ASYMPTOTIC SQUARE PACKING PROBLEMS

by

Rory McClenagan

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Abstract

In this thesis, we study two problems that belong to the family of infinite square packing problems. Our first result establishes a multidimensional generalization of a result of Terrance Tao that proved a weak form of the Meir-Moser packing problem. More precisely, we show that if $\frac{1}{d} < t < \frac{1}{d-1}$, and n_0 is sufficiently large depending on t, then the d-dimensional cubes of sidelength n^{-t} for $n \geqslant n_0$ can perfectly pack a d-dimensional cube of volume $\sum_{n=n_0}^{\infty} \frac{1}{n^{dt}}$. Our second result proves that the wasted space generated when packing a square of sidelength x by unit squares is bounded by $O(x^{3/5})$.

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Acknowledgements

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In writing [10], I would like to thank Professor Terence Tao for the discussion on his blog post describing his work in [14] and for answering my many questions. I would also like to thank Professor Tao's research team members Jaume de Dios, Dr. Rachel Greenfeld, and Dr. Jose Madrid for helpful comments.

Indeed, Professor Tao has often been my source for mathematical inspiration over nearly a decade. His ability to express complex and deep ideas in a beautiful and elegant manner is hard to match and is one of the key reasons why I love mathematics so much to this day.

Finally, I would like to thank my family. My parents' encouragement and examples of hard work and my brother's willingness to listen to my latest mathematical

idea have been invaluable.

Chapter 1

Introduction

1.1 Some History and Overview of Existing Literature

Packing problems are concerned with configuring a collection of packing shapes into some larger container such that the interiors of the packing shapes do not overlap. In general, the goal is to maximize the proportion of the container that is packed.

For instance, consider the problem of arranging circles inside of a larger container (see Figure 1.1). We can see that the "hexagonal" packing on the left is more dense than the packing on the right. If we define the *packing density* of a particular packing as the ratio of the packed area to the total area of the container, then we could say that the hexagonal packing pictured in Figure 1.1 has a higher packing density than the packing on the right.

Indeed, a classical finite packing problem is to determine the smallest circle that can still contain a fixed number n of unit circles and estimate the corresponding packing density of such a configuration. These problems tend to be surprisingly difficult, especially for larger fixed n. Natural extensions to differently shaped packing containers (e.g., squares or equilateral triangles) have also been studied.

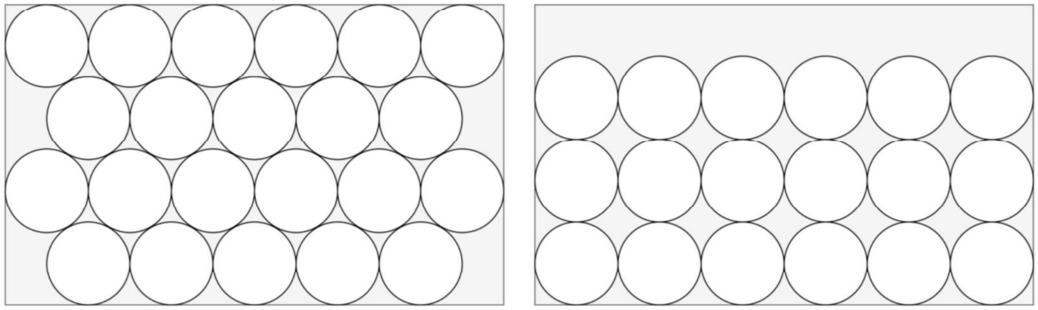

Figure 1.1: A comparison of a packing with higher packing density (left) and lower packing density (right).

See [6] for a review of this topic.

We are not necessarily restrained to instances where the number of packing objects is finite. For instance, in the above example, we can consider the asymptotic behaviour of the packing density when packing circles into asymptotically larger containers in $\mathbb{R}^2$. In this scenario, one can compute that the ”hexagonal” packing depicted on the left-hand side of Figure 1.1 would have an asymptotic packing density of $\frac{\pi\sqrt{3}}{6} \approx 0.9$, while the packing arrangement on the right would have an asymptotic packing density of $\frac{\pi}{4} \approx 0.8$.

This introduces the question of whether or not the ”hexagonal” packing has the highest packing density for any configuration of congruent circles. This question can be answered in the affirmative, and is known as Thue’s theorem. If only lattice configurations were considered (where the center points of the circles are arranged in a lattice), the hexagonal packing can easily be shown to be the densest packing, a result that is usually attributed to Lagrange (see [1]) all the way back in 1773. Thue claimed to have proven the general result in 1890, but his proof was considered incomplete. It was not until 1940 that Fejes Tóth proved that the hexagonal packing was the densest of *all* possible packings [15].

We can also consider cases where the the packing objects are unequally sized. In these situations we are often concerned with attaining a *perfect packing*, one in which the packing shapes completely cover the container and include no gaps

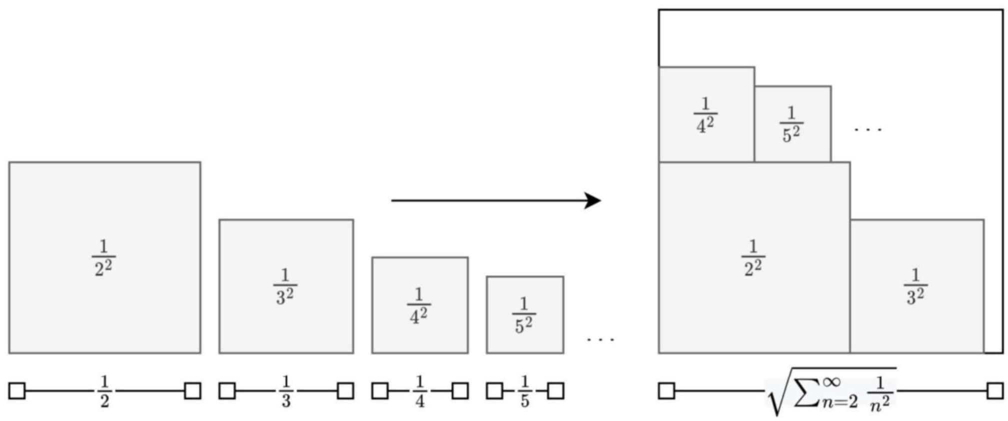

Figure 1.2: One of the classic square-packing problem posed by Meir and Moser.

(namely, the packing density is 1). Let us take a look at a concrete example.

In [11], Meir and Moser introduced two closely related packing problems that remain open to this day. The first asked whether rectangles of dimensions $\frac{1}{n} \times \frac{1}{n+1}$ for $n \geqslant 1$ can perfectly pack a square of area 1. The second asked whether squares of dimensions $\frac{1}{n} \times \frac{1}{n}$ for $n \geqslant 2$ can perfectly pack a square (or rectangle) of area $\frac{\pi^2}{6} - 1$ (see Figure 1.2).

Currently, no solution to either conjecture exists. However, we can instead try to perfectly pack the squares of sidelength $1/n^t$ for $n \geqslant n_0$, with some fixed $n_0 \geqslant 2$ and $t > 1/2$, into a square of area $\sum_{n=n_0}^{\infty} \frac{1}{n^{2t}}$. For reasons that are not immediately obvious, this becomes harder as $t \to 1^-$, and is obviously equivalent to the original problem when $t = 1$ and $n_0 = 2$. Januszewski and Zielonka [7] verified this for $1/2 < t \leqslant 2/3$ and $n_0 = 1$. In 2022, Tao [14] proved that one could extend the range for t to the open interval $1/2 < t < 1$ for some large n_0 that only depends on t.

We can also consider the multidimensional generalization of this problem, where we are attempting to pack d-dimensional bricks of sidelength n^{-t} for $n \geqslant n_0$ and for $\frac{1}{d} < t < \frac{1}{d-1}$ into a d-dimensional cube of volume $\sum_{n=n_0}^{\infty} \frac{1}{n^{dt}}$, a result which we will prove in this thesis (see Theorem 1.1, which was published in [10]).

We can also consider an imperfect infinite packing problem where we try to optimize the packing density. Consider a large square $S(x)$ of sidelength x. Pack

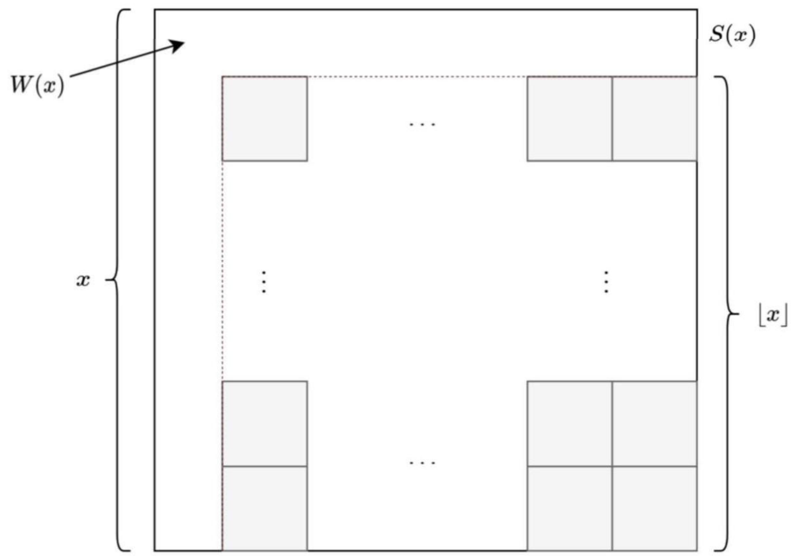

Figure 1.3: The trivial packing of $S(x)$ by squares of unit sidelength. The wasted space $W(x)$ is $O(x)$ if $\{x\} = x - \lfloor x \rfloor$ is bounded away from 0.

$S(x)$ by non-overlapping unit squares, and let $W(x)$ represent the remaining unpacked area, or wasted space, of $S(x)$. Our goal is to pack $S(x)$ in such a way that we minimize this wasted space and determine the optimal bound for $W(x)$.

If we pack $S(x)$ in the trivial manner where all of the unit squares have parallel sides (see Figure 1.3), then $W(x) \ll x\{x\}$, where $\{x\}$ represents the fractional part of x. This bound is only $O(x)$ when $\{x\}$ is bounded away from 0.

If one instead packs the squares at slight angles, the wasted space can be decreased. Indeed, determining the right order of magnitude of $W(x)$ (when x is bounded away from an integer) remains an open problem. It was first attacked by Erdös and Graham in 1975 (see [4]), in which they proved $W(x) \ll x^{7/11}$. While Chung and Graham in [3] introduced a new packing algorithm that they claimed reduced the wasted space bound to $O(x^{3/5})$, we found that their proof contains an error, and it only reproves the original bound $O(x^{7/11})$. We introduce a more sophisticated algorithm that allows us to attain the bound $O(x^{3/5})$ (see Theorem 1.2).

1.2 Main Results

In this thesis, we will extend Tao's work in [14] to its multidimensional setting to demonstrate the perfect packing of cubes of nearly harmonic sidelengths (note that this result is published in [10]):

Theorem 1.1. *If* $\frac{1}{d} < t < \frac{1}{d-1}$, *and* n_0 *is sufficiently large depending on* t, *then the cubes of sidelength* n^{-t} *for* $n \geqslant n_0$ *can perfectly pack a cube of volume* $\sum_{n=n_0}^{\infty} \frac{1}{n^{dt}}$.

In addition, we will improve the state of the art bound on the wasted space generated when packing a large square by unit squares in the following result:

Theorem 1.2. *The wasted space in packing the square* $S(x)$ *by unit squares is bounded by*

$$W(x) = O(x^{3/5}).$$

1.3 Organization of the Thesis

This thesis is organized as follows. In Chapter 2, we give a detailed overview of the progress that has been made towards the Meir-Moser problem. We discuss the methodologies used in these results including a detailed look at Tao's argument in [14]. We end by summarizing some of the difficulties that arise when extending Tao's result to its multidimensional setting. We prove this multidimensional analogue, Theorem 1.1, in Chapter 3.

In Chapter 4, we give a history of the upper and lower bounds on the wasted space generated in the unit square packing problem. We end by discussing the techniques that have been used to date, their intuitions, and shortcomings, including an explanation of where the mistake was made in [3]. This sets us up for the proof of Theorem 1.2 in Chapter 5.

1.4 Conventions and Notation

Throughout this work, we will use the standard asymptotic notation $X = O(Y)$, $X \ll Y$, and $Y \gg X$ to refer to the relation $X \leqslant C|Y|$. In a specific section we will note if we are allowing the constant C to depend on auxiliary parameters. We will explicitly use the notation $X = O_M(Y)$ or $X \ll_M Y$ to signify that the corresponding constant C is allowed to depend on the parameter M. We use $X \asymp Y$ if $X \ll Y$ and $Y \ll X$. We use $X = \Omega(Y)$ to refer to the case when $\sup|X/Y| > 0$. Finally, we use the notation $X = o(Y)$ if $X/Y \to 0$. All of these notations hold with respect to some explicit or implicit limiting behaviour defined in context.

Chapter 2

An Overview of the Meir-Moser Problem and its Variants

In this chapter we will focus on the square version of the Meir-Moser problem (see [11]), as most of the methods will generalize in a straightforward manner to the rectangular version. Recall that this problem asked whether squares of dimensions $\frac{1}{n} \times \frac{1}{n}$ for $n \geqslant 2$ can perfectly pack a square (or rectangle) of area $\frac{\pi^2}{6} - 1$ (see Figure 1.2). By perfectly pack, we mean that the squares are packed in such a way that there is no wasted space, and the packing density is 1.

While there is still no complete solution to the Meir-Moser problem, various partial results have been established. In Section 2.1, we will discuss progress that has been made towards the weaker, imperfect packing, version of the problem where we are trying to pack the same squares into a rectangle that has slightly larger area. In Section 2.2, we look at a different way of weakening the problem, where we perfectly pack squares of sidelength $1/n^t$ for some $t < 1$ instead of squares of sidelength $1/n$. We focus on the result [14] of Tao, explaining some of the motivation behind his work. We end in Section 2.3 by discussing the difficulties behind extending to the $t = 1$ case and the problems that arise when we try to generalize Tao's result to higher dimensions.

2.1 Imperfect Packing Variants

We will begin by discussing the problem of trying to pack the same squares of sidelength $1/n$ for $n \geqslant 2$ into a a rectangle R of area $\frac{\pi^2}{6} - 1 + \varepsilon$ for some $\varepsilon > 0$, a weaker "imperfect" version of the original problem.

In 1997, Paulhus [12] claimed that one could take $\varepsilon = \frac{1}{1244918662}$. While this proof did contain an error, this error was later fixed. Since the basic structure of the corrected proof remains the same, we still discuss some of the ideas that Paulhus used. Paulhus followed a three stage packing method in his proof to pack the squares into a rectangle R_ε with dimensions $\frac{1}{2} \times 2(\frac{\pi^2}{6} - 1 + 2\varepsilon)$. First, he developed an efficient first-phase packing algorithm that allowed him to pack a large number of squares into the rectangle R (of dimensions $\frac{1}{2} \times 2(\frac{\pi^2}{6} - 1)$) testable via computer simulations up to some large N_1. For some large $N_2 > N_1$, the squares of width $\frac{1}{n}$ for $n \in \{N_1, \dots, N_2\}$ were packed into a small sliver of a rectangle adjoined to R of width 2ε using a second-phase packing algorithm, and the infinite number of squares of width $\frac{1}{n}$ for n larger than N_2 were packed into one of the remaining rectangles in this packing using a third-phase algorithm (that had to be proven mathematically since it concerned an infinite number of squares).

The first-phase packing algorithm Paulhus used was a greedy algorithm that packed one square at a time from largest to smallest, which we now describe. Let $S_1, S_2, \dots$ be the squares we are packing into R, ordered such that their widths are in descending order. Suppose that at some point we have packed R by $S_1, \dots, S_n$. Subdivide the remaining space $R \setminus \{S_1, \dots, S_n\}$ into a collection of rectangles $\mathcal{R}$. Let $R_0 \in \mathcal{R}$ be the smallest width rectangle that fits S_{n+1}. Place this next smallest square S_{n+1} inside R_0 such that the corner of S_{n+1} aligns with one of the corners of R_0.

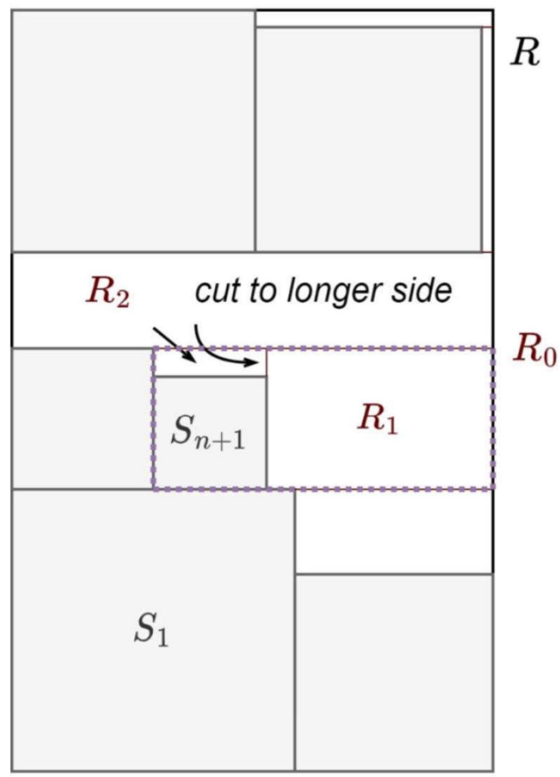

Figure 2.1: An example of Paulhus' first-phase algorithm.

Partition R_0 into two new rectangles R_1 and R_2 by cutting from the free corner of S_{n+1} to the longer side of R. We can then iterate this algorithm by replacing $\mathcal{R}$ with $\mathcal{R} \setminus \{R_0\} \cup \{R_1, R_2\}$. This packing algorithm is pictured in Figure 2.1.

Paulhus applied his algorithm to pack the first 10^9 squares. The second phase algorithm explicitly describes the packing of the squares from 10^9 to 2761408695, and can be verified directly. The third-phase of the algorithm necessitates packing the infinite number of remaining squares. Paulhus appeals to a lemma that states that all of the squares from n to ∞ can be packed in a rectangle R_0 of width w and height h, as long as

$$n \geqslant \frac{h+1}{wh}.$$

To get a heuristic sense of this lemma, consider the case when h is much larger than w. Then $\frac{h+1}{h} \approx 1$, and so the lemma would claim that as long as n was significantly larger than $\frac{1}{w}$, we could pack the squares of width $\frac{1}{n}, \frac{1}{n+1}, \ldots$. In other words, rearranging the inequality, as long as R_0 was long and narrow, the lemma would only require that the rectangle R_0 be a little wider than the very first square we would pack of width $\frac{1}{n}$.

On the other hand, if R_0 is sufficiently small and it is not too narrow, we would have $h + 1 \approx 1$ and the lemma would only require n to be a little larger than $1/wh$. Since the squares we are packing have total area $\sum_{i=n}^{\infty} \frac{1}{i^2} \asymp \frac{1}{n}$, this lemma would roughly be saying that we would only need the area of R_0 to be a little larger than the area of all of the squares we were packing.

After applying the first phase of the Paulhus' algorithm computationally, Paulhus confirmed that there was an empty space with dimensions $w = h = 0.00001903$, which allowed him to pack the squares with $n \geqslant 2761408695$ by this lemma.

Paulhus proved the lemma by performing a naive packing, placing the squares along the long-edge of R_0 in rows. However, it was pointed out in 2017 (see [8]) that this proof contained an obvious error. The naive packing method that Paulhus used does not actually yield the necessary lemma.

Two years later, Grzegorek and Januszewski showed in [5] that if the lemma was constrained to the particular case in which Paulhus actually applied it, the result was still true, provided the underlying packing method and proof in the lemma was changed to a more sophisticated one, thus providing a final rigorous proof of the following result:

Theorem 2.1. *The squares of sidelength $1/n$ for $n \geqslant 2$ can be packed into a square of area $\frac{\pi^2}{6} - 1 + 1/1244918662$.*

Grzegorek and Januszewski relied on packing along the length of R_0, but performed this packing more efficiently by packing in "birows" instead of rows. Each birow consisted of a row of squares with widths descending from left to right and then a row of squares with widths descending from right to left. This pattern was

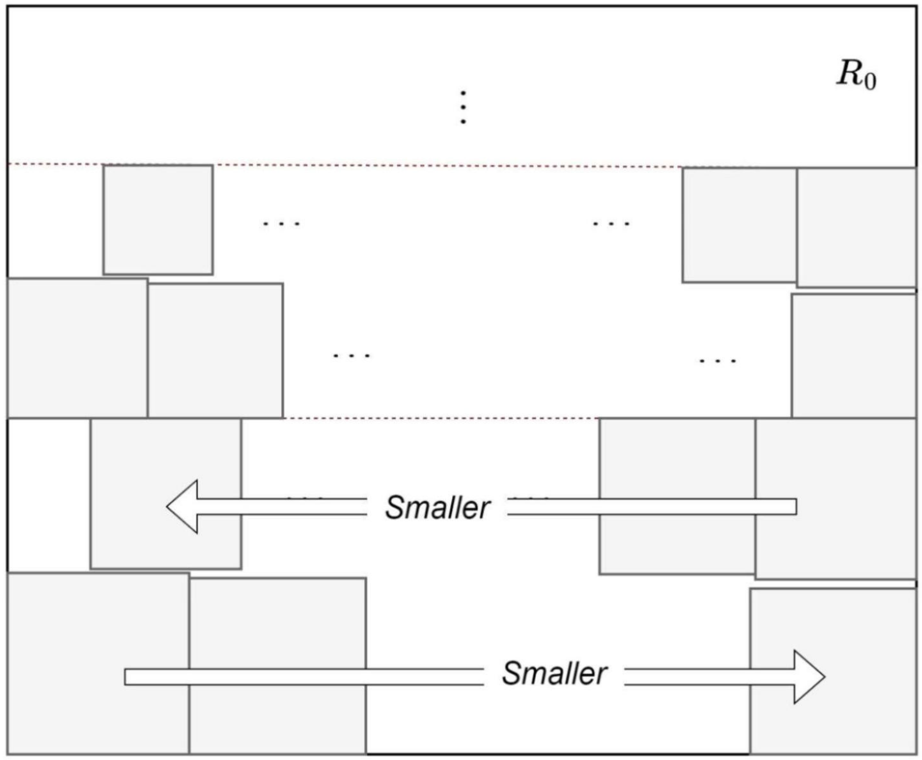

Figure 2.2: Grzegorek and Januszewski's "birow" packing method used to correct the error in the proof of Paulhus's lemma.

then replicated over all of R_0. This packing configuration is illustrated in Figure 2.2.

To date, [5] still holds the best result for the smallest ε in this particular variant of the Meir-Moser packing problem.

2.2 Perfect Packing Variants for $t < 1$

Another way of weakening the Meir-Moser conjecture is to instead try to perfectly pack the squares of sidelength $1/n^t$ for $n \geqslant n_0$, with some fixed $n_0 \geqslant 2$ and $t > 1/2$, into a square of area $\sum_{n=n_0}^{\infty} \frac{1}{n^{2t}}$. This becomes harder as $t \to 1^-$, and is obviously equivalent to the original problem when $t = 1$ and $n_0 = 2$. We will touch upon this issue later in the section.

Januszewski and Zielonka [7] verified this for $1/2 < t \leqslant 2/3$ and $n_0 = 1$. In

2022, Tao [14] proved that one could extend the range for t to the open interval $1/2 < t < 1$ for some large n_0 that only depends on t. More precisely, Tao proved the following theorem.

Theorem 2.2. *Let $1/2 < t < 1$, and suppose that n_0 is a natural number that is sufficiently large depending on t. Then squares of sidelength n^{-t} for $n \geqslant n_0$ can perfectly pack a square of area $\sum_{n=n_0}^{\infty} \frac{1}{n^{2t}}$.*

Let us take a closer look at the ideas behind Tao's proof of Theorem 2.2. Fix $1/2 < t < 1$. We are trying to pack the squares S_n of sidelength $1/n^t$ for $n \geqslant n_0$ into a square S of area $\sum_{n \geqslant n_0} \frac{1}{n^{2t}}$. Suppose that we already packed $\mathcal{S}_{n_1} = \{S_{n_0}, \dots, S_{n_1 - 1}\}$ into S. Partition the remaining space we have to pack, namely $S \setminus \mathcal{S}_{n_1}$, into a collection of rectangles $\mathcal{R}$ (see Figure 2.3). A sufficient condition to be able to pack the next square S_{n_1} is the existence of a rectangle $R \in \mathcal{R}$ with width $w(R) \geqslant \frac{1}{(n_1)^t}$.

Now, Tao's innovation was to develop an "efficient" packing algorithm that kept the perimeter of $\mathcal{R}$ small. This works because as long as the total perimeter of $\mathcal{R}$ is small enough there must be a rectangle $R \in \mathcal{R}$ that is wide enough to pack the next square S_{n_1}. More concretely, define the perimeter of the collection of rectangles $\mathcal{R}$ by

$$\operatorname{perim}(\mathcal{R}) = \sum_{R \in \mathcal{R}} 2\left(w(R) + h(R)\right),$$

where $w(R)$ is the width of R and $h(R)$ is the height of R. Define the area of $\mathcal{R}$ by

$$\operatorname{area}(\mathcal{R}) = \sum_{R \in \mathcal{R}} w(R)h(R),$$

and observe that it is bounded by

$$\operatorname{area}(\mathcal{R}) \leqslant \sup_{R \in \mathcal{R}} w(R) \sum_{R \in \mathcal{R}} h(R) \leqslant \frac{1}{2}\left(\sup_{R \in \mathcal{R}} w(R)\right) \operatorname{perim}(\mathcal{R}).$$

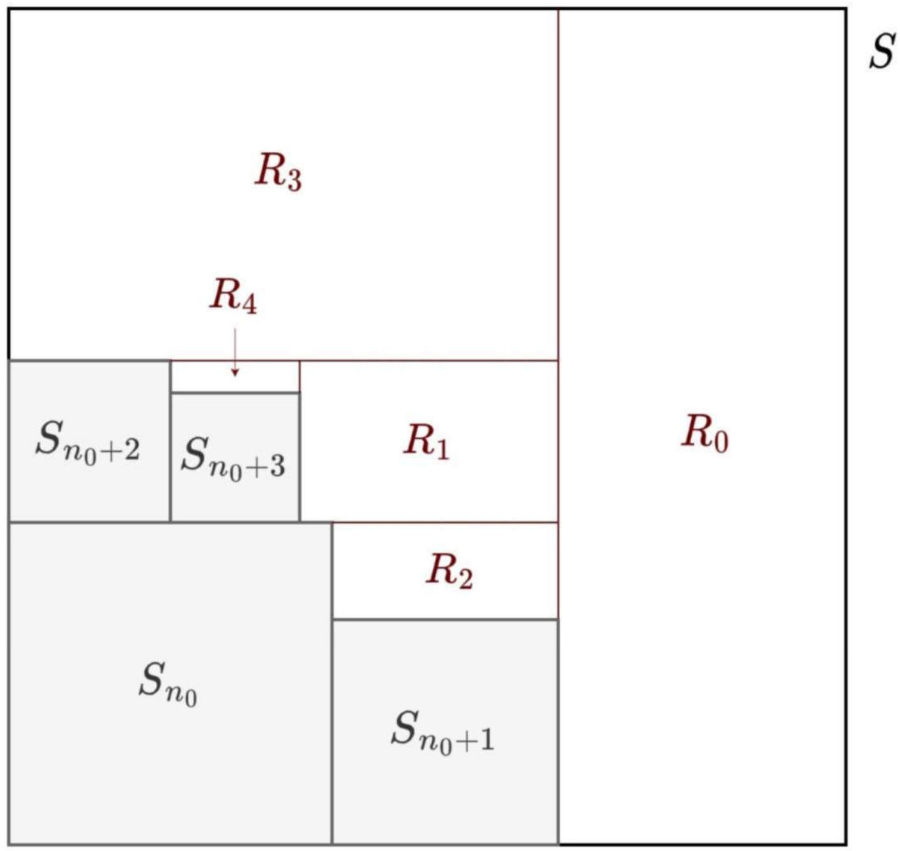

Figure 2.3: Packing S by squares $\mathcal{S}_{n_1}$ with $n_1 = n_0 + 4$. The remaining space has been partitioned into a collection of rectangles $\mathcal{R} = \{R_0, R_1, \ldots, R_4\}$.

Thus, there always exists a rectangle $R \in \mathcal{R}$ satisfying

$$w(R) \geqslant 2\left(\frac{\text{area}(\mathcal{R})}{\text{perim}(\mathcal{R})}\right). \tag{2.1}$$

How large of a rectangle R do we want to find? Well, at worse, we would want to find a rectangle $R \in \mathcal{R}$ such that $w(R) \geqslant \frac{1}{n_1^t}$. To satisfy this condition, we would need the perimeter of $\mathcal{R}$ to be bounded by $n_1^t \text{area}(\mathcal{R})$. Observe that

$$\text{area}(\mathcal{R}) = \sum_{n \geqslant n_0} \frac{1}{n^{2t}} - \sum_{n_0 \leqslant n < n_1} \frac{1}{n^{2t}} \sim \left(\frac{1}{2t-1}\right) \frac{1}{n_1^{2t-1}}, \quad \text{as } n_1 \to \infty.$$

Thus, for large n_1, to ensure the existence of a rectangle with width at least $\frac{1}{n_1^t}$, it would suffice to require that

$$\operatorname{perim}(\mathcal{R}) \leqslant \left(\frac{1}{2t-1}\right)(n_1)^{1-t}. \tag{2.2}$$

Thus, we can only increase the perimeter of $\mathcal{R}$ at a rate of about $p(t)(n_1)^{-t}$, where

$$p(t) = \frac{1-t}{2t-1}.$$

Note that $p(t)$ is decreasing for $t > 1/2$, tends to infinity as $t \to 1/2^+$ and $p(1) = 0$.

The next step would be to come up with some sort of packing algorithm such that for each square of width n_1^{-t} that we packed, the perimeter of $\mathcal{R}$ only increased by about $p(t)n_1^{-t}$. This immediately illustrates the critical nature of the point $t = 1$, since that scenario would require the perimeter of $\mathcal{R}$ not to increase at all, and to remain bounded. However, as we move t towards $1/2$, this problem becomes easier since $p(t)$ gets larger. This, in turn, allows us to pack a little less efficiently since we have a larger perimeter allotment to play with.

If we are only packing a single square in our rectangle R, then the best case scenario is to place the square so it is aligned with the corner of the rectangle (see Figure 2.4). This means that R is replaced by two rectangles, which have total perimeter $\operatorname{perim}(R) + 2(w(R) - n_0^{-t})$. Now, unless $w(R)$ happens to be very close to n_1^{-t}, there is no particular reason that this quantity would be bounded by $p(t)n_1^{-t}$, at least when t is close to 1 (if t was sufficiently close to $1/2$, then $p(t)$ tends to infinity, making this bound much easier to attain).

If we wish to push the range of t up to 1, we would need a more sophisticated technique. The solution is to require our rectangle to be wider so that we can pack batches of squares at the same time in an efficient manner, instead of packing a single square at a time. For some large fixed M, let us instead require that $w(R) \geqslant$

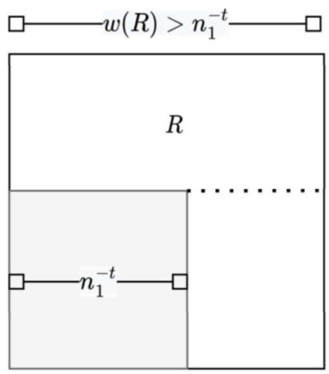

Figure 2.4: A simplistic packing algorithm.

Mn_1^{-t}. This would replace the perimeter bound (2.2) with

$$\text{perim}(\mathcal{R}) \leqslant \frac{1}{M}\left(\frac{1}{2t-1}\right)(n_1)^{1-t}, \tag{2.3}$$

and the maximal rate of increase for the perimeter of $\mathcal{R}$ to be $\frac{1}{M}p(t)n_1^{-t}$, where $p(t) = \frac{1-t}{2t-1}$.

Our rectangle R now has width at least Mn_1^{-t}. In fact, if we define $M_1 = \left\lfloor \frac{w(R)}{n_1^{-t}} \right\rfloor$ and $M_2 = \left\lfloor \frac{h(R)}{n_1^{-t}} \right\rfloor$ then clearly $M_2 \geqslant M_1 \geqslant M$, and we should easily be able to pack R with M_1M_2 squares of sidelengths $n_1^{-t}, (n_1+1)^{-t}, \ldots, (n_2-1)^{-t}$ where $n_2 = n_1 + M_1M_2$. If we, for now, enforce that $M_1 \asymp M_2 \asymp M$, then $n_1 \asymp n_2$, and this means that we are allowed to increase the perimeter of $\mathcal{R}$ by at most $p(t)Mn_1^{-t}$. We now have to find such a packing algorithm.

A comparison of two packing algorithms when $M_1 = 3$ and $M_2 = 4$ is given in Figure 2.5. A naive packing algorithm is attempted on the left-hand side. Due to the ”slivers” in between the squares, the perimeter of $\mathcal{R}$ is increased by an amount approximately equal to the total perimeter of the M_1M_2 squares, namely about $M^2n_1^{-t}$, which is clearly not good enough for our purposes. However, on the right-hand side of Figure 2.5, we have arranged the squares in a near-lattice type structure. Here, the only substantial increase in perimeter is due to the rectangles that will be formed against the $M_1 + M_2 - 1$ *boundary squares* (in red). Thus, the perime-

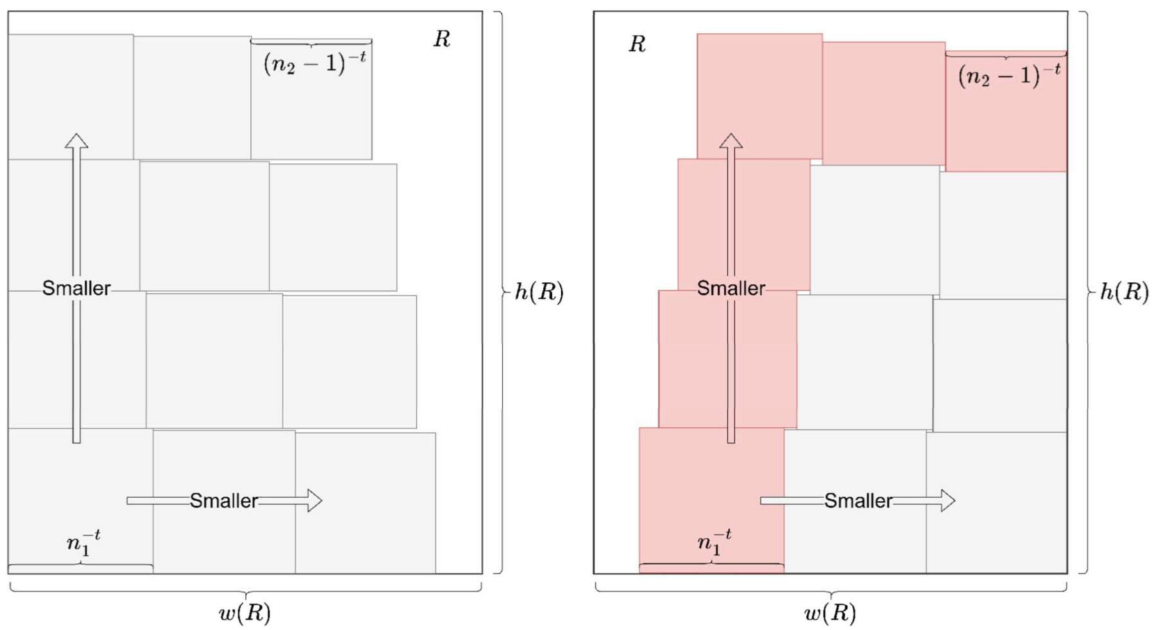

Figure 2.5: A lower quality (left) and higher quality (right) packing algorithm.

ter is only increased by about Mn_1^{-t}, as desired.

Now, even though we have found an efficient packing algorithm that allows us to only increase the perimeter of $\mathcal{R}$ at a rate of $\frac{1}{M}n_1^{-t}$, this is not quite sufficient, since our maximum allotted rate of increase is actually $\frac{p(t)}{M}n_1^{-t}$. Although $p(t)$ is going to be fixed for a particular t, this still makes the induction challenging especially as $t \to 1^-$.

The solution is to introduce a somewhat artifical concept of *weighted perimeter* of a collection of rectangles $\mathcal{R}$:

$$\text{perim}_\delta(\mathcal{R}) = \sum_{R\in\mathcal{R}} w(R)^\delta h(R),$$

for some $\delta > 0$. Clearly, this definition coincides with the definition of unweighted perimeter (up to an order of magnitude) when $\delta = 0$. Then, since

$$\text{area}(\mathcal{R}) = \sum_{R\in\mathcal{R}} w(R)h(R) \leqslant \left(\sup_{R\in\mathcal{R}} w(R)^{1-\delta}\right)\text{perim}_\delta(\mathcal{R}),$$

we can say that there always exists a rectangle $R \in \mathcal{R}$ satisfying

$$w(R) \geqslant \left(\frac{\text{area}(\mathcal{R})}{\text{perim}_\delta(\mathcal{R})} \right)^{\frac{1}{1-\delta}}. \tag{2.4}$$

Why does this help us? Well, now, to ensure there is a rectangle of width Mn_1^{-t}, we only need

$$\text{perim}_\delta(\mathcal{R}) \leqslant \frac{1}{M^{1-\delta}} \left(\frac{1}{2t-1} \right) (n_1)^{1-(1+\delta)t}, \tag{2.5}$$

replacing the old perimeter bound (2.3). This allows for a rate of increase in the weighted perimeter of $\mathcal{R}$ amounting to $\frac{1}{M^{1-\delta}} p_\delta(t) n_1^{-(1+\delta)t}$, where

$$p_\delta(t) = \frac{1-(1+\delta)t}{2t-1}.$$

Now, the weighted perimeter of a single rectangle with sidelength $\asymp n_1^{-t}$ is going to be $n_1^{-(1+\delta)t}$, which means for each square we add in the packing algorithm depicted in Figure 2.5, the weighted perimeter is only going to increase by an average amount of $\frac{1}{M} n_1^{-(1+\delta t)}$. However, this is less than our allowed weighted perimeter rate of increase by a multiplicative factor of $M^\delta p_\delta(t)$. If we choose $\delta = 1-t$, then $\delta > 0$ while $(1+\delta)t < 1$. Thus, since t is fixed and M can be chosen large, the factor $M^\delta p_\delta(t)$ can be made large enough so that the desired inequality is satisfied.

The intuition behind weighted perimeter is that smaller rectangles are weighted a little less than larger rectangles. This means that even though the unweighted perimeter of the little rectangles formed after packing R is approximately the same or a little larger than unweighted perimeter of R itself, the weighted perimeter is smaller.

2.3 Extensions and Generalizations

The major obstacle preventing us from extending the arguments in Section 2.2 to $t = 1$ is the fact that the allowable perimeter of the rectangles $\mathcal{R}$, computed in (2.2), would now have to remain bounded throughout the entire packing algorithm. This is in contrast to the case when $t < 1$, where for each square of width $\frac{1}{n^t}$ that we pack, we can increase the total perimeter of $\mathcal{R}$ by about $O\left(\frac{1}{n^t}\right)$. For $t = 1$, our current packing algorithm would still increase the perimeter of $\mathcal{R}$ by about most $O_M(1)\frac{1}{n}$ for each square of width $\frac{1}{n}$ that we pack. This would mean that the perimeter of $\mathcal{R}$ would grow in an unbounded manner. Thus, extending to $t = 1$ would require a much more subtle approach.

Extending Tao's ideas to higher dimensions is feasible. However, while the packing arrangement in Figure 2.5 itself is generalizable to higher dimensions, Tao's method of *explicitly* verifying that this packing is legal and efficient becomes much more difficult in three dimensions or more. Our contribution is to introduce the notion of "snugness"; this allows us to perform this portion of the argument in an elegant fashion which does not become too complex in the higher-dimensional setting. This is the content of Chapter 3 in which we extend Theorem 2.2 to the d-dimensional setting.

Chapter 3

Perfectly Packing a Cube by Cubes of Nearly Harmonic Sidelength

Let d be an integer greater than 1. We define a *brick* to be a closed d-dimensional hyperrectangle and use the term *cube* to refer to a brick with equal sidelengths. We define a *packing* of a finite or infinite collection of bricks $\mathcal{B}$ to be a particular configuration of the bricks in $\mathbb{R}^d$ such that the interior of the bricks are disjoint and the facets of the bricks are parallel to the coordinate hyperplanes. A *packing of* $\mathcal{B}$ *in a solid* $\Omega \subset \mathbb{R}^d$ is a packing of $\mathcal{B}$ such that every brick is contained in Ω. The packing is *perfect* if the measure $m(\Omega \setminus \mathcal{B})$ is 0. In this case the sum of the volumes of the bricks must be equal to $m(\Omega)$.

In this chapter, we extend Tao's work [14] in the 2-dimensional case to the d-dimensional case of cubes:

Theorem 1.1. *If* $\frac{1}{d} < t < \frac{1}{d-1}$*, and* n_0 *is sufficiently large depending on* t*, then the cubes of sidelength* n^{-t} *for* $n \geqslant n_0$ *can perfectly pack a cube of volume* $\sum_{n=n_0}^{\infty} \frac{1}{n^{dt}}$.

The reader is reminded that the content of this chapter appears in [10]. To prove Theorem 1.1, we apply an inductive-type argument similar to that used by Tao in [14]. Initially, we suppose that we can pack a finite set of cubes $\mathcal{C}$ of sidelength

n^{-t} for $n_0 \leqslant n < n_0'$ into our single cube S. As long as $S \setminus \mathcal{C}$ can be partitioned into bricks $\mathcal{B}$ with small enough total surface area, then we can find a brick $B \in \mathcal{B}$ which is wide enough to pack the next cube of sidelength $(n_0')^{-t}$. We pack B by cubes $\mathcal{C}'$ of sidelength n^{-t} for $n_0' \leqslant n < n_0''$ in some efficient manner. By efficient, we mean that the remaining space $B \setminus \mathcal{C}'$ can be partitioned into bricks $\mathcal{B}'$ with small enough total surface area. In the next iteration, we choose a wide brick from $\mathcal{B} \setminus \{B\} \cup \mathcal{B}'$ and pack it efficiently. We proceed recursively until we have packed an arbitrarily large finite number of cubes into S. Theorem 1.1 would then follow from a compactness argument.

This type of argument reduces the problem to finding a general technique for packing cubes efficiently into some brick, in essence, forming the inductive step in the above argument. Up until now, we have followed Tao's argument in [14] closely. However, while it is fairly straightforward to generalize the standard two-dimensional packing algorithm used in [14] to the higher-dimensional case, Tao's method of *explicitly* verifying that this packing is legal and efficient becomes much more difficult in three dimensions or more. Our innovation is to introduce the notion of "snugness" (see Section 3.1); this allows us to perform this portion of the argument in an elegant fashion which does not become too complex in the higher-dimensional setting.

In Section 3.1, we introduce our notation and prove some simple lemmas. In Section 3.2, we reduce the proof of Theorem 1.1 to a more general result, Proposition 3.4, which can be proved via induction. The inductive step of this argument is furnished by Theorem 3.5 which provides a general and efficient method for packing a brick by cubes. This result is proved in Section 3.3.

3.1 Preliminary Lemmas and Notation

Note that we are using standard asymptotic notation, although we will implicitly assume that all of our constants are allowed to depend upon the dimension d. Note that we also use some non-standard asymptotic notation. If $\vec{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, then we use $\vec{x} + O(X)$ to refer to a vector $(x_1 + O(X), \ldots, x_d + O(X))$, and analogously for little-o notation. Similarly, if $B = [B_1, B_1'] \times \cdots \times [B_d, B_d']$ is a brick positioned in $\mathbb{R}^d$, then we use $B + O(X)$ to refer to a brick $[B_1 + O(X), B_1' + O(X)] \times \cdots \times [B_d + O(X), B_d' + O(X)]$, and analogously for little-o notation.

Let $i, j \in \{1, 2, \ldots, d\}$. Given a brick B, we will denote its sidelengths by $w_i(B)$, ordered so that $w_i(B) \leqslant w_j(B)$ for any $i \leqslant j$. We say that the *width* of B is the smallest sidelength, and denote it by $w(B) := w_1(B)$. Clearly, $w_i(B) = w_j(B)$ for every i and j if and only if B is a cube. We define the *volume* of a single brick B to be

$$\text{vol}(B) := w_1(B) w_2(B) \ldots w_d(B).$$

We define the *eccentricity* of a brick as

$$\text{ecc}(B) := \frac{\text{vol}(B)}{w(B)^d} \geqslant 1.$$

Note that $\text{ecc}(B) = 1$ if and only if B is a cube.

Let $\mathcal{B}$ be a collection of bricks. Define the *volume* of $\mathcal{B}$ to be

$$\text{vol}(\mathcal{B}) := \sum_{B \in \mathcal{B}} w_1(B) w_2(B) \ldots w_d(B).$$

Define the *unweighted surface area* of $\mathcal{B}$ to be

$$\mathrm{surf}(\mathcal{B}) := 2\sum_{B\in\mathcal{B}}\sum_{1\leqslant i_1<i_2<\cdots<i_{d-1}\leqslant d} w_{i_1}(B)w_{i_2}(B)\dots w_{i_{d-1}}(B)$$
$$\asymp \sum_{B\in\mathcal{B}} w_2(B)w_3(B)\dots w_d(B).$$

For $0 \leqslant \delta < 1$, define the *weighted surface area* of $\mathcal{B}$ to be

$$\mathrm{surf}_\delta(\mathcal{B}) := \sum_{B\in\mathcal{B}} w_1(B)^\delta w_2(B)\dots w_d(B).$$

Clearly, $\mathrm{surf}_0(\mathcal{B}) \asymp \mathrm{surf}(\mathcal{B})$. Weighted surface area is roughly speaking a version of unweighted surface area which weights high eccentricity bricks a little less than low eccentricity bricks. We can use the inequality $w(B) \leqslant (w_2(B)w_3(B)\dots w_d(B))^{\frac{1}{d-1}}$ for any brick B, to derive the crude bound

$$\mathrm{surf}_\delta(\mathcal{B}) \ll (\mathrm{surf}(\mathcal{B}))^{1+\frac{\delta}{d-1}}, \tag{3.1}$$

for a finite collection of bricks $\mathcal{B}$.

A solid $S \subset \mathbb{R}^d$ is called *simple* if it is connected and can be formed as a union of a finite collection of bricks. A packing of a finite collection of bricks $\mathcal{B}$ in a simple solid S is called *ε-snug* for some $\varepsilon > 0$ if, for every brick $B \in \mathcal{B}$, the portion of ∂B which does not intersect the boundary of another brick or the boundary of S has surface area $\ll (w\varepsilon)^{d-1}$ and the portion of ∂S which does not intersect the boundary of any brick in $\mathcal{B}$ also has surface area $\ll (w\varepsilon)^{d-1}$. Here w is the width of the widest brick in $\mathcal{B}$. The *size discrepancy* of a finite collection of bricks $\mathcal{B}$ is

$$\mathrm{sd}(\mathcal{B}) = \frac{\max_{B\in\mathcal{B}} w(B)}{\min_{B\in\mathcal{B}} w(B)} - 1.$$

The following lemma gives a criterion for the existence of a brick of a certain minimum width in terms of an elegant relationship between volume and weighted surface area.

Lemma 3.1. *Let* $0 \leqslant \delta < 1$. *For any finite collection of bricks* $\mathcal{B}$, *there exists a brick with width at least* $\left(\frac{\mathrm{vol}(\mathcal{B})}{\mathrm{surf}_\delta(\mathcal{B})}\right)^{\frac{1}{1-\delta}}$.

Proof. By definition,

$$\mathrm{vol}(\mathcal{B}) = \sum_{B \in \mathcal{B}} w_1(B) w_2(B) \dots w_d(B) \leqslant \Big(\sup_{B \in \mathcal{B}} w(B)^{1-\delta} \Big) \mathrm{surf}_\delta(\mathcal{B}).$$

This implies that $(\sup_{B \in \mathcal{B}} w(B))^{1-\delta} \geqslant \frac{\mathrm{vol}(\mathcal{B})}{\mathrm{surf}_\delta(\mathcal{B})}$, giving the desired result. □

This illustrates the principle behind using weighted surface area. If we were to use unweighted surface area, namely setting $\delta = 0$, then to guarantee the existence of a brick of width w, we would need an upper bound on the surface area of the form $w^{-1}\mathrm{vol}(\mathcal{B})$. However, if $\delta > 0$, then we only need a weaker bound of the form $w^{-(1-\delta)}\mathrm{vol}(\mathcal{B})$.

The following result follows from a compactness argument (see, for example, [9]).

Lemma 3.2. *Let* $\mathcal{B}$ *be an, at most, countable collection of bricks and let* $\Omega \subset \mathbb{R}^d$ *be compact. Suppose that an arbitrarily large, but finite, number of bricks from* $\mathcal{B}$ *can be packed into* Ω. *Then* $\mathcal{B}$ *in its entirety can be packed into* Ω. *Furthermore, if* $\mathrm{vol}(\mathcal{B}) = m(\Omega)$, *then this packing is perfect.*

The following lemma states that if a packing of bricks is sufficiently snug, then the region between the bricks has negligible surface area.

Lemma 3.3. *Suppose that a finite collection of bricks* $\mathcal{B}$, *where the widest brick has width* w, *has a* ε*-snug packing in a brick* B, *for some* $\varepsilon > 0$. *Then,* $B \setminus \mathcal{B}$ *can be partitioned into bricks with weighted surface area* $\ll C_{|\mathcal{B}|} \nu w^{d-1+\delta}$, *where* $C_{|\mathcal{B}|}$ *is a constant that depends on* $|\mathcal{B}|$ *and* $\nu \to 0$ *as* $\varepsilon \to 0$.

Proof. Partition $B \setminus \mathcal{B}$ into a finite number of bricks $\mathcal{B}'$. The maximum number of bricks in $\mathcal{B}'$ can be bounded by a constant dependent upon $|\mathcal{B}|$. By the definition of snugness, we know that the true surface area, A, (in the sense of the $(d-1)$-dimensional Lebesgue measure) of the solid $\cup\mathcal{B}'$ could not exceed $(\varepsilon w)^{d-1}(|\mathcal{B}|+1)$. The result follows from the crude bound $\mathrm{surf}(\mathcal{B}') \ll A|\mathcal{B}'|$ and (3.1). □

3.2 Initial Reductions

In this section we prove the higher-dimensional analogue of Proposition 2.1 in [14] which will allow us to deduce Theorem 1.1.

Proposition 3.4. *Fix $\frac{1}{d} < t < \frac{1}{d-1}$ and δ depending on t, such that $0 < \delta < 1$ and $(d-1)t + \delta t < 1$. Choose a scale M sufficiently large and choose N_0 sufficiently large depending on M. Let $n_{max} \geqslant n_0 \geqslant N_0$, and suppose that $\mathcal{B}$ is a family of bricks with volume*

$$\mathrm{vol}(\mathcal{B}) = \sum_{n=n_0}^{\infty} \frac{1}{n^{dt}}, \tag{3.2}$$

weighted surface area bound

$$\mathrm{surf}_\delta(\mathcal{B}) \ll \frac{1}{M^{1-\delta/2}} \sum_{n=1}^{n_0-1} \frac{1}{n^{(d-1)t+\delta t}}, \tag{3.3}$$

and height bound

$$\sup_{B\in\mathcal{B}} w_d(B) \ll 1. \tag{3.4}$$

Then one can pack $\bigcup_{B\in\mathcal{B}} B$ by cubes of sidelength n^{-t} for $n_0 \leqslant n < n_{max}$.

First we see how we can derive our main result from Proposition 3.4.

Proof of Theorem 1.1. Fix $\delta = \frac{1}{d-1} - t$, and note that it easily satisfies the necessary conditions. Take $\mathcal{B} = \{C\}$ where C is the cube of volume $\sum_{n=n_0}^{\infty} \frac{1}{n^{dt}}$, having sidelength $\ll n_0^{1/d-t}$ (since $t > 1/d$). Observe that (3.3) is satisfied, since

$$\mathrm{surf}_\delta(\mathcal{B}) \ll n_0^{(1/d-t)(d-1+\delta)} \ll n_0^{1-dt} \ll \frac{1}{M^{1-\delta/2}} n_0^{1-(d-1)t-\delta t} \ll \frac{1}{M^{1-\delta/2}} \sum_{n=1}^{n_0-1} \frac{1}{n^{(d-1)t+\delta t}},$$

recalling that $(d-1)t + \delta t < 1$ and $0 < \delta < 1$. We also have used the fact that $\frac{n_0^{t-\delta t}}{M^{1-\delta/2}} \gg 1$ since $n_0 \geqslant N_0$, which is sufficiently large depending upon M. Since (3.2) and (3.4) are trivially satisfied, we can then apply Proposition 3.4 to conclude that C can be packed by cubes of sidelength n^{-t} for $n_0 \leqslant n < n_{max}$, and the result follows from Lemma 3.2. □

The inductive step in the proof of Proposition 3.4 requires us to pack a brick by a collection of cubes. We isolate this result as a corollary to the following more general theorem:

Theorem 3.5. *Fix $0 \leqslant \delta < 1$. Let $M = M_1 \leqslant M_2 \leqslant \cdots \leqslant M_d$ be natural numbers, and $\mathcal{C}$ be a family of $M_* = M_1 M_2 \ldots M_d$ cubes with maximum width w and with size discrepancy ε, for some $\varepsilon > 0$. Let S be a brick with dimensions $S_1 \times S_2 \times \cdots \times S_d$ satisfying $M_i w \leqslant S_i \leqslant M_i w + O(w)$ for $i \in \{1, 2, \ldots, d\}$. Then, there exists a packing of $\mathcal{C}$ in S such that $S \setminus \mathcal{C}$ can be partitioned into bricks $\mathcal{B}$ satisfying*

$$\mathrm{surf}_\delta(\mathcal{B}) \ll \frac{M_*}{M} w^{d-1+\delta} + C_M \nu w^{d-1+\delta},$$

where C_M is some constant dependent upon M and $\nu \to 0$ as $\varepsilon \to 0$.

We will prove this theorem in Section 3.3. For now, we use it to derive the corollary we need in the proof of Proposition 3.4:

Corollary 3.5.1. *Fix $\frac{1}{d} < t < \frac{1}{d-1}$ and δ depending on t, such that $0 < \delta < 1$ and $(d-1)t + \delta t < 1$. Choose a scale M sufficiently large and choose N_0 sufficiently large depending on M. Suppose that S is a brick satisfying the width bound $M n_0^{-t} \leqslant w(S) \leqslant$*

$Mn_0^{-t} + O(n_0^{-t})$ *for some* $n_0 \geqslant N_0$ *and satisfying the eccentricity bound* $\mathrm{ecc}(S) = o(n_0)$. *Then we can find* $n_0' \geqslant n_0$ *with* $n_0' - n_0 \asymp \mathrm{ecc}(S)M^d$, *such that* S *can be perfectly packed by cubes of sidelength* n^{-t} *for* $n_0 \leqslant n < n_0'$ *and a collection of bricks* $\mathcal{B}$ *satisfying the weighted surface area bound*

$$\mathrm{surf}_\delta(\mathcal{B}) \ll \frac{1}{M} \sum_{n=n_0}^{n_0'-1} \frac{1}{n^{(d-1)t+\delta t}}.$$

Proof. Let the parameters be chosen as in the corollary, and use the notation $S = S_1 \times S_2 \times \cdots \times S_d$ such that $S_1 \leqslant S_2 \leqslant \cdots \leqslant S_d$. Thus, $Mn_0^{-t} \leqslant S_1 \leqslant Mn_0^{-t} + O(n_0^{-t})$. Define $M_i = \lfloor S_i / n_0^{-t} \rfloor$ for $i \in \{1, 2, \ldots, d\}$, so that $M_1 \asymp M$. Choose $n_0' = n_0 + M_*$. Note that

$$M_* \asymp \mathrm{ecc}(S)M^d, \tag{3.5}$$

as required. Let $\mathcal{C}$ be the collection of cubes of sidelength n^{-t} for $n_0 \leqslant n < n_0'$. The size discrepancy is $\frac{n_0'^t}{n_0^t} - 1$. This can be made arbitrarily small as long as N_0 is chosen to be sufficiently large compared with M. This makes the second term in the bound of Theorem 3.5 negligible with respect to the first. Thus, we can apply Theorem 3.5 to get a packing of $\mathcal{C}$ in S such that $S \setminus \mathcal{C}$ can be partitioned into bricks $\mathcal{B}$ satisfying

$$\mathrm{surf}_\delta(\mathcal{B}) \ll \frac{M_*}{M}(n_0^{-t})^{d-1+\delta} \ll \frac{M_*}{M} \frac{(1 + \mathrm{sd}(\mathcal{C}))^{d-1+\delta}}{(n_0')^{(d-1)t+\delta t}} \ll \frac{1}{M} \sum_{n=n_0}^{n_0'-1} \frac{1}{n^{(d-1)t+\delta t}},$$

since $\mathrm{sd}(\mathcal{C}) \to 0$. This completes the proof. □

Observe that the power of M in the weighted surface area bound of the corollary is independent of δ. This fact allows us to loosen our weighted surface area bound (3.3) by a factor of $M^{\delta/2}$, which is enough to let us complete the inductive step of the proof of Proposition 3.4, illustrating the advantage of working with weighted surface area (see also the discussion after Lemma 3.1).

We now use this corollary to prove Proposition 3.4. Our proof closely mirrors the proof of Proposition 2.1 in [14] except for higher dimensions. However, for the reader's convenience, we include it here.

Proof of Proposition 3.4. We prove this via downward induction on n_0. Fix $n_{max} \geqslant N_0$. Clearly, the result holds if $n_0 = n_{max}$. Fix some $n_0 \leqslant n_{max}$, and assume the result holds with n_0 replaced by any strictly larger integer up to n_{max}. We show that the result will then hold for n_0.

Since $t > 1/d$, (3.2) implies that $\mathrm{vol}(\mathcal{B}) \asymp n_0^{1-dt}$. Furthermore, since $(d-1)t + \delta t < 1$, we have $\mathrm{surf}_\delta(\mathcal{B}) \ll M^{-(1-\delta/2)} n_0^{1-(d-1)t-\delta t}$. Thus, Lemma 3.1 implies the existence of a brick $B' \in \mathcal{B}$ satisfying

$$w(B') \gg \left(\frac{n_0^{1-dt}}{M^{-(1-\delta/2)} n_0^{1-(d-1)t-\delta t}} \right)^{\frac{1}{1-\delta}} = M^{\frac{1-\delta/2}{1-\delta}} n_0^{-t}.$$

Since $\frac{1-\delta/2}{1-\delta} > 1$ for $0 < \delta < 1$, then as long as we take M sufficiently large, we can drop the implied constant and conclude that $w(B') \geqslant M n_0^{-t}$. Partition B' into two bricks B and $B' \setminus B$, so that $M n_0^{-t} \leqslant w(B) \leqslant M n_0^{-t} + O(n_0^{-t})$. By the height bound, (3.4), $\mathrm{ecc}(B) \ll M^{-(d-1)} n_0^{(d-1)t} = o(n_0)$, which means that we can apply Corollary 3.5.1 to pack B by cubes of sidelengths n^{-t} for $n_0 \leqslant n < n_0'$ with $n_0' - n_0 \gg M^d$ and a collection of bricks $\mathcal{B}_0$ satisfying

$$\mathrm{surf}_\delta(\mathcal{B}_0) \ll \frac{1}{M} \sum_{n=n_0}^{n_0'-1} \frac{1}{n^{(d-1)t+\delta t}}. \tag{3.6}$$

Now, if $n_0' \geqslant n_{max}$, then we are done, as we have packed every cube of sidelengths n^{-t} for $n_0 \leqslant n < n_{max}$. Otherwise, suppose that $n_0' < n_{max}$. Since n_0' is strictly larger than n_0, it makes sense to now apply our inductive hypothesis, replacing n_0 by n_0' (which is strictly larger than n_0), and replacing $\mathcal{B}$ by $\mathcal{B}' = (\mathcal{B} \setminus \{B'\}) \cup \{B' \setminus B\} \cup \mathcal{B}_0$. First, however we have to check to assure that the conditions of the proposition are met. Observe,

$$\mathrm{vol}(\mathcal{B}') = \sum_{n=n_0}^{\infty} \frac{1}{n^{dt}} - \sum_{n=n_0}^{n_0'-1} \frac{1}{n^{dt}} = \sum_{n=n_0'}^{\infty} \frac{1}{n^{dt}},$$

and so $\mathcal{B}'$ has the required total volume (3.2). Clearly, $\mathcal{B}'$ satisfies the height bound (3.4). Finally, by (3.6), (3.3), and the fact that $\mathrm{surf}_\delta\{B' \setminus B\} \leqslant \mathrm{surf}_\delta\{B'\}$, we have

$$\begin{aligned}\mathrm{surf}_\delta(\mathcal{B}') &\leqslant \mathrm{surf}_\delta(\mathcal{B} \cup \mathcal{B}_0) \\ &\ll \frac{1}{M^{1-\delta/2}} \sum_{n=1}^{n_0-1} \frac{1}{n^{(d-1)t+\delta t}} + \frac{1}{M} \sum_{n=n_0}^{n_0'-1} \frac{1}{n^{(d-1)t+\delta t}} \\ &\ll \frac{1}{M^{1-\delta/2}} \sum_{n=1}^{n_0'-1} \frac{1}{n^{(d-1)t+\delta t}},\end{aligned}$$

and thus $\mathcal{B}'$ satisfies the weighted surface bound (3.3). Thus, we can apply the inductive hypothesis for n_0' and pack $\bigcup_{B\in\mathcal{B}'} B$ by the remaining cubes of sidelength n^{-t} for $n_0' \leqslant n < n_{max}$, which in turn implies that we can pack $\bigcup_{B\in\mathcal{B}} B$ by cubes of sidelength n^{-t} for $n_0 \leqslant n < n_{max}$. □

All that remains is proving Theorem 3.5 which provides a general and efficient brick-packing algorithm.

3.3 Efficient Brick-Packing Algorithm

Proof of Theorem 3.5. Note that, without loss of generality, we can assume that $S_i = M_i w$ for $i \in \{1, 2, \ldots, d\}$. To see this, suppose that S' is a cube that contains S and instead satisfies $M_i w \leqslant S_i' \leqslant M_i w + O(w)$. Then, $S' \setminus S$ can be partitioned into a $O(1)$ bricks, each of which contributes an allowable weighted surface area $\ll \frac{M_*}{M} w^{d-1+\delta}$.

To explicitly define our packing, we position S in $\mathbb{R}^d$ as

$$[0, M_1 w] \times [0, M_2 w] \times \cdots \times [0, M_d w].$$

Index $\mathcal{C}$ as $\{C_n\}_{n=0}^{M_*-1}$ from largest width to smallest width. We use the notation $w_n := w(C_n)$. By construction, $w_n \leqslant w_m$ if and only if $n \geqslant m$. We further index $n = 0, 1, \ldots, M_* - 1$ by $n_{\vec{i}} = n_{i_1, i_2, \ldots, i_d}$, where

$$n_{i_1, i_2, \ldots, i_d} := i_1 + i_2 M_1 + i_3 M_1 M_2 + \cdots + i_d M_1 M_2 \ldots M_{d-1},$$

for $i_k = 0, \ldots, M_k - 1$ (with $k = 1, 2, \ldots, d$). We use the notation $C_{\vec{i}} := C_{n_{\vec{i}}}$ and $w_{\vec{i}} := w_{n_{\vec{i}}}$. Position each $C_{\vec{i}}$ in S as

$$C_{\vec{i}} := [x_{\vec{i}}^1, x_{\vec{i}}^1 + w_{\vec{i}}] \times [x_{\vec{i}}^2, x_{\vec{i}}^2 + w_{\vec{i}}] \times \cdots \times [x_{\vec{i}}^d, x_{\vec{i}}^d + w_{\vec{i}}],$$

where for any $k \in \{1, 2, \ldots, d\}$, we define

$$x_{\vec{i}}^k = \sum_{i_k'=0}^{M_k-1} w_{i_1, \ldots, i_{k-1}, i_k', 0, \ldots, 0} - \sum_{i_k'=i_k}^{M_k-1} w_{i_1, \ldots, i_{k-1}, i_k', i_k+1, \ldots i_d}$$

(see Figure 3.1).

We will verify that this is a legal packing shortly. Note that each $(x_{\vec{i}}^1, x_{\vec{i}}^2, \ldots, x_{\vec{i}}^d)$ is asymptotically fixed at a lattice point as $\mathrm{sd}(\mathcal{C}) \ll \varepsilon$, namely

$$(x_{\vec{i}}^1, x_{\vec{i}}^2, \ldots, x_{\vec{i}}^d) = (w i_1, w i_2, \ldots, w i_d) + O_M(\varepsilon w). \tag{3.7}$$

Collect the subset of cubes from $\mathcal{C}$ which form its exterior "shell":

$$\tilde{\mathcal{C}} = \{C_{\vec{i}} \in \mathcal{C} : i_k = 0 \text{ or } i_k = M_k - 1, \text{ for some } k \in \{1, 2, \ldots, d\}\}.$$

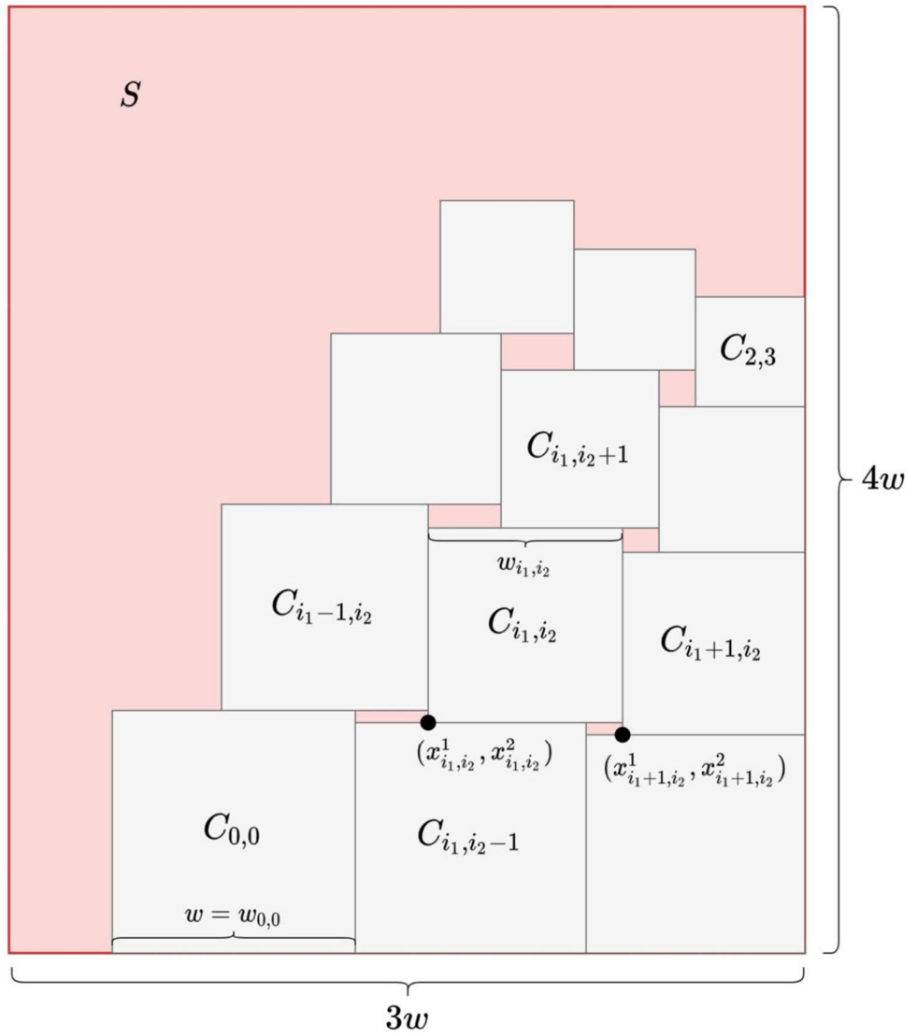

Figure 3.1: The packing of the cubes $C_{\vec{i}}$ in S. Here, $d = 2$, $M_1 = 3$, $M_2 = 4$, $M_* = 12$, and $i_1 = i_2 = 1$. Note that the diagram is not to scale.

Let B be the smallest brick containing $\mathcal{C} \setminus \tilde{\mathcal{C}}$. By (3.7), B has dimensions

$$[w, (M_1 - 1)w] \times [w, (M_2 - 1)w] \times \cdots \times [w, (M_d - 1)w] + o(1).$$

Define the simple solid $K = B \cup \mathcal{C} = B \cup \tilde{\mathcal{C}}$ (see Figure 3.2). Observe that

$$S \setminus K = (S \setminus B) \setminus \tilde{\mathcal{C}}.$$

Observe that $S \setminus B$ can be partitioned into $O(M_*/M)$ bricks $\mathcal{B}'$ each with dimensions

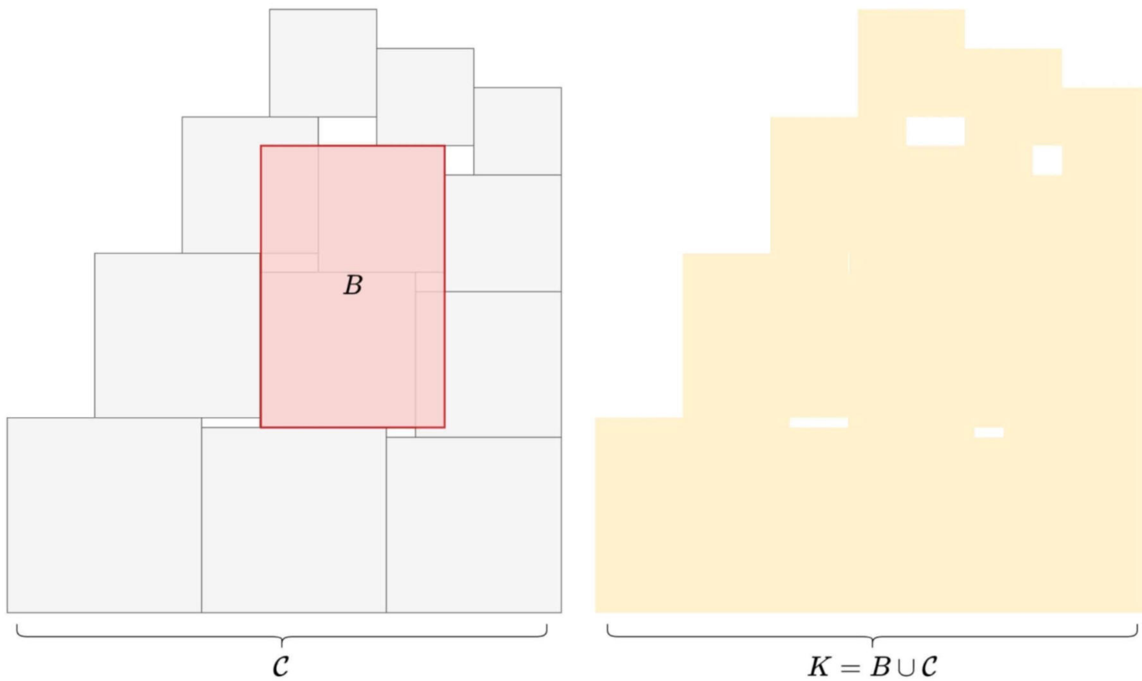

Figure 3.2: The simple solid K is constructed from B and $\mathcal{C}$.

$O(w)$. Each brick $B' \in \mathcal{B}'$ intersects at most $O(1)$ cubes in $\tilde{\mathcal{C}}$. This means that we can partition $B' \setminus \tilde{\mathcal{C}}$ into $O(1)$ bricks, each with weighted surface area less than that of B', which is $\ll w^{d-1+\delta}$. Thus, $S \setminus K$ can be partitioned into bricks with allowable weighted surface area $\ll \frac{M_*}{M} w^{d-1+\delta}$. It then suffices to show that $K \setminus \mathcal{C}$ can be partitioned into bricks with weighted surface area $\ll \frac{M_*}{M} w^{d-1+\delta}$.

By Lemma 3.3, it suffices to show that $\mathcal{C}$ is a packing that is $O_M(\varepsilon)$-snug in K. First, observe that all of the cubes are inside of S. This follows from the bound $w_{\vec{i}} \leqslant w$. Thus, we have to check that none of the cubes' interiors overlap, and that every cube in $\mathcal{C} \setminus \tilde{\mathcal{C}}$ is touching the $2d$ adjacent cubes (the cubes in $\tilde{\mathcal{C}}$ are already touching K by construction).

Define π_k to be the projection operator onto the x_k axis for every $k \in \{1, 2, \ldots, d\}$. Let $\tilde{E}$ be the collection of $2^d - 1$ vectors $\vec{e} = (e_1, e_2, \ldots, e_d)$ such that every $e_k \in \{0, 1\}$ but $\vec{e} \neq \vec{0}$. Let $\vec{e}_k$ be the kth unit vector, namely $(0, \ldots, 0, 1, 0, \ldots, 0)$, with a 1 in the kth component and let $E \subset \tilde{E}$ be the collection of such d unit vectors. Define I be the collection of $\vec{i} = (i_1, i_2, \ldots, i_d)$ such that $i_k \in \{0, 1, \ldots, M_k - 2\}$ for $k \in \{1, 2, \ldots, d\}$. By symmetry and the asymptotic positioning of the cubes (3.7), we only have to worry about checking overlap on "adjacent" cubes, reducing the proof to showing the following two claims:

(i) Let $\vec{e} \in E$ and $\vec{i} \in I$. Then for at least one $k \in \{1, 2, \ldots, d\}$ we have that $\pi_k(C_{\vec{i}}) \cap \pi_k(C_{\vec{i}+\vec{e}})$ is *exactly* one point.

(ii) Let $\vec{e} \in \tilde{E} \setminus E$ and $\vec{i} \in I$. Then for at least one $k \in \{1, 2, \ldots, d\}$, we have that $\pi_k(C_{\vec{i}}) \cap \pi_k(C_{\vec{i}+\vec{e}})$ is *at most* one point.

To see why this is sufficient to complete the proof, observe that as long as the boundary of the cubes are touching, the asymptotic positioning of the cubes (3.7) ensures that the non-overlapping boundary will have area $O_M(\varepsilon w)$, meaning that (i) will imply that the packing is $O_M(\varepsilon)$-snug. Clearly, (ii) implies that none of the cubes' interiors overlap, and thus our packing of $\mathcal{C}$ is valid.

To see (i), note that the construction of the C_i immediately implies that for every $k \in \{1, 2, \ldots, d\}$, we have

$$\pi_k(C_{\vec{i}+\vec{e}_k}) \cap \pi_k(C_{\vec{i}}) = \{x_{\vec{i}}^k + w_{\vec{i}}\}.$$

Now we show (ii). Fix some $\vec{e} = (e_1, e_2, \ldots, e_d) \in \tilde{E} \setminus E$. Let k be the smallest index such that the component e_k is nonzero. Clearly, $k \in \{1, 2, \ldots, d-1\}$. Recall that

$$x_{\vec{i}}^k = \sum_{i'_k=0}^{M_k-1} w_{i_1,\ldots,i_{k-1},i'_k,0,\ldots,0} - \sum_{i'_k=i_k}^{M_k-1} w_{i_1,\ldots,i_{k-1},i'_k,i_k+1,\ldots i_d}.$$

By the ordering of $w_{\vec{i}}$, we have that $w_{\vec{i}+\vec{e}} \leqslant w_{\vec{i}+\vec{e}_k}$. Thus, $x_{\vec{i}+\vec{e}} \geqslant x_{\vec{i}+\vec{e}_k} = x_{\vec{i}} + w_{\vec{i}}$. This shows that $\pi_k(C_{\vec{i}+\vec{e}}) \cap \pi_k(C_{\vec{i}})$ is at most a singleton, as desired. This completes the proof. □

Chapter 4

An Overview of the Unit Square Packing Problem

We now discuss the second type of packing problems that is studied in this thesis. Consider a large square $S(x)$ of sidelength x. Pack $S(x)$ by non-overlapping unit squares, and let $W(x)$ represent the remaining unpacked area, or wasted space, of $S(x)$. Our goal is to pack $S(x)$ in such a way that we minimize this wasted space and determine the optimal bound for $W(x)$.

There are a few differences between this style of packing problems and the Meir-Moser type problems. While both problems are "asymptotic" in nature (in the sense that they are concerned with packing an infinite number of squares), in this style of problem, the squares we are packing are all the same size. Consequently, we are no longer interested in finding a perfect packing, and so the concept of the perimeter of the wasted space is lost. Instead we are more concerned about packing squares at very slight inclines that will allow us to minimize the total wasted space.

Recall that Erdös and Graham studied this problem in 1975 (see [4]) and proved that you could bound $W(x)$ as low as $O(x^{7/11})$. We will give a version of their proof in Section 4.1, since it introduces many important ideas.

Erdös and Graham were unable to prove any asymptotic lower bound on $W(x)$, and stated that they could not even rule out the possibility that $W(x) = O(1)$. They did state that perhaps the proper correct bound was $O(x^{1/2})$. This prompted Roth and Vaughan to analyze the problem, and in 1978 (see [13]), they showed that

$$W(x) > 10^{-100}\sqrt{x(x - \lfloor x \rfloor)}.$$

Thus, we know that upper bound for $W(x)$ can be no better than $O(x^{1/2})$.

In the other direction, Montgomery improved the upper bound to $W(x) = O(x^{\frac{3-\sqrt{3}}{2}})$, according to personal communication (see, for example, [3]). In 2009, Chung and Graham [2] improved it further to $W(x) = O(x^{\frac{3+\sqrt{2}}{7}} \log x)$. Recently, in 2020, they published a result that states that $W(x) = O(x^{3/5})$ (see [3]). Unfortunately, while studying their paper, we found an error in their work, which brings the best known bound back to $W(x) = O(x^{\frac{3+\sqrt{2}}{7}} \log x)$. We will discuss this error later in this chapter.

In Section 4.1, we give an overview of the proof in [4] bounding $W(x) = O(x^{7/11})$. In Section 4.2, we discuss the difficulties that arise when attempting to improve the upper bound on $W(x)$ beyond $O(x^{7/11})$, and the intuition behind the ideas in our proof. We also discuss heuristics for what the proper order of asymptotic growth of $W(x)$ might be.

4.1 Bounding the Wasted Space by $O(x^{7/11})$

We will now give an overview of Erdös and Graham's proof in [4] that $W(x) = O(x^{7/11})$. We begin by packing $S(x)$ in the trivial manner starting from the lower right-hand corner of $S(x)$. We will leave rectangles of width h unpacked at the top and left-hand sides of $S(x)$ (see Figure 4.1).

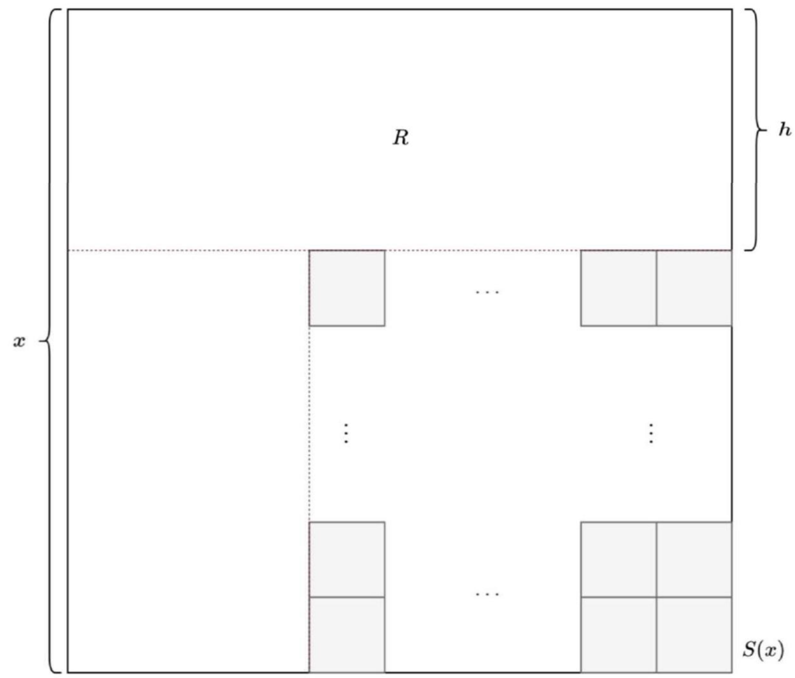

Figure 4.1: We pack $S(x)$ in a trivial manner leaving rectangles of width h at the top and left portion of $S(x)$. We call one of these unpacked rectangles R.

Without loss of generality, we will pack only one of the rectangles, which we call R. We will assume that the short h-length side of R is aligned with the vertical coordinate axis (see Figure 4.1). We then pack R by near-vertical stacks of unit squares of length n inclined so that they touch both sides of R (see Figure 4.2). Choose $n \in \mathbb{Z}^+$ so that $1 < n - h \ll 1$. If θ is the angle of inclination of these stacks, observe that

$$\sec(\theta) = \frac{n + \tan\theta}{h}.$$

Taylor series expansion then gives

$$\theta^2 = \frac{2(n-h)}{h} + O\left(\frac{\theta}{h} + \theta^4\right).$$

Thus, when $n - h \asymp 1$,

$$\theta \asymp \frac{1}{\sqrt{h}}. \tag{4.1}$$

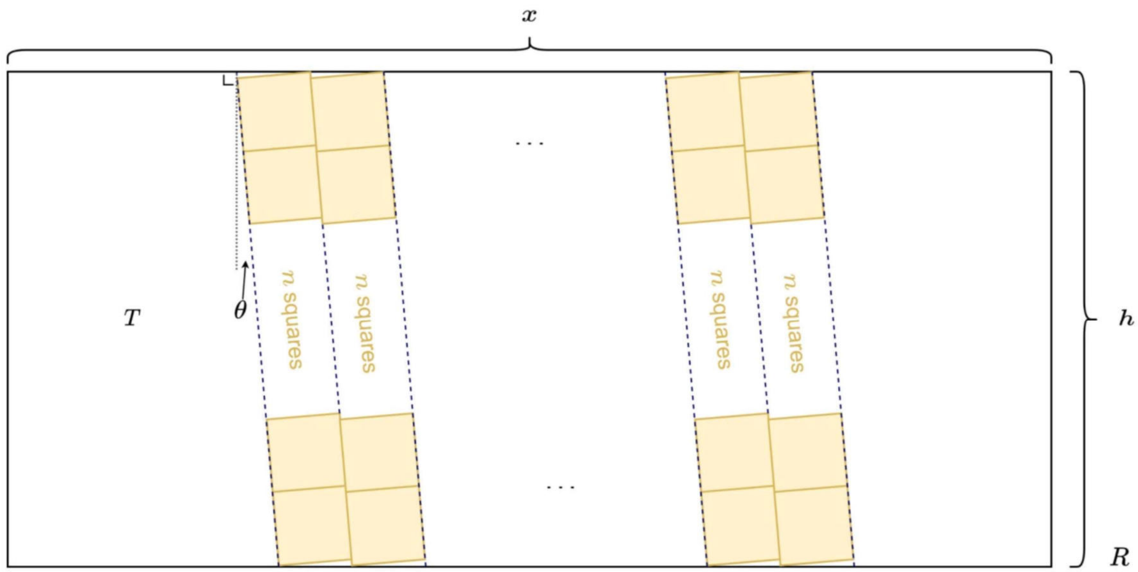

Figure 4.2: We pack R by n-length parallel stacks of unit squares inclined at an angle θ such that each stack touches both the top and the bottom of R. We call one of the unpacked trapezoids formed T.

This allows us to bound the error term by $O(\frac{1}{h^{3/2}})$, which gives a more precise expression for θ:

$$\theta = \sqrt{\frac{2(n-h)}{h}} + O\left(\frac{1}{h}\right). \tag{4.2}$$

Pack all of R in such a manner except for a narrow trapezoid on each end of height h. Without loss of generality, we will pack only one such trapezoid, which we call T. Note that we can fix the width of T (namely, the width of its smaller side), which we call w, to the nearest $O(1)$ by changing the number of n-length stacks that we pack during this stage, and will do so at a later time. We will assume that the h-length side of T is aligned with the vertical coordinates axis. We will call this the *vertical wall* of T. We will call the opposing side the *inclined wall* of T.

We begin by partitioning T into a collection of sub-trapezoids. Let $m = \lfloor \theta^{-1} \rfloor$. Partition T into trapezoids $T_1, \ldots, T_k$ of width m starting at the lower end of T, and working our way up to the upper end (see Figure 4.3). Naturally, there will be a

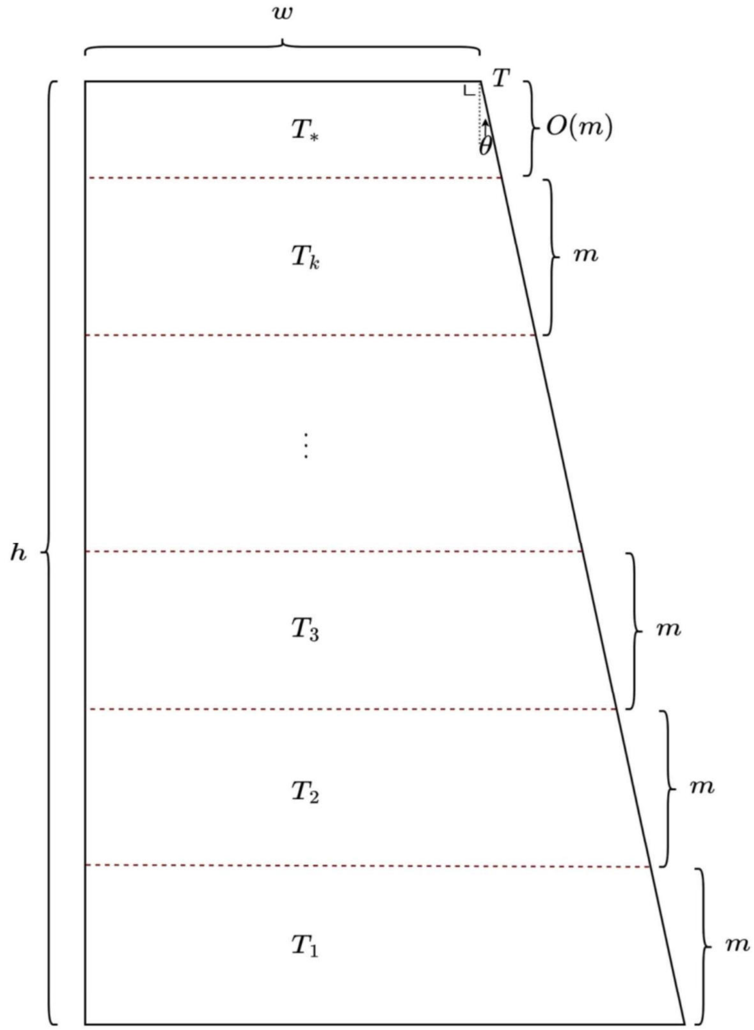

Figure 4.3: We partition T into $\asymp \theta h$ sub-trapezoids $T_1, \ldots, T_k$.

left-over trapezoid T_* of width $O(m)$ at the top of T, which we will just pack in the trivial manner.

Now, for each of the $k \asymp \theta h$ sub-trapezoids, we will pack the left-hand side of the given T_i with vertical stacks of length m, leaving an unpacked trapezoid T_i' on the right-hand side of T_i that has lower width W_i and upper width w_i (see Figure 4.4). Pack the T_i by vertical stacks so that W_i is within a distance 1 of w.

We then pack T_i' by near-horizontal stacks of length n_i, where n_i is some integer greater than W_i such that $n_i - W_i \asymp 1$. We will place the stacks so that they are touching both of the opposing sides of T_i' and are touching each other on the right-

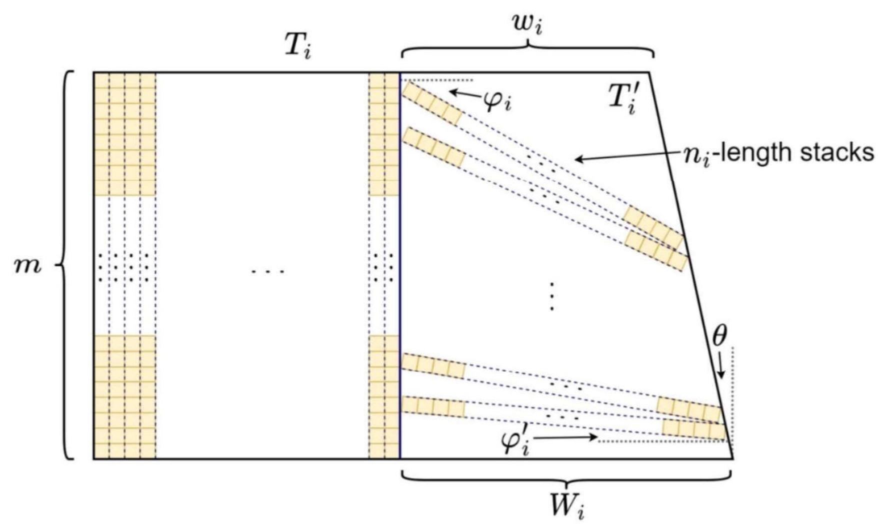

Figure 4.4: We pack each T_i trivially, except for the sub-trapezoid T_i' of width $w_i \sim w$, which we then pack by angled near-horizontal stacks.

hand side, starting as close as we can to the top of T_i' and continuing until we reach its bottom (see Figure 4.4). Observe that the angles at which we place the stacks will depend upon how wide T_i' is in that location, and will vary between some φ_i' and φ_i. However, recall that from (4.1), we have

$$\varphi_i, \varphi_i' \asymp \frac{1}{\sqrt{w}}. \tag{4.3}$$

Now, since $\theta^{-1} \sim m$, we have $W_i - w_i \sim 1$. By construction, we also have that $w_i, W_i \sim w$. Combining these observation with (4.2) gives

$$\varphi_i - \varphi_i' \sim \sqrt{\frac{2}{w}} \tag{4.4}$$

This completes the packing algorithm. We now have to determine the wasted space that has been generated. Let us first focus on a particular T_i' (see Figure 4.4). The wasted space at the top and bottom of T_i' is $O(w^{3/2})$ due to (4.3). The *total* wasted space generated by the "sliver" triangles between the near-horizontal stacks is also $O(w^{3/2})$ from (4.4). There are also small triangles on the right and left side of each near-horizontal stack which have area $O(\frac{1}{\theta\sqrt{w}})$ courtesy of (4.3) and our definition of m.

Now, recall that there are a total of θh such sub-trapezoids T_i, and so the total wasted space generated by packing $T_1, \ldots, T_k$ is

$$\ll \frac{h}{\sqrt{w}} + \theta h w^{3/2}.$$

Since $\theta \asymp \frac{1}{\sqrt{h}}$, the wasted space becomes bounded by

$$\ll \frac{h}{\sqrt{w}} + w^{3/2}\sqrt{h}.$$

Note that if we pack T_* trivially, we will generate an additional waste of $O(\sqrt{h}+w)$, which is clearly absorbed into the above terms. Now, if we equalize these two expressions, it motivates the choice of $w \sim h^{1/4}$, which gives an upper bound on the wasted space generated when packing T of $O(h^{7/8})$.

Now, we also have to account for the wasted space generated by packing R with the θ-inclined vertical stacks (see Figure 4.2). Once again, from (4.1), this is just $O(\frac{x}{\sqrt{h}})$. Thus, the total wasted space is

$$W(x) \ll h^{7/8} + \frac{x}{\sqrt{h}}.$$

This motivates the choice of $h \sim x^{8/11}$, and gives the desired result $W(x) = O(x^{7/11})$.

4.2 Issues Around Improved Upper and Lower Bounds

The above proof demonstrates the basic form that general solutions of this problem must adhere to. There are two main sources of wasted space generated by a "constant-width packing algorithm" and a "variable-width packing algorithm".

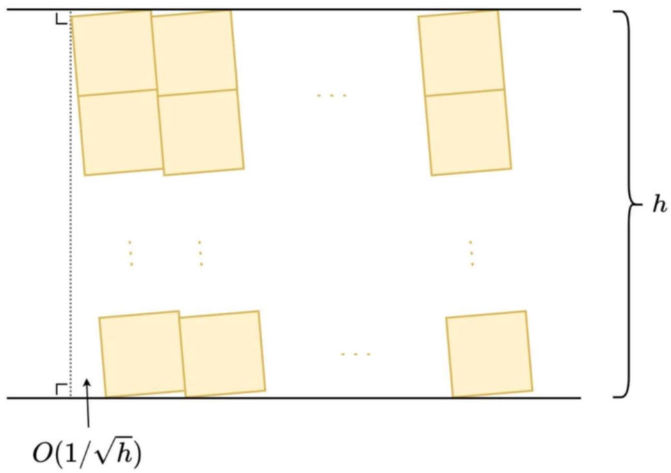

Figure 4.5: Constant-width packing algorithms are straightforward since they can be packed by stacks with a single inclination which ensures there are no gaps between the stacks. The algorithm shown here generates a wasted space of $O(x/\sqrt{h})$.

At first, we need some sort of packing algorithm to fill the distance between opposing sides of a rectangle R of width h and height $\asymp x$ (see Figure 4.2). Of course, h is a parameter that we can optimize for in the proof.

The packing methods we can use in this situation seem fairly straightforward since the distance between the opposing walls is obviously constant (we will refer to this a "constant-width packing algorithm"). Thus, a simple packing of near-vertical stacks inclined at an angle $O(1/\sqrt{h})$ is natural and will only generate wasted space at the top and bottom of the stacks and not between them (see Figure 4.5). The larger h becomes, the less wasted space that will be generated by the constant-width packing algorithm as the angle of the near-vertical stacks becomes slighter.

Second, no matter what variation on the above algorithm that we use, we are generally going to generate a trapezoid T whose width can be chosen arbitrarily and whose height in the above example would be h. We then need some sort of packing algorithm that can pack the region between the inclined and vertical wall. Note that any attempted optimization where we pack between the top and bottom parallel walls during any step tends to only add additional redundancy since any such optimization could generally be performed when we fixed the width of T.

What distinguishes the required algorithm in this setting to the constant-width packing algorithm is that the distance between the two walls is no longer constant. Thus, we refer to this as a "variable-width packing algorithm". We need some way of increasing the width covered by each "stack" of squares as we move down T. This is a hard problem because each stack is going to consist of an integer number of unit squares, which will need to be rotated by some small amount in order to cover a non-integer width.

Now, the larger h becomes the slighter the angle of the stacks in the constant-width packing algorithm (and the smaller the wasted space). However, larger h means that the length of the inclined wall will also increase, leading to more wasted space generated during the variable-width packing algorithm.

What are some natural candidates for a variable-width packing algorithm? Let us focus on a single near-horizontal slice of T with height about $\sqrt{h}$, meaning the width between the vertical wall and the inclined wall would change by about 1 moving from top to bottom over this slice (see Figure 4.6). Naturally, there would be about $\sqrt{h}$ such slices that we would need to pack. While in practice, we would need a method to deal with transitions between the slices as well as the top and bottom of T, we will ignore these concerns for now and isolate just the variable-space packing algorithm element by analyzing how we would pack a single such slice.

If we try to pack this slice in the trivial manner *without* modifying the angles of each successive stack, we would generate a lot of wasted space due to the large triangle created against the inclined wall (see Figure 4.6). Over one slice this area is $O(\sqrt{h})$, and so the total wasted space generated over T would be $O(h)$. Assuming a wasted space of $O(x/\sqrt{h})$ generated during the constant-width packing algorithm, this would yield a total wasted space of $O(x^{2/3})$.

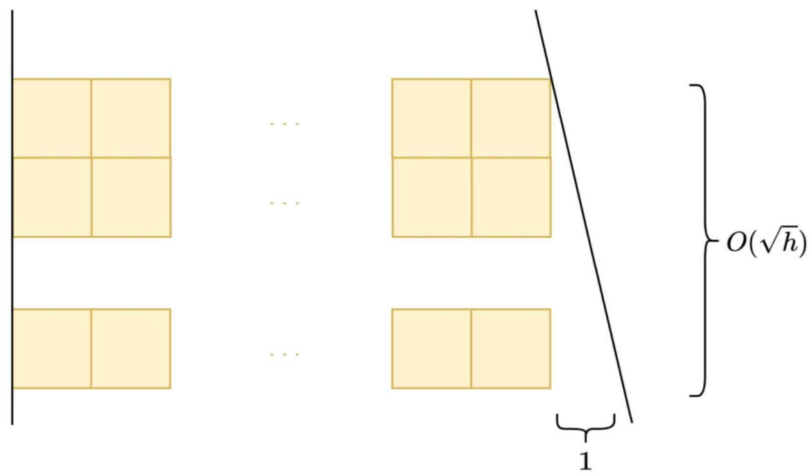

Figure 4.6: Variable-width packing algorithms are more difficult because of the changing angle required if we want the stacks to touch both walls simultaneously. If we ignored this and used the trivial packing algorithm pictured above, we will generate a wasted space of $O(h)$.

If we instead rotate each stack like we did in the previous section (see Figure 4.7), the total amount we would have to rotate the top stack compared to the bottom stack would be about $\frac{1}{\sqrt{w}}$. Thus the area of the long triangles between the stacks would amount to $O(w^{3/2})$. Furthermore, even if the first stack was inclined at a small angle, $\Omega(\sqrt{h})$) of the $\sqrt{h}$ stacks would still necessarily be inclined at $\Omega(1/\sqrt{w})$. Thus, there will always be small triangles on the left and right-hand sides of the stacks that will have a total area of $O(\frac{\sqrt{h}}{\sqrt{w}})$. Equalizing the wasted space here yields a choice of $w \sim h^{1/4}$ and a wasted space generated of $O(h^{3/8})$. Over the entire trapezoidal region, the wasted space accumulated would be $O(h^{7/8})$, which would yield the bound $W(x) = O(x^{7/11})$.

Making improvements to the variable-width packing algorithm beyond this is challenging. There are two possible points where we can try to reduce the wasted space. First, there are the small triangles at the left and right ends of each of the stacks. Second there are the long sliver triangles between each of the stacks themselves. The first source of wasted space is difficult to reduce because no matter what packing algorithm we use we are always going to need to have $\Omega(1)$ pro-

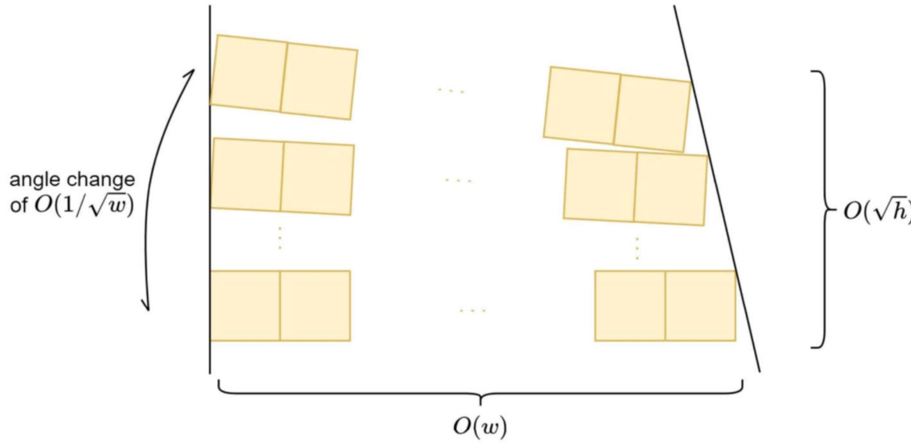

Figure 4.7: If we rotate each successive stack and choose $w \sim h^{1/4}$, then the total wasted space generated will be $O(h^{7/8})$.

portion of the squares inclined at an angle that is at least $\Omega(1/\sqrt{w})$ if we want the horizontal stacks to touch both walls. The second source of wasted space is where we can make an improvement if we are careful.

Currently, we are taking stacks and rotating them somewhat to decrease the amount of horizontal space that they cover. Now, if we are packing a "constant-width" space, this works great as we can snugly fit these inclined stacks next to each other. However, inefficiencies arise when we are packing a "variable-width" space. The necessary angle change of the stacks generates long sliver triangles with base $O(w)$. The key innovation here is to realize that modifying the inclination of the stacks is not the only way to modify the space covered by these stacks. Instead, we can apply a "shear" to the stacks modifying the inclination of each square individually as supposed to the inclination of the entire stack (see Figure 4.8).

Note that while the net change in angles of the squares in the top stack compared to squares in the bottom stack is still $O(1/\sqrt{w})$, the triangles between each of the stacks now only have area $\sqrt{w}$ (compared to the old $w^{3/2}$). Now, there are unfortunately additional small rectangular regions that are generated in such a pack-

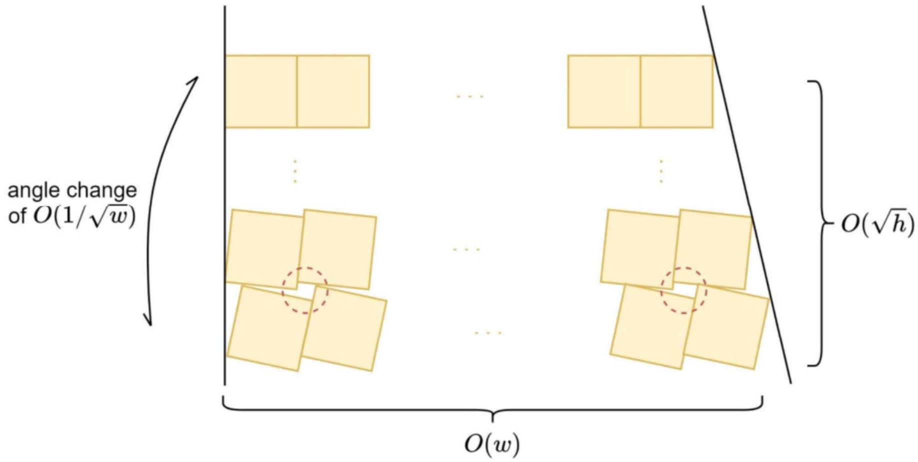

Figure 4.8: If we "shear" the stacks instead of "tilt" the stacks, the triangles of wasted space generated are no longer long "sliver" triangles. Unfortunately, there are small rectangular regions generated in this packing regimen as indicated by the circled regions.

ing (see Figure 4.8). However, if we ignore these for a moment, we can see that the wasted space on the left and right ends is still $O(\frac{\sqrt{h}}{\sqrt{w}})$, and so equalizing the two terms yields a choice of $w = \sqrt{h}$ and a wasted space generated of $O(h^{1/4})$. The wasted space generated over the entire trapezoid would then be $O(h^{3/4})$, yielding a total wasted space of $O(x^{3/5})$.

Now, unfortunately the small rectangular regions are rather problematic and will increase the wasted space generated. The intuition behind the solution to this problem is to slide the bottom stacks to the left to close these rectangular gaps, and then to rotate the entire configuration counterclockwise. There are a couple of more technical difficulties that arise when we implement this technique which are beyond the scope of this section, but that we will cover in Chapter 5 to attain the full bound $O(x^{3/5})$.

If we only consider the constant-packing portion of the algorithm, it is easy to see that a choice of $h = x$ only generates a wasted space of $\sqrt{x}$, at least when we pack the squares in parallel stacks (see Figure 4.9). This may be why Erdös and Graham conjectured a proper order of magnitude of $\sqrt{x}$ for the wasted space in

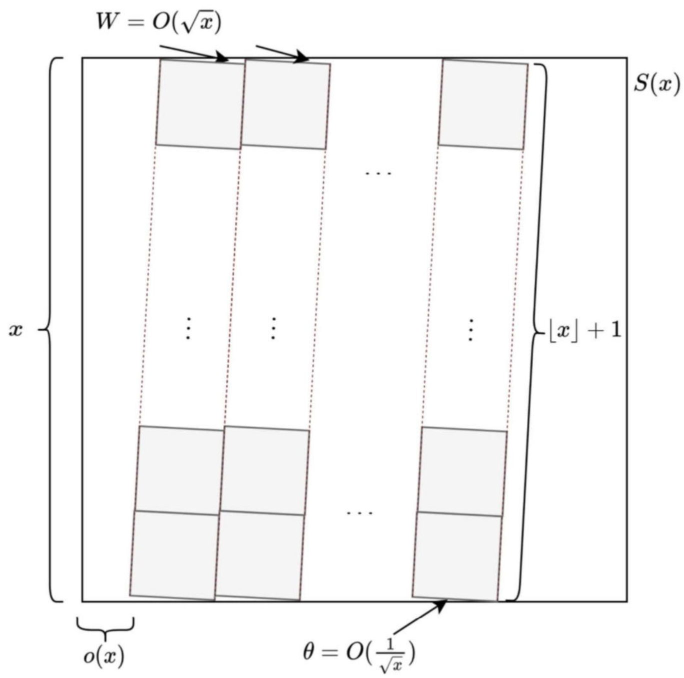

Figure 4.9: Packing the majority of $S(x)$ introducing wasted space $O(\sqrt{x})$.

their initial paper in 1975 (see [4]). However, it seems infeasible that it would be possible to find a variable-width packing of necessary strength to support a choice of $h = x$. This would require a variable-width packing algorithm that packed a trapezoid of height h while only generate wasted space $O(h^{1/2})$.

Note that a proof claiming to improve the $O(x^{7/11})$ bound to $O(x^{3/5})$ will necessarily require some argument similar to our own. In [3], Chung and Graham developed a packing algorithm similar to the one employed in [4], but where they improved the packing of the transitions between successive T_k (see Figure 4.3).

Unfortunately, the amount of angle change that is necessary to lengthen a w-length near-horizontal stack by $O(1)$ (see Figure 4.7) is on the order of $O(1/\sqrt{w})$, as shown in (4.4). As we have just shown, this will limit the attainable wasted space bound to $O(x^{7/11})$ no matter how efficient the transitions are packed. Chung and Graham instead essentially claimed that this angle change was of the order of magnitude $w^{-3/2}$.

For example, in the first paragraph of page 696, they claim that the "[...] tilt of the lower stack is $\sim \frac{c}{\sqrt{x^{2/5}+x^{-1/5}}}$" while the "tilt of the upper stack is $\sim \frac{c}{x^{1/5}} + O(x^{-2/5})$" (an error term which is consequently dropped from their argument in the first equation on page 696). On page 696, they then bound this discrepancy by $O(x^{-4/5})$. With our notation, w would be of order of magnitude $x^{2/5}$ and they are increasing the width covered by the w-length consecutive stacks by $\sim \frac{1}{\sqrt{w}}$. Thus, they are claiming that the discrepancy between these angles is $O(1/w^2)$, when in reality it is $O(1/w)$.

Unfortunately the constant c they are using is not *actually* fixed between these two stacks of differing angles and is dependent on the fractional part of w which changes by the same amount as w, namely $O(1/\sqrt{w})$, as shown in (4.2). At an intuitive level, the mistake lies in differentiating the approximation $\theta \asymp \sqrt{\frac{n-w}{w}}$ (for some integer n larger than w) by holding $n-w$ constant instead of allowing it to depend upon w (see 4.2).

The final observation we make is that we strongly believe that the proper order of growth of the wasted space is $x^{3/5}$. We would expect that the argument that would be necessary to prove this claim would be quite sophisticated, as it would at the very least require a much more general version of the complex argument in [13].

Chapter 5

Efficient Packing of Unit Squares

Let $S(x)$ denote a square of sidelength x for some large x. Pack $S(x)$ as efficiently as possible by squares of unit sidelength with disjoint interiors. Let $W(x)$ denote the minimum amount of area left uncovered in $S(x)$ by any such packing. In this chapter, we will prove the following result:

Theorem 1.2. *The wasted space in packing the square* $S(x)$ *by unit squares is bounded by*

$$W(x) = O(x^{3/5}).$$

We will prove Theorem 1.2 in several stages. First, we pack all of $S(x)$ using "stacks" of unit squares except for a finite number of trapezoidal regions T in Section 5.1. We then pack such a generic trapezoidal region using two different "sub-algorithms", described in in sections 5.2 and 5.3. In section 5.4, we show how these can be combined to form a packing of T. We then optimize our parameters, and demonstrate that the wasted space generated by our packing is $O(x^{3/5})$.

5.1 Packing $S(x)$ Using Stacks of Unit Squares

We begin by packing $S(x)$ in a trivial manner by placing unit squares snugly with sides parallel to $S(x)$ starting at one corner until the only unpacked region is two rectangles with width h (see Figure 4.1). Note that we can fix h to the nearest $O(1)$ by changing the number of squares that we pack during this stage, which is something we will do at the end of the proof.

Without loss of generality, we will pack only one of the rectangles, which we call R. We will assume that the short h-length side of R is aligned with the vertical coordinate axis. We then pack R by near-vertical stacks of unit squares of length n inclined so that they touch both sides of R (see Figure 4.2). Choose $n \in \mathbb{Z}^+$ so that $1 < n - h \ll 1$. If θ is the angle of inclination of these stacks, then observe that

$$\sec(\theta) = \frac{n + \tan\theta}{h}.$$

Taylor series expansion then gives

$$\theta^2 = \frac{2(n-h)}{h} + O\left(\frac{\theta}{h} + \theta^4\right).$$

Thus,

$$\theta \asymp \frac{1}{\sqrt{h}}. \tag{5.1}$$

Pack all of R in such a manner except for a narrow trapezoid on each end of height h. Without loss of generality, we will pack only one such trapezoid, which we call T. Note that we can fix the width of T (namely, the width of its smaller side), which we call w, to the nearest $O(1)$ by changing the number of n-length stacks that we pack during this stage, and will do so at a later time. We will assume that the h-length side of T is aligned with the vertical coordinates axis. We will call this the *vertical wall* of T. We will call the opposing side the *inclined wall* of T.

Except for a portion that we pack trivially at the top and bottom of T, we will pack T in a collection of near horizontal strips that will each have a height of about $2\theta^{-1}$. Each such strip will be packed using two packing algorithms. Such packings will be replicated all the way down the length of T. When we describe these two packing algorithm in sections 5.2 and 5.3, we will describe them abstractly and not yet fix where we are vertically within T, but instead will only refer to the two sides of T as the vertical and inclined walls. We will combine these two packing algorithms together formally in section 5.4.

5.2 The First Packing Algorithm

First, place a near-horizontal stack H_0 inclined at an angle of φ containing m unit squares so that its left edge touches the vertical wall and its right edge is some distance $\varepsilon \geqslant 0$ from the inclined wall (see Figure 5.1). We now place a small rectangle of width ε along the inclined wall stretching over the entire vertical region that will be covered during the first packing algorithm. This region will not be packed. Place a vertical stack V_0 inclined at an angle of θ with m squares so that it is sung against the ε-width rectangle and parallel to the inclined wall.

Next place a horizontal stack H_1 snugly against H_0 containing $m-1$ squares such that it touches the vertical wall. Similarly, place a vertical stack V_1 containing $m-1$ squares snugly against V_0 and touching H_1. We then continue in a similar manner to place the stacks H_i and V_i containing $m-i$ squares for $i = 0, \dots, m-1$.

Label the points on the first two horizontal and vertical stacks as in Figure 5.1. Construct the line GE, noting that this must be vertical by construction. We are going to fix φ in such a way that the angle $\angle EDC$ formed is a right-angle. This means that there is a line formed from E to the bottom left corner of V_0 inclined at angle θ, just like the inclined wall, which allows us to iteratively pack each

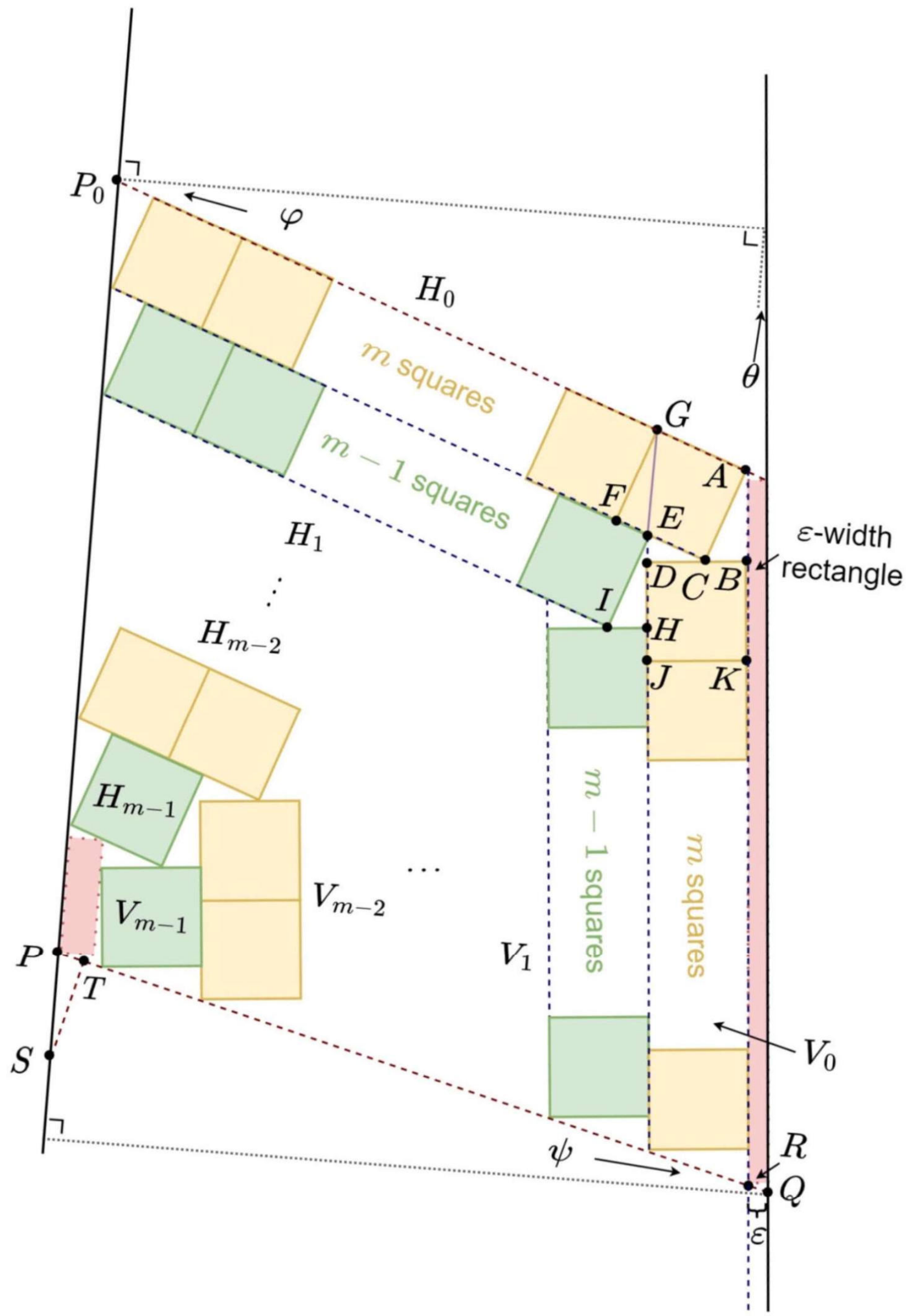

Figure 5.1: The first packing algorithm.

consecutive H_i and V_i in the same manner as H_0 and V_0.

Estimating φ

Observe that

$$\angle FGE = \varphi, \quad \angle ECD = \varphi + \theta, \quad \text{and} \quad \angle CAB = \varphi + \theta,$$

and so

$$EC = 1 - \tan\varphi \quad \text{and} \quad DC = 1 - \sin(\varphi + \theta). \tag{5.2}$$

Thus, $\angle EDC$ is a right angle if and only if

$$(1 - \tan\varphi)\cos(\varphi + \theta) = 1 - \sin(\varphi + \theta). \tag{5.3}$$

Fix φ to be the solution of (5.3). We now want to determine an approximation for φ in terms of θ. Applying Taylor series expansions gives

$$\left(1 - \varphi + O(\varphi^3)\right)\left(1 - \frac{\varphi^2}{2} - \varphi^2\theta^2 - \frac{\theta^2}{2} + O\left((\varphi+\theta)^4\right)\right) = 1 - \varphi - \theta + O\left((\varphi+\theta)^3\right).$$

Aggregating lower order terms gives

$$\theta - \frac{\varphi^2}{2} - \varphi^2\theta^2 - \frac{\theta^2}{2} + \frac{\varphi\theta^2}{2} = O\left((\varphi+\theta)^3\right).$$

This implies that $\varphi^2 + O(\varphi^3) = 2\theta + O(\theta^2)$. It follows that $\varphi^2 \asymp \theta$ allowing us to write $\varphi = \sqrt{2\theta + O(\theta^{3/2})}$. Taylor series expansion gives

$$\varphi = \sqrt{2\theta} + O(\theta) \quad \text{and} \quad \theta = \frac{1}{2}\varphi^2 + O(\varphi^3). \tag{5.4}$$

Estimating ψ

Now, we have fixed φ (and estimated it in terms of θ) such that the packing setup shown in Figure 5.1 is feasible and can be iterated for each H_i and V_i. This automatically fixes the angle ψ formed at the bottom of this region that we are packing (see Figure 5.1). We now want to estimate this angle. Observe that $\angle HKJ = \psi + \theta$, so

$$DH = 1 - \tan(\psi + \theta). \tag{5.5}$$

On the other hand since IEH and ECD are similar, so

$$ED + DH = \frac{DC}{EC}.$$

Since $\angle ECD = \varphi + \theta$, we can rearrange this as

$$DH = \cos(\varphi + \theta) - EC \sin(\varphi + \theta).$$

Combining this with (5.2) gives

$$DH = \cos(\varphi + \theta) - (1 - \tan\varphi)\sin(\varphi + \theta)$$

Comparing this with (5.5) allows us to see that ψ is the unique solution of

$$1 - \tan(\psi + \theta) = \cos(\varphi + \theta) - (1 - \tan\varphi)\sin(\varphi + \theta). \tag{5.6}$$

We can Taylor expand both sides of this equation, simplifying our error term using (5.4):

$$1 - \psi - \theta = 1 - \frac{1}{2}\varphi^2 - (\varphi + \theta) + \varphi^2 + O(\varphi^3 + \psi^3).$$

Solving for ψ gives

$$\psi = \varphi - \frac{1}{2}\varphi^2 + O(\varphi^3 + \psi^3).$$

It follows that $\psi + O(\psi^3) = \varphi + O(\varphi^2)$ and so $\psi \asymp \varphi$, allowing us to simplify the error term. Combining this observation with (5.4), we get

$$\psi = \varphi - \theta + O(\varphi^3). \tag{5.7}$$

Thus, we have "lost" about θ in the inclination of the top portion of the region we are packing compared to the bottom.

Now, at this stage, the only quantity we have fixed is φ, which was chosen in such a way so that we could perform our first packing algorithm. In the second packing algorithm we will want to start by packing $m+1$ square along the length PR compared to the m squares we packed at the beginning of the first packing algorithm (note that R is chosen to be the point on $\overline{PQ}$ such that the distance from R to the inclined wall is ε). To do this, we need to choose m large enough so that the first packing algorithm takes up sufficient vertical space, thus and forces PR to be able to fit $m+1$ squares along it.

Estimating m

We want to choose m such that $\overline{PR}$ is just long enough to fit a near-horizontal stack of $m+1$ squares inclined at an angle of ψ. Choose points S on the vertical wall and T on $\overline{PR}$ such that $ST = 1$ and $\angle PTS$ is a right-angle (equivalently, $\angle TSP = \psi$). Define m to be the least integer such that $TR \geqslant m+1$. Note that the distance between the inclined and the vertical walls increases at a rate of $O(\theta)$. Thus, increasing m by 1 will increase TR by $O(\theta)$. Thus, our definition of m automatically gives

$$TR = m + 1 + O(\theta).$$

It is important to note that this definition of m (and the length TR) is *independent* of ε, although of course TQ depends on ε:

$$TQ = m + 1 + O(\theta + \varepsilon). \tag{5.8}$$

We now want to find an approximate expression for m in terms of our known quantities. Observe that the horizontal distance from the vertical wall to the lower-left corner of V_{m-1} is the same as the horizontal distance from F to J, which is $O(\varphi)$. Thus, the distance from P to the lower-left corner of V_{m-1} is also $O(\varphi)$, we have:

$$TR = \frac{m}{\cos(\psi + \theta)} + O(\varphi)$$

However, once again, we know that the difference between TR and $m + 1$ is at most $O(\theta)$. Thus,

$$\frac{m}{\cos(\psi + \theta)} - m = 1 + O(\varphi).$$

Upon Taylor expansion, the left-hand side becomes $\frac{1}{2}\psi^2 m + O(\psi\theta m)$. Applying (5.7) and (5.4) thus gives

$$\theta m + O(\varphi^3 m) = 1 + O(\varphi).$$

Clearly this implies that $m \asymp \theta^{-1}$, which allows us to simplify the error term, giving us an approximate expression for m:

$$m = \theta^{-1} + O(\varphi^{-1}). \tag{5.9}$$

Summary

This completes the analysis of the first packing stage. We make two important observations. First, observe that the quantities φ, ψ and m are all fixed based solely on θ, independent of ε. Second, note that the angle φ at the top of the packed squares differs from the angle ψ at the bottom of the packed squares by about θ (see 5.7). This is the reason we cannot simply iterate this first packing algorithm all the way down the trapezoid T. Indeed, if we were to reset and apply the first packing algorithm successively each application would generate a triangle of height $O(m)$ and angle $\varphi - \psi \sim \theta$ having area θ^{-1}. This is an order of magnitude larger than the area of the other wasted space that is generated during the algorithm we just described. We instead have to pair this first algorithm with another algorithm that "undoes" the angle change of θ. We do this in the next section.

5.3 The Second Packing Algorithm

The second packing algorithm will begin at the bottom of where the first packing algorithm left off, and proceed down the trapezoid T. Place a horizontal stack H_0' inclined at an angle of ψ up against $\overline{PQ}$ that contains $m+1$ squares and so that its left edge touches the vertical wall and its right side is some distance of ε' from the inclined wall (see Figure 5.2). Note that our choice of m in the last section is what ensures that we have enough room. We are going to place a small rectangle of width ε' along the inclined wall and will not pack that region during the second packing algorithm. Note that from (5.8), we have

$$\varepsilon' \ll \varepsilon + \theta. \tag{5.10}$$

Next, pack a vertical stack V_0' with m' (an integer which is less than m which we

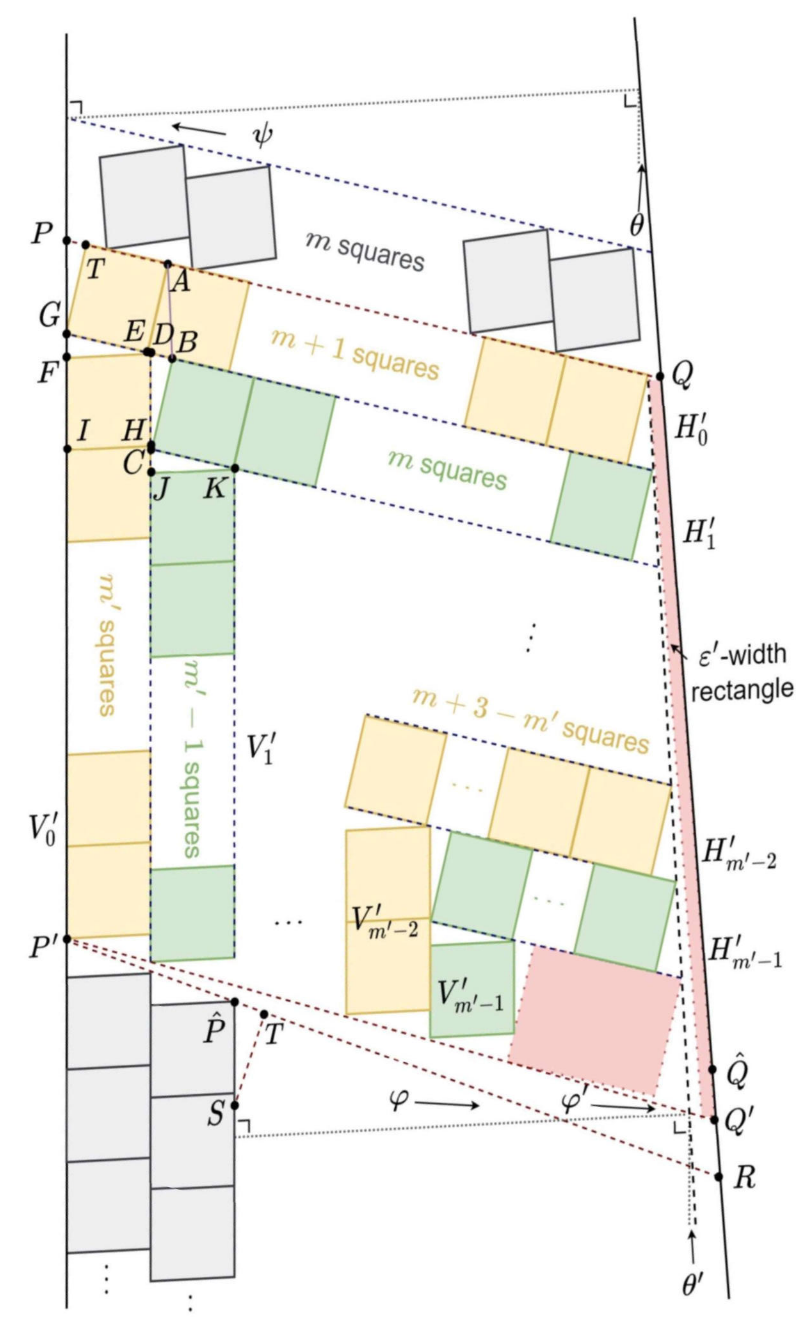

Figure 5.2: The second packing algorithm.

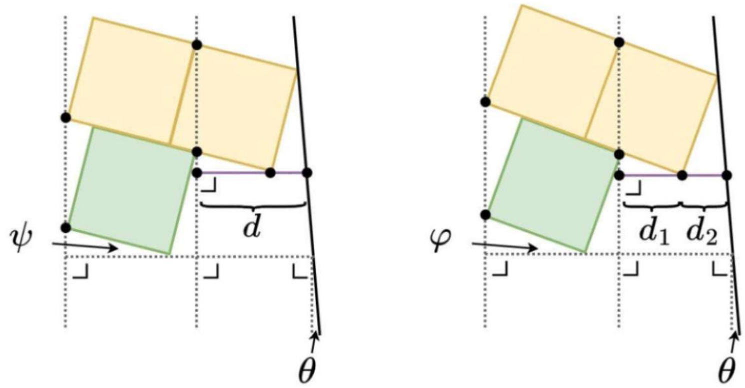

Figure 5.3: Demonstrating that $\theta' < \theta$.

will fix later) squares against the vertical wall so that it touches H'_0. Choose some θ' such that we can fit another horizontal stack H'_1 with m squares snugly against H'_0 so that it touches both V'_0 and a new inclined wall of angle θ' (we will prove the feasibility of such an arrangement shortly). Label the points on these stacks as in Figure 5.2.

Estimating θ'

First we prove that $\theta' \leqslant \theta$ by ensuring that there is enough room to pack H'_1 when $\theta' = \theta$ (if the horizontal stacks were too large we would be forced to choose $\theta' < \theta$). However, this is feasible if and only if the distance d depicted in Figure 5.3 is greater than or equal to 1 (otherwise we would not be able to slide in the vertical stack V'_0). But, since $\varphi > \psi$, then $d_1 + d_2 > d$. Clearly, though, d_1 is greater than DC in the first packing algorithm (see Figure 5.1) and d_2 is greater than CB. Thus, $d > d_1 + d_2 > DB = 1$, implying that $\theta' \leqslant \theta$.

We now give an upper bound on the discrepancy between θ' and θ.

Observe that

$$\angle EAB = \psi + \theta', \quad \angle BCD = \psi, \quad \text{and} \quad \angle GDF = \psi.$$

Thus,

$$ED + DB = \tan(\psi + \theta'), \quad 1 + ED = \frac{1}{\cos\psi}, \quad \text{and} \quad DB = \tan\psi.$$

We can then define θ' to be the unique solution to

$$\sec\psi - 1 + \tan\psi = \tan(\psi + \theta'). \tag{5.11}$$

Taylor series expansion gives

$$\theta' = \frac{1}{2}\psi^2 + O(\psi^3).$$

Comparing with (5.4) and (5.7) gives

$$0 \leqslant \theta - \theta' \ll \varphi^3. \tag{5.12}$$

We then can continue to pack the $m + 1 - i$-square stacks H_i for $i = 0, 1, \dots, m' - 1$ in the same manner. Similarly, we will pack the $m' - i$-square stacks V_i for $i = 0, 1, \dots, m' - 1$.

Estimating φ'

To determine the angle φ', note that $\varphi' = \angle JIH$ (see Figure 5.2). However the triangle CKJ is congruent to the triangle CDB, which implies that $CJ = DB$. Furthermore, the triangle CDB is congruent to the triangle FGD, which implies that $ED = HC$. Thus the triangle JIH is congruent to the triangle BAE, which then implies that $\varphi' = \psi + \theta'$. From (5.7) and (5.12), this implies that

$$\varphi - \varphi' \ll \varphi^3. \tag{5.13}$$

Note that φ' is fixed independent of ε.

Estimating m'

We want to choose m' such that if we packed a new horizontal stack of length m at an angle of φ, there would still be just enough room for two vertical stacks up against the vertical wall. Note that these stacks would continue to the bottom of the trapezoid T. This setup would allow us to once again apply the first packing algorithm.

Label the points as in the bottom of Figure 5.2. We are assuming here that $\varphi \geqslant \varphi'$. The case for $\varphi < \varphi'$ is analogous. Note that we have chosen points S on the right-hand side of the two vertical stacks and T on $\overline{PR}$ such that $ST = 1$ and $\angle\hat{P}TS$ is a right-angle (equivalently, $\angle TS\hat{P} = \varphi$). Thus, we will want to choose m' to be the least integer such that $TR \geqslant m$. Of course, we will have to show that such an m' exists and is less than or equal to m, and we will have to bound the discrepancy between m and m'. For now, it is easy to see that $m' \asymp m \asymp \theta^{-1}$.

Note that m' will have to depend upon ε, since we are attempting to close the gap caused by the ε'-width rectangle. This is important as the discrepancy $TR - m$ will generate the ε for the next iteration of the first and second packing algorithms, and we do not want these errors aggregating over each iteration of this pair of packing algorithms.

Let $\hat{Q}$ be the point on the inclined wall such that $\overline{P'\hat{Q}}$ is parallel to $\overline{PQ}$ (namely, inclined at an angle of ψ). From the sine rule, and using the fact that $m', m \ll \theta^{-1}$, we have

$$P'\hat{Q} - PQ = PP'\frac{\sin\theta}{\sin(\pi/2 - \psi - \theta)} = (m' + O(1))\frac{\sin\theta}{\cos(\psi + \theta)} = m'\theta + O(\theta).$$

Again, by the sine rule,

$$P'R = P'\hat{Q}\left(\frac{\sin(\pi/2+\theta+\psi)}{\sin(\pi/2-\theta-\varphi)}\right) = \left(PQ + m'\theta + O(\theta)\right)\left(\frac{\cos(\theta+\psi)}{\cos(\theta+\varphi)}\right)$$

Observe that since $\varphi - \psi = \theta + O(\varphi^3)$ from (5.7), we have

$$\frac{\cos(\theta+\psi)}{\cos(\theta+\varphi)} = \frac{\cos(\theta+\varphi)\cos(\theta+O(\varphi^3)) + \sin(\theta+\varphi)\sin(\theta+O(\varphi^3))}{\cos(\theta+\varphi)} = 1+\theta\varphi+O(\varphi^4).$$

Thus, since $m' \ll m \ll \varphi^{-2}$ and $PQ = m + \varphi + O(\varphi^2+\varepsilon)$ (from (5.8)), we have

$$\begin{aligned} P'R &= \left(m+1+\varphi+m'\theta+O(\varphi^2+\varepsilon)\right)\left(1+\theta\varphi+O(\varphi^4)\right) \\ &= m+1+\varphi+m'\theta+\theta\varphi m+O(\varphi^2+\varepsilon) \end{aligned}$$

We then have

$$TR = P'R - 2(1-\cos\varphi) - \sin(\varphi) = m + m'\theta + \theta\varphi m - 1 + O(\varphi^2+\varepsilon). \tag{5.14}$$

Now, observe that it is always possible to find a $m' \leqslant m$ such that $TR \geqslant m$ as long as $\varepsilon \ll \varphi^2$, since taking $m' = m$ would make $TR \sim m + \varphi$ (larger than m), so such an m' must exist. We are implicitly using the fact that we can always fix TR to the nearest θ (the slope of the inclined wall). We can summarize these observations with the following equation

$$0 \leqslant TR - m \ll \theta, \tag{5.15}$$

noting that the asymptotic is independent of ε. Substituting this into (5.14), and once again assuming that $\varepsilon = O(\theta)$, gives

$$\theta m' + \theta\varphi m - 1 = O(\theta).$$

This allows us to solve for m':

$$m' = \theta^{-1} + O(\varphi m).$$

Combining this with (5.9) gives

$$m - m' \ll \varphi^{-1}. \tag{5.16}$$

Summary

We have now developed two packing algorithms. When applied back to back, the end of the second packing algorithm is the setup for the initiation of the first packing algorithm. Note that the only parameter that will vary throughout this iterative process is m', as it is dependent upon ε.

5.4 Proof of Theorem 1.2

Packing T

We can now apply our two packing algorithms to complete the proof of Theorem 1.2. Recall that we had not yet fixed the width of the trapezoid T. Now, the inclined wall of T was inclined at an angle $\theta \asymp h^{-1/2}$ (see (5.1)). Observe that with this choice of θ, we must have $m, m' \asymp h^{1/2}$, and $\varphi, \psi, \varphi' \asymp h^{-1/4}$. We now fix the width of the upper edge of T so that the first m-length horizontal stack inclined at an angle of φ is within $O(\sqrt{h})$ of the top of T, thereby implying that T has a width on the order of $\sqrt{h}$ (see Figure 5.4).

To pack T, we apply our two packing algorithms back to back, packing a "sub-trapezoid" at the top of T of height about $2\theta^{-1} \asymp \sqrt{h}$. Note that our first choice of ε can be 0. We then pack the two vertical stacks along the *entire* vertical wall. We

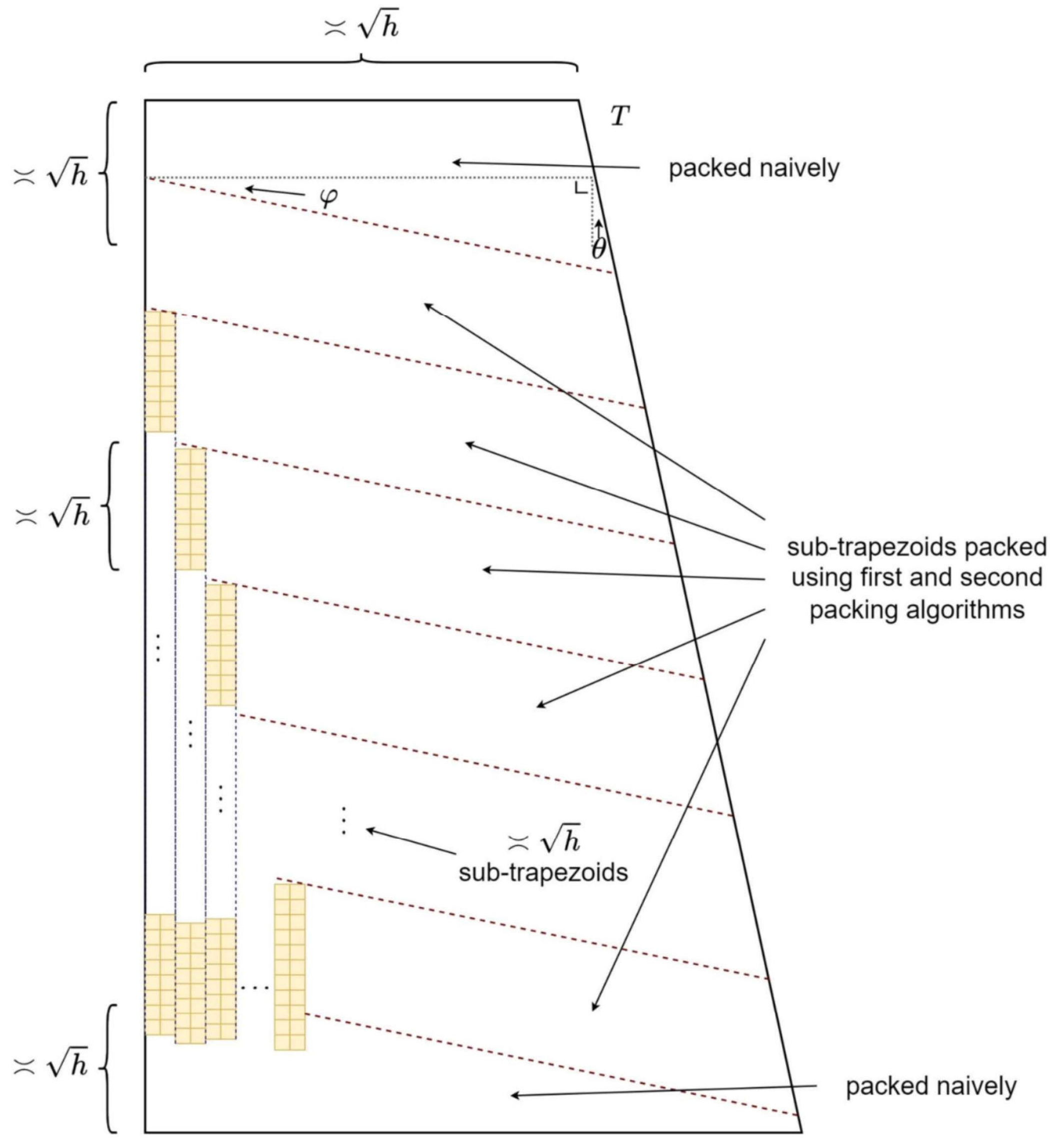

Figure 5.4: Packing the trapezoid T.

repeat this process, packing each sub-trapezoid all the way until we have packed everything except a region at the bottom of T that has height at most $O(\sqrt{h})$ (see Figure 5.4). Note that at each stage, we are using the same φ, ψ, φ' and m, but that the choices for $\varepsilon, \varepsilon'$ and m' will change for each iteration of the packing algorithm. However, we can always ensure that $\varepsilon \ll \theta$, courtesy of (5.15). Note that we will have to apply our algorithm a total of $\asymp \sqrt{h}$ times to pack all of T.

Estimating the wasted space

Let us begin by computing the wasted space from a single iteration of the first and second packing algorithms. Observe that *all* of the rectangles in the wasted space have area at most $O(\frac{1}{\sqrt{\theta}})$. To see this, first observe that H'_{m-1} in Figure 5.2 has length at most $O(\varphi^{-1})$ from (5.16). Second, note that we can always force $\varepsilon \ll \theta$, and thus $\varepsilon' \ll \theta$ (see (5.15) and (5.10)).

Next, note that there are two large sliver triangles of length $O(\theta^{-1})$ and angles $\theta - \theta'$ and $\varphi - \varphi'$ (see Figure 5.2). Thus, from (5.12) and (5.13), the area of these triangles is $\ll \frac{1}{\sqrt{\theta}}$. There is also a small sliver between $\overline{P'Q'}$ and the rectangle of width $\ll \varphi^{-1}$. This has an angle trivially bounded by φ, and so its area is $O(\frac{1}{\sqrt{\theta}})$.

Finally, observe that there are $\ll \theta^{-1}$ $O(1)$-length triangles created through both packing algorithms, and their angles are all $\ll \sqrt{\theta}$, implying that there total contribution is $\ll \frac{1}{\sqrt{\theta}}$.

Thus, the total amount of wasted space generated through these two packing stages is $\ll \frac{1}{\sqrt{\theta}} \ll h^{1/4}$. Since we are applying these packing algorithms $\asymp \sqrt{h}$ times, then the total wasted space contributed by packing the sub-trapezoid regions becomes $O(h^{3/4})$. We pack the regions at the top and bottom of T trivially, which generates an additional wasted space of $O(h^{1/2})$ (see Figure 5.4). Thus, the total wasted space generated by packing T is $O(h^{3/4})$.

Now, the wasted space generated by packing the $O(x)$ vertical stacks inclined at an angle θ in Figure 4.2 is $O(x\theta)$, meaning that our total wasted space is bounded as follows:

$$W(x) \ll \frac{x}{\sqrt{h}} + h^{3/4}.$$

Equalizing the two terms gives a choice of $h \sim x^{4/5}$, yielding the desired result.

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