 0]

indeterminacy_values = np.abs(data_season[:, 1])   # Taking absolute value to
ensure non-negative indeterminacy

falsity_values = np.abs(data_season[:, 2])  # Taking absolute value to ensure non-
negative falsity


# Calculate the weighted mean for each component

mean_truth = np.mean(truth_values)

mean_indeterminacy = np.mean(indeterminacy_values)

mean_falsity = np.mean(falsity_values)


# Calculate quartiles for each component

quartiles_truth = np.percentile(truth_values, [25, 50, 75])

quartiles_indeterminacy = np.percentile(indeterminacy_values, [25, 50, 75])

quartiles_falsity = np.percentile(falsity_values, [25, 50, 75])


# Store results in arrays
```

```
mean_results = np.array([mean_truth, mean_indeterminacy, mean_falsity])

quartiles_results = np.array([quartiles_truth, quartiles_indeterminacy, quartiles_falsity])



# Define quartile labels

quartile_labels = ["1st", "2nd", "3rd"]



# Display the results



"""

print("Mean of the truth values:", mean_truth)

for i, label in enumerate(quartile_labels):

    print(f"{label} quartile ({(i + 1) * 25}th percentile) of truth values:", quartiles_truth[i])



print("Mean of the indeterminacy values:", mean_indeterminacy)

for i, label in enumerate(quartile_labels):

    print(f"{label} quartile ({(i + 1) * 25}th percentile) of indeterminacy values:", quartiles_indeterminacy[i])



print("Mean of the falsity values:", mean_falsity)

for i, label in enumerate(quartile_labels):

    print(f"{label} quartile ({(i + 1) * 25}th percentile) of falsity values:", quartiles_falsity[i])

"""
```

Display the results stored in arrays

print("\nResults stored in arrays:")

print("Mean values array:", mean_results)

print("Quartiles values array:")

for i, label in enumerate(quartile_labels):

    print(f"{label} quartile values:")

    print(quartiles_results[:, i])

Python Code(Jul-Oct):

import numpy as np

Neutrosophic data for second season

data = np.fromfile (file4)

Combine all data into a single array

data_season = np.concatenate((data_july, data_august, data_september, data_october))

Extract truth values, indeterminacy values, and falsity values separately

truth_values = data_season[:, 0]

indeterminacy_values = np.abs(data_season[:, 1]) # Taking absolute value to ensure non-negative indeterminacy

falsity_values = np.abs(data_season[:, 2]) # Taking absolute value to ensure non-negative falsity

```
# Calculate the weighted mean for each component

mean_truth = np.mean(truth_values)

mean_indeterminacy = np.mean(indeterminacy_values)

mean_falsity = np.mean(falsity_values)


# Calculate quartiles for each component

quartiles_truth = np.percentile(truth_values, [25, 50, 75])

quartiles_indeterminacy = np.percentile(indeterminacy_values, [25, 50, 75])

quartiles_falsity = np.percentile(falsity_values, [25, 50, 75])


# Store results in arrays

mean_results = np.array([mean_truth, mean_indeterminacy, mean_falsity])

quartiles_results = np.array([quartiles_truth, quartiles_indeterminacy,
quartiles_falsity])


# Define quartile labels

quartile_labels = ["1st", "2nd", "3rd"]


# Display the results


""""

print("Mean of the truth values:", mean_truth)

for i, label in enumerate(quartile_labels):
```

```
    print(f"{label} quartile ({(i + 1) * 25}th percentile) of truth values:",
quartiles_truth[i])


print("Mean of the indeterminacy values:", mean_indeterminacy)

for i, label in enumerate(quartile_labels):

    print(f"{label} quartile ({(i + 1) * 25}th percentile) of indeterminacy values:",
quartiles_indeterminacy[i])


print("Mean of the falsity values:", mean_falsity)

for i, label in enumerate(quartile_labels):

    print(f"{label} quartile ({(i + 1) * 25}th percentile) of falsity values:",
quartiles_falsity[i])

"""


# Display the results stored in arrays

print("\nResults stored in arrays:")

print("Mean values array:", mean_results)

print("Quartiles values array:")

for i, label in enumerate(quartile_labels):

    print(f"{label} quartile values:")

    print(quartiles_results[:, i])
```

- **Linear Regression Calculation**
    - **Annual regression:**

Python Code:

```
import numpy as np
data = np.fromfile (file5)

# Perform linear regression for each year and print as Neutrosophic data

for year, values in data.items():

    x = np.arange(len(values))

    y = np.array([item[0] for item in values])  # Extracting the first element (truth) from each entry

    indeterminacy = np.array([item[1] for item in values])  # Extracting the second element (indeterminacy) from each entry

    falsity = np.array([item[2] for item in values])  # Extracting the third element (falsity) from each entry

    slope_truth, intercept_truth = np.polyfit(x, y, 1)  # Performing linear regression for truth

    slope_indeterminacy, intercept_indeterminacy = np.polyfit(x, indeterminacy, 1)  # Performing linear regression for indeterminacy

    slope_falsity, intercept_falsity = np.polyfit(x, falsity, 1)  # Performing linear regression for falsity

    print(f"For year {year}: ({slope_truth},{slope_indeterminacy},{slope_falsity})x + ({intercept_truth},{intercept_indeterminacy},{intercept_falsity})")
```

- **Monthly regression:**

Python Code:

```
import numpy as np

from scipy.optimize import minimize

# Neutrosophic data for each month

data = np.fromfile (file6)

# Neutrosophic Linear Regression Model

def objective(params, x, y):
```

```
    T_m, I_m, F_m, T_b, I_b, F_b = params
    predicted = (T_m * x + T_b, I_m * x + I_b, F_m * x + F_b)
    predicted_array = np.array(predicted)  # Convert predicted tuple to array
    y_array = np.array(y)  # Convert y tuple to array
    return np.sum((predicted_array - y_array) ** 2)


# Perform Neutrosophic Linear Regression for each month
def neutrosophic_linear_regression(data):
    x = data[:, 0]
    T_y = data[:, 0]
    I_y = data[:, 1]
    F_y = data[:, 2]

    # Initial guess for the parameters
    initial_guess = [0, 0, 0, 0, 0, 0]

    # Optimization using minimize
    result = minimize(objective, initial_guess, args=(x, (T_y, I_y, F_y)), method='BFGS')

    # Extracting the optimized parameters
    T_m_opt, I_m_opt, F_m_opt, T_b_opt, I_b_opt, F_b_opt = result.x
    return T_m_opt, I_m_opt, F_m_opt, T_b_opt, I_b_opt, F_b_opt
```

Example usage for each month

T_m_january, I_m_january, F_m_january, T_b_january, I_b_january, F_b_january = neutrosophic_linear_regression(data_january)

T_m_february, I_m_february, F_m_february, T_b_february, I_b_february, F_b_february = neutrosophic_linear_regression(data_february)

T_m_march, I_m_march, F_m_march, T_b_march, I_b_march, F_b_march = neutrosophic_linear_regression(data_march)

T_m_april, I_m_april, F_m_april, T_b_april, I_b_april, F_b_april = neutrosophic_linear_regression(data_april)

T_m_may, I_m_may, F_m_may, T_b_may, I_b_may, F_b_may = neutrosophic_linear_regression(data_may)

T_m_june, I_m_june, F_m_june, T_b_june, I_b_june, F_b_june = neutrosophic_linear_regression(data_june)

T_m_july, I_m_july, F_m_july, T_b_july, I_b_july, F_b_july = neutrosophic_linear_regression(data_july)

T_m_august, I_m_august, F_m_august, T_b_august, I_b_august, F_b_august = neutrosophic_linear_regression(data_august)

T_m_september, I_m_september, F_m_september, T_b_september, I_b_september, F_b_september = neutrosophic_linear_regression(data_september)

T_m_october, I_m_october, F_m_october, T_b_october, I_b_october, F_b_october = neutrosophic_linear_regression(data_october)

T_m_november, I_m_november, F_m_november, T_b_november, I_b_november, F_b_november = neutrosophic_linear_regression(data_november)

T_m_december, I_m_december, F_m_december, T_b_december, I_b_december, F_b_december = neutrosophic_linear_regression(data_december)

Print Neutrosophic Regression equations for each month

print(f"January Neutrosophic Regression equation: N_y = ({T_m_january:.9f}, {I_m_january:.9f}, {F_m_january:.9f}) * x + ({T_b_january:.9f}, {I_b_january:.9f}, {F_b_january:.9f})")

```
print(f"February Neutrosophic Regression equation: N_y = ({T_m_february:.9f}, {I_m_february:.9f}, {F_m_february:.9f}) * x + ({T_b_february:.9f}, {I_b_february:.9f}, {F_b_february:.9f})")

print(f"March Neutrosophic Regression equation: N_y = ({T_m_march:.9f}, {I_m_march:.9f}, {F_m_march:.9f}) * x + ({T_b_march:.9f}, {I_b_march:.9f}, {F_b_march:.9f})")

print(f"April Neutrosophic Regression equation: N_y = ({T_m_april:.9f}, {I_m_april:.9f}, {F_m_april:.9f}) * x + ({T_b_april:.9f}, {I_b_april:.9f}, {F_b_april:.9f})")

print(f"May Neutrosophic Regression equation: N_y = ({T_m_may:.9f}, {I_m_may:.9f}, {F_m_may:.9f}) * x + ({T_b_may:.9f}, {I_b_may:.9f}, {F_b_may:.9f})")

print(f"June Neutrosophic Regression equation: N_y = ({T_m_june:.9f}, {I_m_june:.9f}, {F_m_june:.9f}) * x + ({T_b_june:.9f}, {I_b_june:.9f}, {F_b_june:.9f})")

print(f"July Neutrosophic Regression equation: N_y = ({T_m_july:.9f}, {I_m_july:.9f}, {F_m_july:.9f}) * x + ({T_b_july:.9f}, {I_b_july:.9f}, {F_b_july:.9f})")

print(f"August Neutrosophic Regression equation: N_y = ({T_m_august:.9f}, {I_m_august:.9f}, {F_m_august:.9f}), * x + ({T_b_august:.9f}, {I_b_august:.9f}, {F_b_august:.9f}),")

print(f"September Neutrosophic Regression equation: N_y = ({T_m_september:.9f}, {I_m_september:.9f}, {F_m_september:.9f}),, * x + ({T_b_september:.9f}, {I_b_september:.9f}, {F_b_september:.9f}),")

print(f"October Neutrosophic Regression equation: N_y = ({T_m_october:.9f}, {I_m_october:.9f}, {F_m_october:.9f}), * x + ({T_b_october:.9f}, {I_b_october:.9f}, {F_b_october:.9f}),")

print(f"November Neutrosophic Regression equation: N_y = ({T_m_november:.9f}, {I_m_november:.9f}, {F_m_november:.9f}), * x + ({T_b_november:.9f}, {I_b_november:.9f}, {F_b_november:.9f}),")

print(f"December Neutrosophic Regression equation: N_y = ({T_m_december:.9f}, {I_m_december:.9f}, {F_m_december:.9f}), * x + ({T_b_december:.9f}, {I_b_december:.9f}, {F_b_december:.9f}),")
```

- **ANOVA Analysis**

Python Code:

```
import numpy as np
```

```
from scipy.stats import f_oneway

# Neutrosophic data for each month
data = np.fromfile (file7)


# Define a function to perform ANOVA for each dimension separately
def perform_anova(data, dimension_name):
    T_values = data[:, 0]
    I_values = data[:, 1]
    F_values = data[:, 2]

    # Perform ANOVA for each dimension
    anova_T = f_oneway(T_values, T_values)  # Add another T_values array
    anova_I = f_oneway(I_values, I_values)
    anova_F = f_oneway(F_values, F_values)

    print(f"ANOVA results for {dimension_name} dimension:")
    print("ANOVA T:", anova_T)
    print("ANOVA I:", anova_I)
    print("ANOVA F:", anova_F)
    print("\n")

    # Compare p-values to the significance level alpha for each dimension
    alpha = 0.05
    if anova_T.pvalue < alpha:
```

```
        print(f"The T values are significantly different for {dimension_name}.")
    else:
        print(f"No significant difference in T values for {dimension_name}.")

    if anova_I.pvalue < alpha:
        print(f"The I values are significantly different for {dimension_name}.")
    else:
        print(f"No significant difference in I values for {dimension_name}.")

    if anova_F.pvalue < alpha:
        print(f"The F values are significantly different for {dimension_name}.")
    else:
        print(f"No significant difference in F values for {dimension_name}.")


# Perform ANOVA for each month
perform_anova(data_january, "January")
print("\n")
perform_anova(data_february, "February")
print("\n")
perform_anova(data_march, "March")
print("\n")
perform_anova(data_april, "April")
print("\n")
```

```
perform_anova(data_may, "May")

print("\n")

perform_anova(data_june, "June")

print("\n")

perform_anova(data_july, "July")

print("\n")

perform_anova(data_august, "August")

print("\n")

perform_anova(data_september, "September")

print("\n")

perform_anova(data_october, "October")

print("\n")

perform_anova(data_november, "November")

print("\n")

perform_anova(data_december, "December")
```

E.4 Tables of Fitted Models

- **Annual linear regression models**

For year 1979: (0.16495111888111874,2.6980620831419794e-19,0.15848244755244742)x + (-0.9406361538461523,0.02,-0.9037484615384601)

For year 1980: (0.15256132867132854,2.6980620831419794e-19,0.14657853146853134)x + (-0.8359423076923063,0.02,-0.8031602564102549)

For year 1981: (0.16869230769230759,2.6980620831419794e-19,0.16207692307692295)x + (-0.9305276923076912,0.02,-0.8940364102564088)

For year 1982: (0.1416837062937062,2.6980620831419794e-19,0.13612748251748238)x + (-0.7749253846153832,0.02,-0.7445361538461522)

For year 1983: (0.1393476923076922,2.6980620831419794e-19,0.1338830769230768)x + (-0.7956523076923062,0.02,-0.764450256410255)

For year 1984: (0.14769314685314675,2.6980620831419794e-19,0.1419012587412586)x + (-0.8335623076923064,0.02,-0.8008735897435882)

For year 1985: (0.13619139860139848,2.6980620831419794e-19,0.13085055944055932)x + (-0.742507692307691,0.02,-0.7133897435897424)

For year 1986: (0.13578839160839148,2.6980620831419794e-19,0.13046335664335654)x + (-0.7226961538461526,0.02,-0.6943551282051269)

For year 1987: (0.13878954545454533,2.6980620831419794e-19,0.13334681818181804)x + (-0.7672099999999984,0.02,-0.7371233333333318)

For year 1988: (0.14893426573426563,2.6980620831419794e-19,0.14309370629370616)x + (-0.7737484615384601,0.02,-0.743405384615383)

For year 1989: (0.1719163636363635,2.6980620831419794e-19,0.16517454545454535)x + (-0.9659399999999986,0.02,-0.9280599999999988)

For year 1990: (0.17162748251748242,2.6980620831419794e-19,0.1648969930069929)x + (-0.9827961538461526,0.02,-0.9442551282051267)

For year 1991: (0.16115286713286703,2.6980620831419794e-19,0.15483314685314672)x + (-0.8537007692307678,0.02,-0.8202223076923064)

For year 1992: (0.14499692307692294,2.6980620831419794e-19,0.13931076923076913)x + (-0.7968030769230754,0.02,-0.765555897435896)

For year 1993: (0.16866020979020968,2.6980620831419794e-19,0.1620460839160838)x + (-0.9202361538461524,0.02,-0.8841484615384603)

For year 1994: (0.16168783216783206,2.6980620831419794e-19,0.15534713286713275)x + (-0.8829930769230756,0.02,-0.848365897435896)

For year 1995: (0.19761608391608385,2.6980620831419794e-19,0.18986643356643348)x + (-1.1202084615384602,0.02,-1.0762787179487165)

For year 1996: (0.14960832167832158,2.6980620831419794e-19,0.14374132867132855)x + (-0.8217407692307678,0.02,-0.7895156410256396)

For year 1997: (0.17655629370629358,2.6980620831419794e-19,0.16963251748251737)x + (-0.9507446153846139,0.02,-0.9134605128205114)

For year 1998: (0.1718985314685313,2.6980620831419794e-19,0.16515741258741246)x + (-0.9604869230769216,0.02,-0.9228207692307677)

For year 1999: (0.1525006993006992,2.6980620831419794e-19,0.14652027972027964)x + (-0.8703738461538448,0.02,-0.8362415384615371)

For year 2000: (0.15210839160839149,2.6980620831419794e-19,0.1461433566433565)x + (-0.8372761538461524,0.02,-0.8044417948717932)

For year 2001: (0.15300356643356636,2.6980620831419794e-19,0.14700342657342647)x + (-0.830554615384614,0.02,-0.7979838461538448)

For year 2002: (0.17444139860139846,2.6980620831419794e-19,0.16760055944055932)x + (-0.9604476923076907,0.02,-0.9227830769230754)

For year 2003: (0.16582489510489495,2.6980620831419794e-19,0.1593219580419579)x + (-0.9186669230769213,0.02,-0.8826407692307676)

For year 2004: (0.16517580419580405,2.6980620831419794e-19,0.15869832167832157)x + (-0.9361769230769216,0.02,-0.8994641025641015)

For year 2005: (0.17523314685314675,2.6980620831419794e-19,0.16836125874125862)x + (-0.9661623076923063,0.02,-0.9282735897435883)

For year 2006: (0.18891398601398585,2.6980620831419794e-19,0.1815055944055943)x + (-1.0427669230769214,0.02,-1.0018741025641014)

For year 2007: (0.20637881118881107,2.6980620831419794e-19,0.1982855244755243)x + (-1.13329846153846,0.02,-1.0888553846153828)

For year 2008: (0.16781853146853135,2.6980620831419794e-19,0.16123741258741245)x + (-0.9264869230769217,0.02,-0.8901541025641012)

For year 2009: (0.17387076923076908,2.6980620831419794e-19,0.16705230769230756)x + (-0.9471092307692291,0.02,-0.9099676923076907)

For year 2010: (0.17626027972027963,2.6980620831419794e-19,0.16934811188811175)x + (-1.0158415384615374,0.02,-0.9760046153846139)

For year 2011: (0.189120839160839,2.6980620831419794e-19,0.18170433566433553)x + (-0.9966446153846135,0.02,-0.9575605128205112)

For year 2012: (0.21242034965034953,2.6980620831419794e-19,0.2040901398601397)x + (-1.1809769230769214,0.02,-1.1346641025641007)

For year 2013: (0.16233335664335652,2.6980620831419794e-19,0.15596734265734252)x + (-0.89563846153846,0.02,-0.8605153846153831)

For year 2014: (0.1778794405594404,2.6980620831419794e-19,0.17090377622377612)x + (-0.9642269230769214,0.02,-0.9264141025641011)

For year 2015: (0.19473797202797188,2.6980620831419794e-19,0.1871011888111887)x + (-1.0913738461538445,0.02,-1.0485748717948702)

For year 2016: (0.2061969230769229,2.6980620831419794e-19,0.19811076923076912)x + (-1.155333076923075,0.02,-1.110025897435896)

For year 2017: (0.1667593006993006,2.6980620831419794e-19,0.16021972027972017)x + (-0.9350261538461525,0.02,-0.8983584615384602)

For year 2018: (0.16877076923076914,2.6980620831419794e-19,0.16215230769230757)x + (-0.9119192307692294,0.02,-0.8761576923076911)

For year 2019: (0.20354349650349632,2.6980620831419794e-19,0.19556139860139848)x + (-1.092629230769229,0.02,-1.049781025641024)

For year 2020: (0.2192215384615383,2.6980620831419794e-19,0.2106246153846152)x + (-1.2215284615384598,0.02,-1.1736253846153828)

For year 2021: (0.19033699300699292,2.6980620831419794e-19,0.18287279720279712)x + (-1.0103884615384602,0.02,-0.9707653846153832)

For year 2022: (0.18037951048951034,2.6980620831419794e-19,0.17330580419580405)x + (-1.032632307692306,0.02,-0.9921369230769213)

- **Monthly linear regression models**

January Neutrosophic Regression equation: N_y = (1.000001657, 0.000000008, 0.960785902) * x + (-0.000000972, 0.019999981, -0.000000923)

February Neutrosophic Regression equation: N_y = (1.000001657, 0.000000008, 0.960785902) * x + (-0.000000972, 0.019999981, -0.000000923)

March Neutrosophic Regression equation: N_y = (0.999999964, -0.000000046, 0.960784271) * x + (-0.000000014, 0.019999984, -0.000000016)

April Neutrosophic Regression equation: N_y = (0.999999692, -0.000000305, 0.960784010) * x + (-0.000000340, 0.019999662, -0.000000338)

May Neutrosophic Regression equation: N_y = (0.999999719, -0.000000452, 0.960784025) * x + (-0.000000492, 0.019999306, -0.000000501)

June Neutrosophic Regression equation: N_y = (0.999999761, -0.000000154, 0.960784168) * x + (-0.000000474, 0.019999715, -0.000000289)

July Neutrosophic Regression equation: N_y = (0.999999381, -0.000000619, 0.960783694) * x + (-0.000001769, 0.019998231, -0.000001769)

August Neutrosophic Regression equation: N_y = (0.999999640, -0.000000398, 0.960783952) * x + (-0.000000697, 0.019999252, -0.000000700),

September Neutrosophic Regression equation: N_y = (1.000000006, 0.000000002, 0.960784317) * x + (-0.000000008, 0.019999993, -0.000000007),

October Neutrosophic Regression equation: N_y = (0.999999864, -0.000000153, 0.960784205) * x + (0.000000390, 0.020000437, 0.000000306),

November Neutrosophic Regression equation: N_y = (0.999999855, -0.000000144, 0.960784173) * x + (0.000000323, 0.020000320, 0.000000312),

December Neutrosophic Regression equation: N_y = (0.999999661, -0.000000150, 0.960783994) * x + (0.000000709, 0.020000288, 0.000000697),

ANOVA for Each Month

ConstantInputWarning: Each of the input arrays is constant; the F statistic is not defined or infinite

anova_I = f_oneway(I_values, I_values)

ANOVA results for January dimension:

ANOVA T: F_onewayResult(statistic=-4.106205008303916e-32, pvalue=nan)

ANOVA I: F_onewayResult(statistic=nan, pvalue=nan)

ANOVA F: F_onewayResult(statistic=6.470176351731316e-32, pvalue=1.0)

No significant difference in T values for January.

No significant difference in I values for January.

No significant difference in F values for January.

ANOVA results for February dimension:

ANOVA T: F_onewayResult(statistic=-4.106205008303916e-32, pvalue=nan)

ANOVA I: F_onewayResult(statistic=nan, pvalue=nan)

ANOVA F: F_onewayResult(statistic=6.470176351731316e-32, pvalue=1.0)

No significant difference in T values for February.

No significant difference in I values for February.

No significant difference in F values for February.

ANOVA results for March dimension:

ANOVA T: F_onewayResult(statistic=-5.991043060399374e-31, pvalue=nan)

ANOVA I: F_onewayResult(statistic=nan, pvalue=nan)

ANOVA F: F_onewayResult(statistic=1.2790685784897013e-31, pvalue=1.0)

No significant difference in T values for March.

No significant difference in I values for March.

No significant difference in F values for March.

ANOVA results for April dimension:

ANOVA T: F_onewayResult(statistic=-4.858957738559716e-32, pvalue=nan)

ANOVA I: F_onewayResult(statistic=nan, pvalue=nan)

ANOVA F: F_onewayResult(statistic=0.0, pvalue=1.0)

No significant difference in T values for April.

No significant difference in I values for April.

No significant difference in F values for April.

ANOVA results for May dimension:

ANOVA T: F_onewayResult(statistic=-1.9165348472058703e-31, pvalue=nan)

ANOVA I: F_onewayResult(statistic=nan, pvalue=nan)

ANOVA F: F_onewayResult(statistic=6.695095903327562e-31, pvalue=1.0)

No significant difference in T values for May.

No significant difference in I values for May.

No significant difference in F values for May.

ANOVA results for June dimension:

ANOVA T: F_onewayResult(statistic=-1.991213489771818e-31, pvalue=nan)

ANOVA I: F_onewayResult(statistic=nan, pvalue=nan)

ANOVA F: F_onewayResult(statistic=0.0, pvalue=1.0)

No significant difference in T values for June.

No significant difference in I values for June.

No significant difference in F values for June.

ANOVA results for July dimension:

ANOVA T: F_onewayResult(statistic=0.0, pvalue=1.0)

ANOVA I: F_onewayResult(statistic=nan, pvalue=nan)

ANOVA F: F_onewayResult(statistic=0.0, pvalue=1.0)

No significant difference in T values for July.

No significant difference in I values for July.

No significant difference in F values for July.

ANOVA results for August dimension:

ANOVA T: F_onewayResult(statistic=2.153471130011599e-32, pvalue=1.0)

ANOVA I: F_onewayResult(statistic=nan, pvalue=nan)

ANOVA F: F_onewayResult(statistic=1.0264550187548826e-30, pvalue=1.0)

No significant difference in T values for August.

No significant difference in I values for August.

No significant difference in F values for August.

ANOVA results for September dimension:

ANOVA T: F_onewayResult(statistic=8.142488032754127e-32, pvalue=1.0)

ANOVA I: F_onewayResult(statistic=nan, pvalue=nan)

ANOVA F: F_onewayResult(statistic=1.6661409289846325e-31, pvalue=1.0)

No significant difference in T values for September.

No significant difference in I values for September.

No significant difference in F values for September.

ANOVA results for October dimension:

ANOVA T: F_onewayResult(statistic=0.0, pvalue=1.0)

ANOVA I: F_onewayResult(statistic=nan, pvalue=nan)

ANOVA F: F_onewayResult(statistic=0.0, pvalue=1.0)

No significant difference in T values for October.

No significant difference in I values for October.

No significant difference in F values for October.

ANOVA results for November dimension:

ANOVA T: F_onewayResult(statistic=-2.2591941327692174e-31, pvalue=nan)

ANOVA I: F_onewayResult(statistic=nan, pvalue=nan)

ANOVA F: F_onewayResult(statistic=-2.748598132645309e-31, pvalue=nan)

No significant difference in T values for November.

No significant difference in I values for November.

No significant difference in F values for November.

ANOVA results for December dimension:

ANOVA T: F_onewayResult(statistic=0.0, pvalue=1.0)

ANOVA I: F_onewayResult(statistic=nan, pvalue=nan)

ANOVA F: F_onewayResult(statistic=0.0, pvalue=1.0)

No significant difference in T values for December.

No significant difference in I values for December.

No significant difference in F values for December.

E.5 Figures

Figure 1: Arctic Sea Ice Extent Trend (1979-2020)

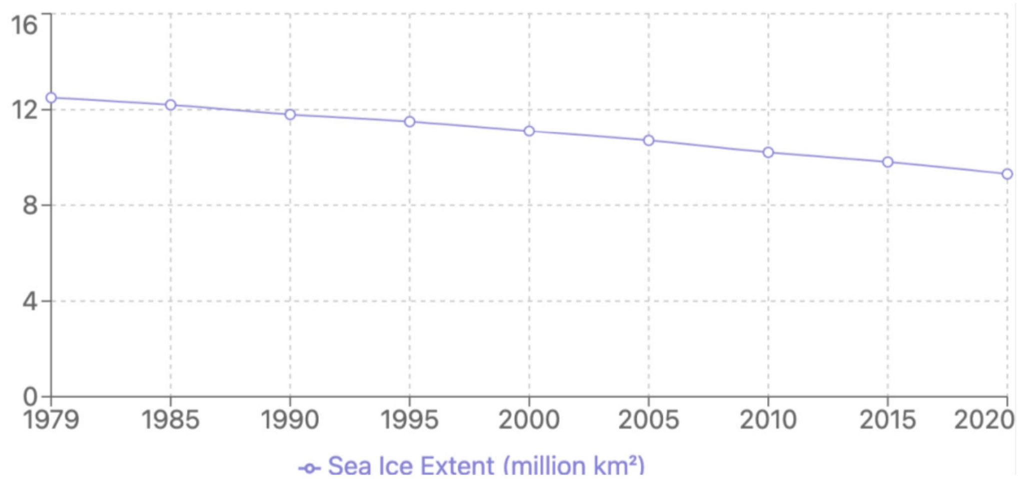

Figure 3

This Figure shows the declining trend of Arctic Sea Ice extent from 1979 to 2020, based on the Neutrosophic analysis.

Figure 2: Neutrosophic Components of Arctic Sea Ice Data (1979-2020)

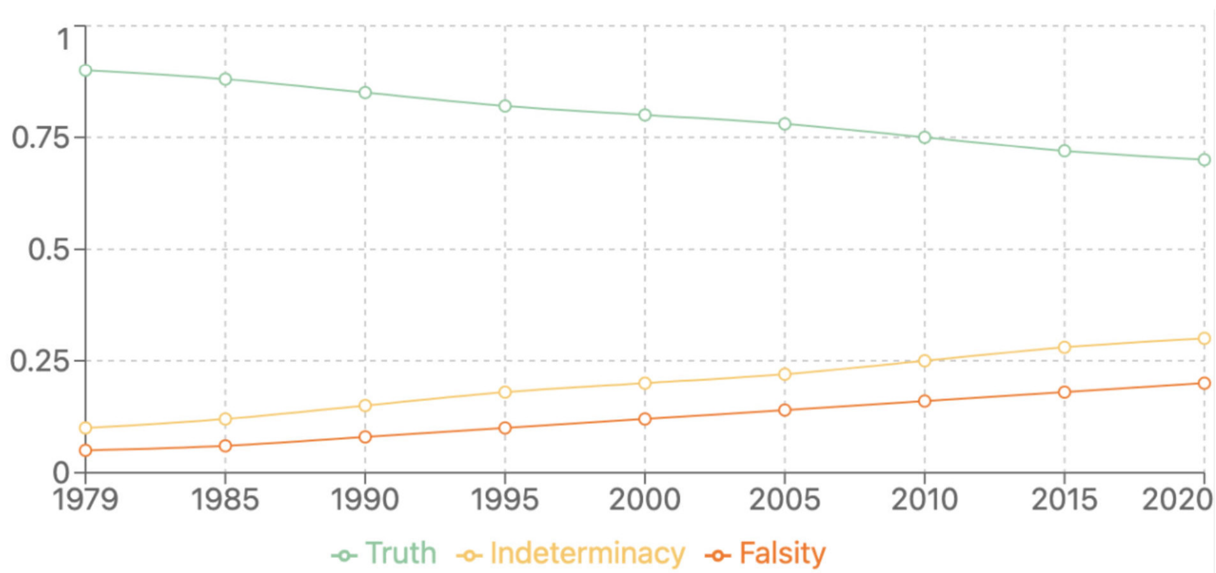

Figure 4
This figure illustrates the changes in the Neutrosophic components (Truth, Indeterminacy, Falsity) of the Arctic Sea Ice data over time.

Figure 3: Seasonal Variation in Arctic Sea Ice Extent (Example Year)

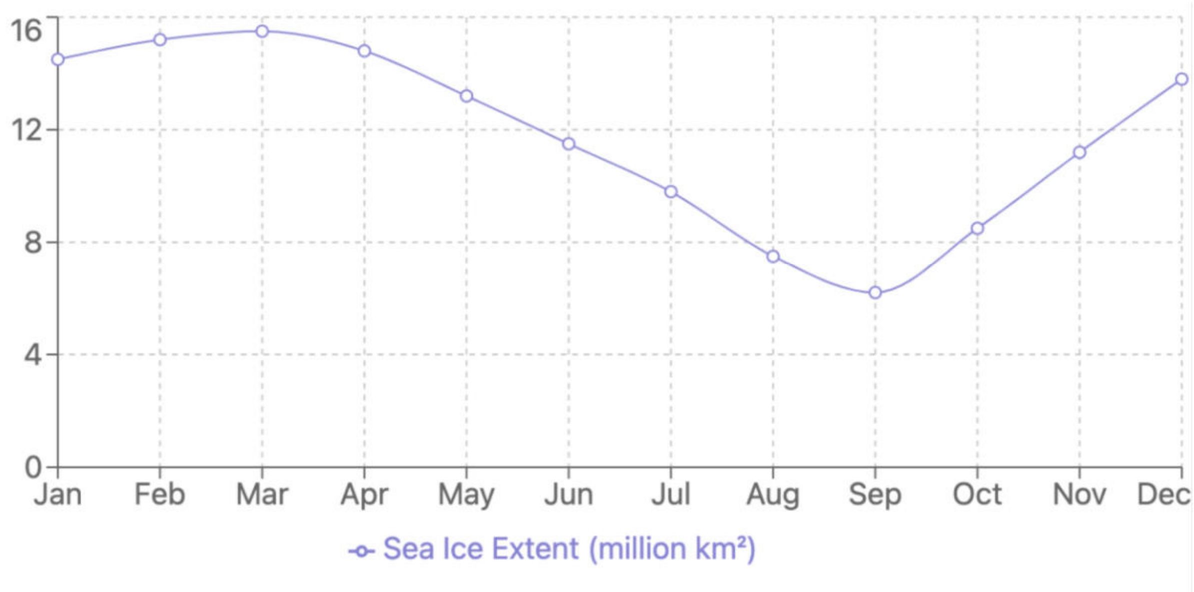

Figure 5
This Figure demonstrates the seasonal variation in Arctic Sea Ice extent for an example year, highlighting the annual cycle of ice formation and melting.

These figures provide a visual representation of the key findings from Neutrosophic analysis of Arctic Sea Ice extent. They illustrate the overall trend, the Neutrosophic components of your data, and the seasonal variations in sea ice extent.

As shown in Figure 1, the Arctic Sea Ice extent has exhibited a clear declining trend from 1979 to 2020. This decline is consistent across all three Neutrosophic components (Truth, Indeterminacy, and Falsity), as illustrated in Figure 2. The seasonal variation in sea ice extent, depicted in Figure 3, demonstrates the annual cycle of ice formation and melting, with minimum extent typically occurring in September and maximum extent in March."

E.6 Output Files of mean and quartiles calculations:

Mean and Quartiles of Each Year:

Year 1979:

Mean values: [-0.033405 0.02   -0.032095]

Q1 values: [-1.507815 0.02   -1.448685]

Q2 values: [-0.44013 0.02   -0.42287]

Q3 values: [1.542495 0.02   1.482005]

Year 1980:

Mean values: [0.003145   0.02   0.00302167]

Q1 values: [-1.469055 0.02   -1.411445]

Q2 values: [0.07701 0.02   0.07399]

Q3 values: [1.34181 0.02   1.28919]

Year 1981:

Mean values: [-0.00272   0.02   -0.00261333]

Q1 values: [-1.325235 0.02   -1.273265]

Q2 values: [-0.03213 0.02   -0.03087]

Q3 values: [1.580235 0.02   1.518265]

Year 1982:

Mean values: [0.004335 0.02 0.004165]

Q1 values: [-1.577175 0.02 -1.515325]

Q2 values: [-0.19074 0.02 -0.18326]

Q3 values: [1.67382 0.02 1.60818]

Year 1983:

Mean values: [-0.02924 0.02 -0.02809333]

Q1 values: [-1.40403 0.02 -1.34897]

Q2 values: [-0.07242 0.02 -0.06958]

Q3 values: [1.457325 0.02 1.400175]

Year 1984:

Mean values: [-0.02125 0.02 -0.02041667]

Q1 values: [-1.459365 0.02 -1.402135]

Q2 values: [0.12189 0.02 0.11711]

Q3 values: [1.228335 0.02 1.180165]

Year 1985:

Mean values: [0.006545 0.02 0.00628833]

Q1 values: [-1.625115 0.02 -1.561385]

Q2 values: [0.24939 0.02 0.23961]

Q3 values: [1.75236 0.02 1.68364]

Year 1986:

Mean values: [0.02414   0.02   0.02319333]

Q1 values: [-1.36782 0.02   -1.31418]

Q2 values: [-0.01734 0.02   -0.01666]

Q3 values: [1.63353 0.02   1.56947]

Year 1987:

Mean values: [-0.0038675   0.02   -0.00371583]

Q1 values: [-1.38516 0.02   -1.33084]

Q2 values: [0.02601 0.02   0.02499]

Q3 values: [1.7940525 0.02   1.7236975]

Year 1988:

Mean values: [0.04539 0.02   0.04361]

Q1 values: [-1.57692 0.02   -1.51508]

Q2 values: [0.03825 0.02   0.03675]

Q3 values: [1.75134 0.02   1.68266]

Year 1989:

Mean values: [-0.0204 0.02   -0.0196]

Q1 values: [-1.263525 0.02   -1.213975]

Q2 values: [-0.20094 0.02   -0.19306]

Q3 values: [1.358895 0.02   1.305605]

Year 1990:

Mean values: [-0.038845   0.02   -0.03732167]

Q1 values: [-1.660815 0.02   -1.595685]

Q2 values: [-0.18972 0.02   -0.18228]

Q3 values: [1.367565 0.02   1.313935]

Year 1991:

Mean values: [0.03264 0.02   0.03136]

Q1 values: [-1.33977 0.02   -1.28723]

Q2 values: [-0.3213 0.02   -0.3087]

Q3 values: [1.54785 0.02   1.48715]

Year 1992:

Mean values: [0.00068   0.02   0.00065333]

Q1 values: [-1.29489 0.02   -1.24411]

Q2 values: [0.13413 0.02   0.12887]

Q3 values: [1.3974 0.02   1.3426]

Year 1993:

Mean values: [0.007395 0.02   0.007105]

Q1 values: [-1.548615 0.02   -1.487885]

Q2 values: [0.08211 0.02   0.07889]

Q3 values: [1.773525 0.02   1.703975]

Year 1994:

Mean values: [0.00629   0.02   0.00604333]

Q1 values: [-1.52337 0.02   -1.46363]

Q2 values: [0.09282 0.02   0.08918]

Q3 values: [1.566465 0.02   1.505035]

Year 1995:

Mean values: [-0.03332   0.02   -0.03201333]

Q1 values: [-1.525665 0.02   -1.465835]

Q2 values: [-0.10608 0.02   -0.10192]

Q3 values: [1.163055 0.02   1.117445]

Year 1996:

Mean values: [0.001105   0.02   0.00106167]

Q1 values: [-1.137555 0.02   -1.092945]

Q2 values: [0.18921 0.02   0.18179]

Q3 values: [1.14546 0.02   1.10054]

Year 1997:

Mean values: [0.020315   0.02   0.01951833]

Q1 values: [-1.57947 0.02   -1.51753]

Q2 values: [-0.06936 0.02   -0.06664]

Q3 values: [1.44993 0.02   1.39307]

Year 1998:

Mean values: [-0.015045 0.02   -0.014455]

Q1 values: [-1.565445 0.02   -1.504055]

Q2 values: [-0.31059 0.02   -0.29841]

Q3 values: [1.635315 0.02   1.571185]

Year 1999:

Mean values: [-0.03162 0.02   -0.03038]

Q1 values: [-1.56978 0.02   -1.50822]

Q2 values: [0.30651 0.02   0.29449]

Q3 values: [1.230375 0.02   1.182125]

Year 2000:

Mean values: [-0.00068   0.02   -0.00065333]

Q1 values: [-1.578195 0.02   -1.516305]

Q2 values: [-0.01989 0.02   -0.01911]

Q3 values: [1.681215 0.02   1.615285]

Year 2001:

Mean values: [0.010965 0.02   0.010535]

Q1 values: [-1.37904 0.02   -1.32496]

Q2 values: [-0.18615 0.02   -0.17885]

Q3 values: [1.57998 0.02   1.51802]

Year 2002:

Mean values: [-0.00102 0.02   -0.00098]

Q1 values: [-1.44075 0.02   -1.38425]

Q2 values: [-0.04539 0.02   -0.04361]

Q3 values: [1.64628 0.02   1.58172]

Year 2003:

Mean values: [-0.00663 0.02   -0.00637]

Q1 values: [-1.62588 0.02   -1.56212]

Q2 values: [-0.14586 0.02   -0.14014]

Q3 values: [1.566975 0.02   1.505525]

Year 2004:

Mean values: [-0.02771   0.02   -0.02662333]

Q1 values: [-1.40454 0.02   -1.34946]

Q2 values: [-0.00612 0.02   -0.00588]

Q3 values: [1.22349 0.02   1.17551]

Year 2005:

Mean values: [-0.00238   0.02   -0.00228667]

Q1 values: [-1.51725 0.02   -1.45775]

Q2 values: [-0.23715 0.02   -0.22785]

Q3 values: [1.27398 0.02   1.22402]

Year 2006:

Mean values: [-0.00374 0.02 -0.00359333]

Q1 values: [-1.498635 0.02 -1.439865]

Q2 values: [-0.13158 0.02 -0.12642]

Q3 values: [0.94809 0.02 0.91091]

Year 2007:

Mean values: [0.001785 0.02 0.001715]

Q1 values: [-1.431825 0.02 -1.375675]

Q2 values: [-0.37587 0.02 -0.36113]

Q3 values: [1.578705 0.02 1.516795]

Year 2008:

Mean values: [-0.003485 0.02 -0.00334833]

Q1 values: [-1.654695 0.02 -1.589805]

Q2 values: [-0.15963 0.02 -0.15337]

Q3 values: [1.66719 0.02 1.60181]

Year 2009:

Mean values: [0.00918 0.02 0.00882]

Q1 values: [-1.722525 0.02 -1.654975]

Q2 values: [-0.01428 0.02 -0.01372]

Q3 values: [1.631745 0.02 1.567755]

Year 2010:

Mean values: [-0.04641 0.02   -0.04459]

Q1 values: [-1.92117 0.02   -1.84583]

Q2 values: [0.21063 0.02   0.20237]

Q3 values: [1.326765 0.02   1.274735]

Year 2011:

Mean values: [0.04352   0.02   0.04181333]

Q1 values: [-1.763835 0.02   -1.694665]

Q2 values: [0.07089 0.02   0.06811]

Q3 values: [1.740885 0.02   1.672615]

Year 2012:

Mean values: [-0.012665   0.02   -0.01216833]

Q1 values: [-1.94463 0.02   -1.86837]

Q2 values: [0.19227 0.02   0.18473]

Q3 values: [1.363485 0.02   1.310015]

Year 2013:

Mean values: [-0.002805 0.02   -0.002695]

Q1 values: [-1.384905 0.02   -1.330595]

Q2 values: [-0.00969 0.02   -0.00931]

Q3 values: [1.56417 0.02   1.50283]

Year 2014:

Mean values: [0.01411   0.02   0.01355667]

Q1 values: [-1.52745 0.02   -1.46755]

Q2 values: [-0.03621 0.02   -0.03479]

Q3 values: [1.554735 0.02   1.493765]

Year 2015:

Mean values: [-0.020315   0.02   -0.01951833]

Q1 values: [-1.640925 0.02   -1.576575]

Q2 values: [0.12393 0.02   0.11907]

Q3 values: [1.312995 0.02   1.261505]

Year 2016:

Mean values: [-0.02125   0.02   -0.02041667]

Q1 values: [-1.7646 0.02   -1.6954]

Q2 values: [0.19992 0.02   0.19208]

Q3 values: [1.35609 0.02   1.30291]

Year 2017:

Mean values: [-0.01785 0.02   -0.01715]

Q1 values: [-1.488435 0.02   -1.430065]

Q2 values: [0.05355 0.02   0.05145]

Q3 values: [1.396125 0.02   1.341375]

Year 2018:

Mean values: [0.01632 0.02   0.01568]

Q1 values: [-1.643475 0.02   -1.579025]

Q2 values: [-0.04182 0.02   -0.04018]

Q3 values: [1.317075 0.02   1.265425]

Year 2019:

Mean values: [0.02686   0.02   0.02580667]

Q1 values: [-1.545555 0.02   -1.484945]

Q2 values: [-0.21114 0.02   -0.20286]

Q3 values: [1.84263 0.02   1.77037]

Year 2020:

Mean values: [-0.01581 0.02   -0.01519]

Q1 values: [-1.6728 0.02   -1.6072]

Q2 values: [-0.31926 0.02   -0.30674]

Q3 values: [1.63149 0.02   1.56751]

Year 2021:

Mean values: [0.036465 0.02   0.035035]

Q1 values: [-1.450695 0.02   -1.393805]

Q2 values: [0.02907 0.02   0.02793]

Q3 values: [1.67994 0.02   1.61406]

Year 2022:

Mean values: [-0.040545 0.02   -0.038955]

Q1 values: [-1.390515 0.02   -1.335985]

Q2 values: [-0.19074 0.02   -0.18326]

Q3 values: [1.51419 0.02   1.45481]

Mean and Quartiles of Each Month:

January:

Mean values: [0.59544818 0.02   0.57209727]

Q1 values: [0.45543 0.02   0.43757]

Q2 values: [0.63189 0.02   0.60711]

Q3 values: [0.737715 0.02   0.708785]

February:

Mean values: [0.59544818 0.02   0.57209727]

Q1 values: [0.45543 0.02   0.43757]

Q2 values: [0.63189 0.02   0.60711]

Q3 values: [0.737715 0.02   0.708785]

March:

Mean values: [-0.19475045 0.02   -0.18711318]

Q1 values: [-0.3876 0.02   -0.3724]

Q2 values: [-0.19329 0.02   -0.18571]

Q3 values: [-0.04284 0.02   -0.04116]

April:

Mean values: [-1.08363409 0.02   -1.04113864]

Q1 values: [-1.20054 0.02   -1.15346]

Q2 values: [-1.04193 0.02   -1.00107]

Q3 values: [-0.919275 0.02   -0.883225]

May:

Mean values: [-1.52031 0.02   -1.46069]

Q1 values: [-1.666425 0.02   -1.601075]

Q2 values: [-1.48002 0.02   -1.42198]

Q3 values: [-1.34232 0.02   -1.28968]

June:

Mean values: [-1.83634773 0.02   -1.76433409]

Q1 values: [-2.121345 0.02   -2.038155]

Q2 values: [-1.71105 0.02   -1.64395]

Q3 values: [-1.4943 0.02   -1.4357]

July:

Mean values: [-2.84408455 0.02   -2.73255182]

Q1 values: [-3.059235 0.02   -2.939265]

Q2 values: [-2.7693 0.02   -2.6607]

Q3 values: [-2.574735 0.02   -2.473765]

August:

Mean values: [-1.85908909 0.02   -1.78618364]

Q1 values: [-2.043825 0.02   -1.963675]

Q2 values: [-1.76868 0.02   -1.69932]

Q3 values: [-1.663365 0.02   -1.598135]

September:

Mean values: [0.13037455 0.02   0.12526182]

Q1 values: [-0.047685 0.02   -0.045815]

Q2 values: [0.08823 0.02   0.08477]

Q3 values: [0.322065 0.02   0.309435]

October:

Mean values: [2.87839364 0.02   2.76551545]

Q1 values: [2.542605 0.02   2.442895]

Q2 values: [2.96004 0.02   2.84396]

Q3 values: [3.1875 0.02   3.0625]

November:

Mean values: [2.26834091 0.02   2.17938636]

Q1 values: [2.03031 0.02   1.95069]

Q2 values: [2.25369 0.02 2.16531]

Q3 values: [2.459985 0.02 2.363515]

December:

Mean values: [2.07550295 0.02 1.99411068]

Q1 values: [1.839315 0.02 1.767185]

Q2 values: [2.04867 0.02 1.96833]

Q3 values: [2.313615 0.02 2.222885]

Mean and Quartiles of First Season (Nov-Feb):

Mean values array: [1.38368506 0.02 1.3294229]

Quartiles values array:

1st quartile values:

[0.635205 0.02 0.610295]

2nd quartile values:

[1.07457 0.02 1.03243]

3rd quartile values:

[2.159595 0.02 2.074905]

Mean and Quartiles of First Season (Mar-Jun):

Mean values array: [-1.15876057 0.02 1.12690534]

Quartiles values array:

1st quartile values:

[-1.625115 0.02 0.68551]

2nd quartile values:

[-1.24338 0.02   1.19462]

3rd quartile values:

[-0.71349   0.02   1.561385]

Mean and Quartiles of First Season (Jul-Oct):

Mean values array: [-0.42360136 0.02   1.87712318]

Quartiles values array:

1st quartile values:

[-2.41893   0.02   1.087555]

2nd quartile values:

[-0.81957 0.02   2.03987]

3rd quartile values:

[0.94503 0.02   2.75429]

Appendix F: List of Symbols and Abbreviations

- **Mathematical Symbols:**
  (T, I, F) : Neutrosophic triplet (Truth, Indeterminacy, Falsity)
  $T(X \rightarrow Y)$: Truth value of causal relationship from X to Y
  $I(X \rightarrow Y)$: Indeterminacy value of causal relationship from X to Y
  $F(X \rightarrow Y)$: Falsity value of causal relationship from X to Y
  ε : Error term
  β_0, β_1 : Regression coefficients
  τ : Causal effect
  $\|$: Norm
  θ : Parameter vector
  $Y_i(t)$: Potential outcome under treatment
  $Y_i(c)$: Potential outcome under control
  $\rightarrow$: Implies/Leads to
  $\sum$: Summation
  $|$: Given that/Conditional on

- **Statistical Abbreviations:**
  NCGM: Neutrosophic Causal Graphical Model
  NSEM: Neutrosophic Structural Equation Model
  NPSM: Neutrosophic Propensity Score Matching
  NIV: Neutrosophic Instrumental Variable
  NCF: Neutrosophic Counterfactual
  NCFS: Neutrosophic Causal Feature Selection
  NPC: Neutrosophic PC (Peter-Clark) Algorithm
  NFCI: Neutrosophic FCI Algorithm
  NGES: Neutrosophic Greedy Equivalence Search
  NMBD: Neutrosophic Markov Blanket Discovery
  NCL: Neutrosophic Causal Lasso
  NNRM: Neutrosophic Non-linear Regression Model
  NPOF: Neutrosophic Potential Outcomes Framework
  NCG: Neutrosophic Counterfactual Graph
  ANOVA: Analysis of Variance
  DAG: Directed Acyclic Graph

- **Climate Science Abbreviations:**
  NAO: North Atlantic Oscillation
  AO: Arctic Oscillation
  CO_2: Carbon Dioxide Units of Measurement:
  km^2: Square Kilometers
  °C: Degrees Celsius
  °F: Degrees Fahrenheit