INTERVAL-VALUED PYTHAGOREAN FUZZY DECISION-MAKING MODELS FOR EVALUATING CHALLENGES OF DIGITAL TRANSFORMATION

by

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Abstract

The aim and objective of the thesis entitled "Interval-valued Pythagorean Fuzzy Decision-Making Models for Evaluating Challenges of Digital Transformation" are as follows:

The first objective is to develop new entropy and divergence measures to handle the uncertainty under interval-valued Pythagorean fuzzy environment to determine the significance degree/weight DT challenges of the manufacturing systems.

The second objective is to develop a hybrid decision-making models to evaluate the DT challenges of the manufacturing systems from interval-valued Pythagorean fuzzy perspective.

And the last objective is to propose a comprehensive framework to evaluate digital transformation challenges in sustainable financial service systems of the banking sector.

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CHAPTER 1

INTRODUCTION

1.1. Background

Digital transformation (DT) has become essential for businesses to remain competitive in today's ever-changing technological landscape (Barrett, Davidson and Vargo, 2015). It is the integration of digital technology into all aspects of business from daily operations to strategic decision making. It includes not only a move from analog to digital instruments but also a culture shift and rethinking of ways a company should work. (Yoo et al., 2010, 2012; Shang et al., 2023). It can significantly improve an organization's productivity by automating manual processes, reducing errors and improving production. For instance, implementing cloud-based solutions can enable employees to access data and team up more efficiently from anywhere in the world. In general, DT is defined as the process by which companies embed technologies across their businesses to drive fundamental change (Yoo et al., 2010). These transformations are long-term efforts to rewire how an organization continuously improves and changes.

In recent years, a rapid transformation has been witnessed in the business world with many companies adopting new technologies to streamline their operations and improve their bottom line (Shang et al., 2023; Wang et al., 2023; Rani et al., 2023). Many businesses can leverage emerging technologies like artificial intelligence, machine learning, big data and the Internet of Things to gain insights into customer behavior, streamline operations and improve decision-making (Wang, 2023). DT has dual functions in that it enables banking organizations to offer new service channels through new electronic platforms (e-banking, virtual banking) and service points (e-branch stores, POS) and also reduces their operating costs by limiting the number of physical stores and staff that they use. Digital innovation could drive a range of industrial organization outcomes. On the one hand, digital technology enables niche providers to reach a target customer base and be economically viable. On the other hand, customer acquisition, funding, "assembly," and switching costs tend to favour larger providers of digital financial services. There are several challenges and drivers that may affect the digital technologies' implementation processes in the financial sectors. Moreover, uncertainty is widely occurred in such types of real-life problems. An alternative "digital technology" is considered "most suitable" to the degree that it is consistent with the economic, environment, social, technical, cultural and political aspects of the society. Thus, the fuzzy multi-criteria decision-making approaches are more appropriate to systematically solve this problem.

1.2. Fuzzy Sets

The concept of vagueness is a long-time challenge for mathematicians and it is a crucial issue in the area of artificial intelligence in computer science. To overcome this situation the concept fuzzy set (FS) was introduced by Zadeh (1965). The theory of FS was directly extended from the crisp set theory in order to handle vagueness and uncertainty. A FS '*F*' is determined over a space '*U*' by a membership function $\mu,$ whose value is between zero and one, for example $\mu(u)=0.4$ means the membership degree of u in F is 0.4 and then $1-\mu(u)=0.6$ is understood that the non-membership degree of u in F. The membership value of an element is zero shows that the element does not belong to the class. The membership value of an element is one shows that element belongs to that class and other values between zero to one indicate the degree of membership to a class. The theory of fuzzy sets proposed by Zadeh (1965) has attracted wide spread attentions in various fields, especially where conventional mathematical techniques are of limited effectiveness, including biological and social sciences, linguistic, psychology, economics, and more generally soft sciences.

1.2.1. Intuitionistic Fuzzy Sets

The main drawback of FS theory is the inconclusive property because the exclusiveness of non-membership function and the ignorance for the possibility of hesitation margin. To overcome the above drawback Atanassov (1986) carefully studied this drawback and proposed a new concept namely intuitionistic fuzzy set (IFS). The idea of IFS is more adjustable and reasonable in dealing with vagueness and fuzzy information which has given deep attention from literature. IFSs accommodate both membership function and non-membership function with hesitation margin. An IFS '*M*' over a finite universal set '*U*' is characterized by a membership function μ and a non-membership function ν, represented by (μ, ν) with the condition $\mu : U \to [0,1]$, $\nu : U \to [0,1]$ and $0 \le \mu + \nu \le 1$. The hesitation margin is defined as $1-\mu-\nu$. For example, $\mu(u)=0.5$ means the membership degree of *u* in *M* is 0.5 and $\nu(u)=0.3$ means the non-membership degree of *u* in *M* is 0.3, then $1-0.5-0.3$ is the degree of hesitation of *u* in *M*. The concept of IFS can be viewed as an alternative approach to define a FS in a situation where available information is not sufficient for the definition of an imprecise concept by means of a conventional FS. There are several situations that can be modelled using IFS but cannot be represented using classical FS theory. For example, suppose voters may be partitioned into three groups of those who vote for, who vote against and who abstain. If we take $\langle u, 0.6, 0.25 \rangle$ as an element of IFS *M* of voting, we can interpret that "the vote for the applicant is 0.6 in favor to 0.25 against with 0.15 nonparticipations". Therefore, IFS is more comprehensive and reasonable than classical FS in describing the uncertainty of an object (Atanassov, 1986).

In 1986, Atanassov (1986) gave the concept of IFS, which is mathematically presented as

Definition 1.1 (Atanassov, 1986). An IFS *M* on a finite universal set $U=\{u_1, u_2, ..., u_s\}$ is defined as

$$M=\{\langle u_i, \mu_M(u_i), \nu_M(u_i)\rangle : u_i \in U\}, \tag{1.1}$$

where $\mu_M : U \to [0, 1]$ denote the membership degree (MD) and $\nu_M : U \to [0, 1]$ denote the non-membership degree (NMD) of an element u_i to *M* in *U*, with the condition

$$0 \le \mu_M(u_i) \le 1,\ 0 \le \nu_M(u_i) \le 1 \text{ and } 0 \le \mu_M(u_i)+\nu_M(u_i) \le 1,\ \forall\, u_i \in U. \tag{1.2}$$

The degree of hesitation of an element $u_i \in U$ to *M* is defined as

$$\pi_M(u_i)=1-\mu_M(u_i)-\nu_M(u_i) \text{ and } 0 \le \pi_M(u_i) \le 1,\ \forall u_i \in U.$$

For convenience, Xu (2007) characterized the intuitionistic fuzzy number (IFN) $\varsigma=(\mu_\varsigma, \nu_\varsigma)$, which satisfies $\mu_\varsigma, \nu_\varsigma \in [0,1]$ and $0 \le \mu_\varsigma + \nu_\varsigma \le 1$.

Atanassov (1986) defined some basic operations over intuitionistic fuzzy sets, which is given in Definition 1.2.

Definition 1.2. For two IFSs *M* and *N*, some operational laws are defined as follows:

(i) $M^c=\{\langle u, \nu_M(u), \mu_M(u)\rangle : u \in U\},\quad N^c=\{\langle u, \nu_N(u), \mu_N(u)\rangle : u \in U\}$,

(ii) $M \cup N=\{\langle u, \max(\mu_M(u), \mu_N(u)), \min(\nu_M(u), \nu_N(u))\rangle : u \in U\}$,

(iii) $M \cap N=\{\langle u, \min(\mu_M(u), \mu_N(u)), \max(\nu_M(u), \nu_N(u))\rangle : u \in U\}$,

(iv) $M+N=\{\langle u, \mu_M(u)+\mu_N(u)-\mu_M(u).\mu_N(u), \nu_M(u).\nu_N(u)\rangle : u \in U\}$,

(v) $M.N=\left\{\left\langle u,\mu_M(u).\mu_N(u),\ \nu_M(u)+\nu_N(u)-\nu_M(u).\nu_N(u)\right\rangle : u\in U\right\}$,

In 2007, Xu (2007) presented some basic aggregation operators to aggregate the individual information, given in Definition 1.3.

Definition 1.3 (Xu, 2007). Let $\varsigma_k=\left(\mu_k,\nu_k\right),\ k=1,2,...,s,$ be the collection of IFNs and $\psi=\left(\psi_1,\psi_2,...,\psi_s\right)^T$ be the weight vector of ς_k, $k=1,2,...,s,$ with $\sum_{k=1}^{s}\psi_k=1$ and $\psi_k\in[0,1]$. Then intuitionistic fuzzy weighted averaging (IFWA) and intuitionistic fuzzy weighted geometric (IFWG) operators are presented as

$$IFWA_{\psi}\left(\varsigma_1,\varsigma_2,...,\varsigma_s\right)=\overset{s}{\underset{k=1}{\oplus}}\psi_k\,\varsigma_k=\left[1-\prod_{k=1}^{s}\left(1-\mu_k\right)^{\psi_k},\prod_{k=1}^{s}\nu_k^{\psi_k}\right], \tag{1.3}$$

$$IFWG_{\psi}\left(\varsigma_1,\varsigma_2,...,\varsigma_s\right)=\overset{s}{\underset{k=1}{\otimes}}\varsigma_k^{\psi_k}=\left[\prod_{k=1}^{s}\mu_k^{\psi_k},\ 1-\prod_{k=1}^{s}\left(1-\nu_k\right)^{\psi_k}\right]. \tag{1.4}$$

In 1994, Chen and Tan (1994) proposed a score function to compare the vague values and used it in multiple criteria decision making problems under vague environment. To calculate the accuracy level of vague values, Hong and Choi (2000) introduced an accuracy function. For the first time, Xu (2007) proposed new score and accuracy functions for IFNs and their applications. As Xu's score function lies between -1 to 1, therefore, Xu et al. (2015) developed a normalized score function to rank the intuitionistic fuzzy values, which lies between 0 to 1.

Definition 1.4 (Xu, 2007). The score and accuracy functions of an IFN $\varsigma=\left(\mu_\varsigma,\nu_\varsigma\right)$ is defined by

$$S\left(\varsigma\right)=\left(\mu_\varsigma-\nu_\varsigma\right) \tag{1.5}$$

$$A\left(\varsigma\right)=\left(\mu_\varsigma+\nu_\varsigma\right), \tag{1.6}$$

respectively. Here, $S(\varsigma) \in [-1,1]$ and $A(\varsigma) \in [0,1]$. As $S(\varsigma) \in [-1,1]$, then Xu et al. (2015) presented a modified score function for IFN, which as

Definition 1.5 (Xu et al., 2015). Consider $\varsigma = (\mu_\varsigma, \nu_\varsigma)$ be an IFN. Then,

$$S^*(\varsigma) = \frac{1}{2}(\mu_\varsigma - \nu_\varsigma + 1) \tag{1.7}$$

is defined as normalized score function for IFN ς. Here, $S^*(\varsigma) \in [0,1]$.

1.2.2. Pythagorean Fuzzy Sets

As a new extension of FS, Yager (2013) gave the idea of Pythagorean fuzzy set (PFS), which is effective and powerful in representing fuzzy information. Similar to IFS (Atanassov, 1986), PFS is also characterized by the MD and the NMD, whose sum of squares is less than or equal to 1. Thus, the PFS is more general than the IFS. In some cases, the PFS can solve the problems that the IFS cannot. In 2013, Yager (2013) illustrated an example that shows the use of the Pythagorean fuzzy sets in situations wherein we cannot use IFSs. An example of this would be a case in which a decision-maker may give his/her assessment for the MD of an object $u \in T$ with 0.8, and allow his/her assessment for the NMD of an object $u \in T$ with 0.5. Since the addition of these two values (= 1.3) > 1, but their square sum is < 1, therefore, the PFS can successfully deal with this example.

Yager (2013) gave the concept of PFS, which is mathematically presented as

Definition 1.6 (Yager, 2013). In mathematical form, a PFS K on a fixed set U is defined as

$$K = \{\langle u_i, (\mu_K(u_i), \nu_K(u_i))\rangle | u_i \in U\}, \tag{1.8}$$

where $\mu_K : U \to [0,1]$ and $\nu_K : U \to [0,1]$ symbolize the degrees of membership and non-membership of an element $u_i \in U$ to K, respectively, such that $0 \le (\mu_K(u_i))^2 + (\nu_K(u_i))^2 \le 1$.

For any $u_i \in U$, the term $\pi_K(u_i)=\sqrt{1-\mu_K^2(u_i)-\nu_K^2(u_i)}$ denotes the Pythagorean fuzzy index or hesitation degree of u_i. Further, $(\mu_K(u_i),\nu_K(u_i))$ is defined as a Pythagorean fuzzy number (PFN) and indicated by $\theta=(\mu_\theta,\nu_\theta)$, where $\mu_\theta,\nu_\theta \in [0,1]$ and $0 \le \mu_\theta^2+\nu_\theta^2 \le 1$. Also, all the PFSs on fixed set U is represented by PFS(U). The key discrimination between IFS and PFS is that both sets have diverse conditions, and if an object u in U is IFN, then it must also be a PFN. Though, not all PFNs are IFNs.

Definition 1.7 (Zhang & Xu, 2014). The score and accuracy functions of a PFN $\theta=(\mu_\theta,\nu_\theta)$ is defined by

$$S_P(\theta)=\left((\mu_\theta)^2-(\nu_\theta)^2\right), \tag{1.9}$$

$$A_P(\theta)=\left((\mu_\theta)^2+(\nu_\theta)^2\right), \tag{1.10}$$

respectively. Here, $S_P(\theta)\in[-1,1]$ and $A_P(\theta)\in[0,1]$. As $S_P(\theta)\in[-1,1]$, then Wu & Wei (2017) presented a modified score function for PFN, which as

Definition 1.8 (Wu & Wei, 2017). Consider $\theta=(\mu_\theta,\nu_\theta)$ be a PFN. Then,

$$S_P^*(\theta)=\frac{1}{2}\left((\mu_\theta)^2-(\nu_\theta)^2+1\right) \tag{1.11}$$

is defined as normalized score function for PFN θ. Here, $S_P^*(\theta)\in[0,1]$.

For any two PFNs $\theta_1=(\mu_{\theta_1},\nu_{\theta_1})$ and $\theta_2=(\mu_{\theta_2},\nu_{\theta_2})$,

(i) If $S_P^*(\theta_1)>S_P^*(\theta_2)$, then $\theta_1>\theta_2$,

(ii) If $S_P^*(\theta_1)=S_P^*(\theta_2)$, then

a. if $A_P(\theta_1)>A_P(\theta_2)$, then $\theta_1>\theta_2$,

b. if $A_P(\theta_1)=A_P(\theta_2)$, then $\theta_1=\theta_2$.

Definition 1.9 (Zhang & Xu, 2014). For any three PFNs $\theta=(\mu_\theta,\nu_\theta)$, $\theta_1=(\mu_{\theta_1},\nu_{\theta_1})$ and $\theta_2=(\mu_{\theta_2},\nu_{\theta_2})$, the following operational laws are defined as follows:

(i) $\theta^c=(\nu_\theta,\mu_\theta)$, where θ^c represents the complement of θ^c,

(ii) $\theta_1\oplus\theta_2=\left(\sqrt{\mu_{\theta_1}^2+\mu_{\theta_2}^2-\mu_{\theta_1}^2\mu_{\theta_2}^2},\nu_{\theta_1}\nu_{\theta_2}\right)$,

(iii) $\theta_1\otimes\theta_2=\left(\mu_{\theta_1}\mu_{\theta_2},\sqrt{\nu_{\theta_1}^2+\nu_{\theta_2}^2-\nu_{\theta_1}^2\nu_{\theta_2}^2}\right)$,

(iv) $\lambda\theta=\left(\sqrt{1-\left(1-\mu_\theta^2\right)^\lambda},(\nu_\theta)^\lambda\right)$, $\lambda>0$,

(v) $\theta^\lambda=\left((\mu_\theta)^\lambda,\sqrt{1-\left(1-\nu_\theta^2\right)^\lambda}\right)$, $\lambda>0$.

Definition 1.10 (Yager, 2014; Zhang & Xu, 2014). Consider $\theta_i=(\mu_{\theta_i},\nu_{\theta_i})(i=1,2,...,s)$ be a set of PFNs and $\rho=(\rho_1,\rho_2,...,\rho_s)^T$ be the related weighting vector of θ_i, satisfying $\rho_i\in[0,1]$ and $\sum_{i=1}^{s}\rho_i=1$. Then the Pythagorean fuzzy weighted averaging (PFWA) and Pythagorean fuzzy weighted geometric (PFWG) operators are given by

$$PFWA(\theta_1,\theta_2,...,\theta_s)=\left(\sqrt{1-\prod_{i=1}^{s}\left(1-\mu_i^2\right)^{\rho_i}},\prod_{i=1}^{s}(\nu_i)^{\rho_i}\right), \tag{1.12}$$

$$PFWG(\theta_1,\theta_2,...,\theta_s)=\left(\prod_{i=1}^{s}(\mu_i)^{\rho_i},\sqrt{1-\prod_{i=1}^{s}\left(1-\nu_i^2\right)^{\rho_i}}\right). \tag{1.13}$$

1.3. Interval-Valued Pythagorean Fuzzy Sets

Definition 1.11 (Peng & Yang, 2015). Let $I[0,1]$ be the set of all closed subintervals of $[0,1]$. In mathematical form, an IVPFS F in the finite universal set $U=\{u_1,u_2,...,u_s\}$ is described as

$$F=\left\{\left\langle u_i,\left(\left[\mu_F^-(u_i),\mu_F^+(u_i)\right],\left[\nu_F^-(u_i),\nu_F^+(u_i)\right]\right)\right\rangle : u_i\in U\right\}, \tag{1.14}$$

where $0\le \mu_F^-(u_i)\le\mu_F^+(u_i)\le 1$, $0\le \nu_F^-(u_i)\le\nu_F^+(u_i)\le 1$ and $0\le\left(\mu_F^+(u_i)\right)^2+\left(\nu_F^+(u_i)\right)^2\le 1.$ Here, $\mu_F(u_i)=\left[\mu_F^-(u_i),\mu_F^+(u_i)\right]$ and $\nu_F(u_i)=\left[\nu_F^-(u_i),\nu_F^+(u_i)\right]$ define the degrees of interval-valued MD and ND of an element u_i to U, respectively.

The function $\pi_F(u_i)=\left[\pi_F^-(u_i),\pi_F^+(u_i)\right]$ denotes the indeterminacy degree of u_i to F, where $\pi_F^-(u_i)=\sqrt{1-\left(\mu_F^+(u_i)\right)^2-\left(\nu_F^+(u_i)\right)^2}$ and $\pi_F^+(u_i)=\sqrt{1-\left(\mu_F^-(u_i)\right)^2-\left(\nu_F^-(u_i)\right)^2}$. For simplicity, Peng and Yang (2015) defined this term $\left(\left[\mu_F^-(u_i),\mu_F^+(u_i)\right],\left[\nu_F^-(u_i),\nu_F^+(u_i)\right]\right)$ as the "interval-valued Pythagorean fuzzy number (IVPFN)" and presented by $\alpha=\left(\left[\mu_\alpha^-,\mu_\alpha^+\right],\left[\nu_\alpha^-,\nu_\alpha^+\right]\right)$ which fulfills $0\le\left(\mu_\alpha^+\right)^2+\left(\nu_\alpha^+\right)^2\le 1.$

Definition 1.12 (Peng and Yang, 2015). Let $\alpha_1=\left(\left[\mu_{\alpha_1}^-,\mu_{\alpha_1}^+\right],\left[\nu_{\alpha_1}^-,\nu_{\alpha_1}^+\right]\right)$, $\alpha_2=\left(\left[\mu_{\alpha_2}^-,\mu_{\alpha_2}^+\right],\left[\nu_{\alpha_2}^-,\nu_{\alpha_2}^+\right]\right)$ and $\alpha_3=\left(\left[\mu_{\alpha_3}^-,\mu_{\alpha_3}^+\right],\left[\nu_{\alpha_3}^-,\nu_{\alpha_3}^+\right]\right)$ be three IVPFNs. Then, the operational laws on IVPFNs are presented as

(i) $$\alpha_1 \oplus \alpha_2 = \left(\begin{array}{l} \left[\sqrt{\left(\mu_{\alpha_1}^-\right)^2 + \left(\mu_{\alpha_2}^-\right)^2 - \left(\mu_{\alpha_1}^-\right)^2 \left(\mu_{\alpha_2}^-\right)^2}, \right. \\ \left. \sqrt{\left(\mu_{\alpha_1}^+\right)^2 + \left(\mu_{\alpha_2}^+\right)^2 - \left(\mu_{\alpha_1}^+\right)^2 \left(\mu_{\alpha_2}^+\right)^2} \right], \left[\nu_{\alpha_1}^- \nu_{\alpha_2}^-, \nu_{\alpha_1}^+ \nu_{\alpha_2}^+ \right] \end{array} \right),$$

(ii) $$\alpha_1 \otimes \alpha_2 = \left(\begin{array}{l} \left[\mu_{\alpha_1}^- \mu_{\alpha_2}^-, \mu_{\alpha_1}^+ \mu_{\alpha_2}^+ \right], \\ \left[\sqrt{\left(\nu_{\alpha_1}^-\right)^2 + \left(\nu_{\alpha_2}^-\right)^2 - \left(\nu_{\alpha_1}^-\right)^2 \left(\nu_{\alpha_2}^-\right)^2}, \sqrt{\left(\nu_{\alpha_1}^+\right)^2 + \left(\nu_{\alpha_2}^+\right)^2 - \left(\nu_{\alpha_1}^+\right)^2 \left(\nu_{\alpha_2}^+\right)^2} \right] \end{array} \right),$$

(iii) $$\lambda \alpha_1 = \left(\left[\sqrt{1 - \left(1 - \left(\mu_{\alpha_1}^-\right)^2\right)^\lambda}, \sqrt{1 - \left(1 - \left(\mu_{\alpha_1}^+\right)^2\right)^\lambda} \right], \left[\left(\nu_{\alpha_1}^-\right)^\lambda, \left(\nu_{\alpha_1}^+\right)^\lambda \right] \right),$$

(iv) $$\left(\alpha_1\right)^\lambda = \left(\left[\left(\mu_{\alpha_1}^-\right)^\lambda, \left(\mu_{\alpha_1}^+\right)^\lambda \right], \left[\sqrt{1 - \left(1 - \left(\nu_{\alpha_1}^-\right)^2\right)^\lambda}, \sqrt{1 - \left(1 - \left(\nu_{\alpha_1}^+\right)^2\right)^\lambda} \right] \right),$$

(v) $$\left(\alpha_1\right)^c = \left(\left[\nu_{\alpha_1}^-, \nu_{\alpha_1}^+ \right], \left[\mu_{\alpha_1}^-, \mu_{\alpha_1}^+ \right] \right).$$

Definition 1.13 (Peng and Yang, 2016). Assume $\alpha = \left(\left[\mu_\alpha^-, \mu_\alpha^+ \right], \left[\nu_\alpha^-, \nu_\alpha^+ \right] \right)$ be an IVPFN. Then

$$\mathrm{S}(\alpha) = \frac{1}{2}\left(\frac{1}{2}\left(\left(\mu_\alpha^-\right)^2 + \left(\mu_\alpha^+\right)^2 - \left(\nu_\alpha^-\right)^2 - \left(\nu_\alpha^+\right)^2 \right) + 1 \right), \tag{1.15}$$

and

$$h(\alpha) = \frac{1}{2}\left(\left(\mu_\alpha^-\right)^2 + \left(\mu_\alpha^+\right)^2 + \left(\nu_\alpha^-\right)^2 + \left(\nu_\alpha^+\right)^2 \right) \tag{1.16}$$

are defined as score function S and accuracy function h of an IVPFN α, respectively.

1.4. Fuzzy Information Measures

Information measures such as entropy, divergence, distance and similarity measures play a significant role in the theory of FS as well as IFS (Mishra, 2016). The word “entropy” was first used to measure an amount of uncertainty in probability distribution of a random variable in an experiment by Shannon (1948). For the first time, Zadeh (1968) developed and described the non-probabilistic entropy of a FS. In FS theory, the entropy is a measure of fuzziness which expresses the amount of average ambiguity in making a decision whether an element belongs to a set or not. De Luca and Termini (1972) firstly gave the axiomatic definition of fuzzy entropy. In the context of IFS, Burillo and Bustince (1996) put forward the concept of entropy on interval-valued fuzzy set and IFS to measure the degree of intuitionism. Firstly, Szmidt and Kacprzyk (2001) defined the axioms of De Luca and Termini entropy on IFS. Based on the IF-entropy, Mishra et al. (2023) defined the axiomatic definition of entropy on IVPFS as follows:

Definition 1.15 (Mishra et al. 2023). A real-valued mapping $E: IVPFS(U) \rightarrow [0,1]$ is said to be IVPF entropy if it holds the following properties:

(s1). $E(F)=0$ iff F is a crisp set,

(s2). $E(F)=1$ iff $\left[\mu_F^-(u_i), \mu_F^+(u_i)\right]=\left[\nu_F^-(u_i), \nu_F^+(u_i)\right]$, for all $u_i \in U$,

(s3). $E(F)=E\left(F^c\right)$,

(s4). $E(F) \leq E(C)$ if $F \subseteq C$ when $\mu_F^-(u_i) \leq \nu_C^-(u_i)$ and $\mu_F^+(u_i) \leq \nu_C^+(u_i)$, for each $u_i \in U$ or $C \subseteq F$ when $\mu_F^-(u_i) \geq \nu_C^-(u_i)$ and $\mu_F^+(u_i) \geq \nu_C^+(u_i)$, for each $u_i \in U$.

Divergence measure is an imperative topic to measure the dissimilarity between objects. In FS theory, Bhandari and Pal (1993) introduced the idea of divergence measure for FSs which is inspired from the concept of divergence measure between probability distributions. Montes et al. (2002) presented the axiomatic definition of fuzzy divergence measure. Like as FSs, Vlachos and Sergiadis (2007) pioneered the notion of divergence measure for IFSs and applied to medical diagnosis, pattern recognition problems and others. Montes et al. (2015) defined the divergence measure for IFSs with some enviable properties. Based on the IF-distance measure, Rani et al. (2023) defined the axiomatic definition of divergence measure on IVPFS as follows:

Definition 1.16 (Rani et al., 2023). Assume $F, C \in IVPFSs(U)$. A real-valued mapping $D: IVPFSs(U) \times IVPFSs(U) \to \mathbb{R}$ is said to be interval-valued Pythagorean fuzzy cross entropy if it holds the following requirements:

(A1). $D(F,C) \geq 0$, i.e. $D(F,C)$ is non-negative,

(A2). $D(F,C) = D(C,F)$,

(A3). $D(F,C) = 0$ if and only if $F = C$.

Distance measure is a term that describes the difference between IFS, and can be considered as a dual concept of similarity measure. As an important content in fuzzy mathematics, distance measures between IFSs have also gained much attention for their wide applications in real world, such as machine learning, image processing, pattern recognition, decision making and so forth. In the context of IFS, Burillo and Bustince (1996) firstly gave the definition of distance measure and further developed the Hamming, Euclidean, normalized Hamming and normalized Euclidean distance measures for IFSs. However, the distance measures proposed by Burillo and Bustince (1996) consider only membership and non-membership degrees. Szmidt and Kacprzyk (2000) presented the geometrical interpretation of intuitionistic fuzzy distance measure and introduced some distance measures by considering all three parameters, i.e., membership, non-membership and hesitation degrees. Wang and Xin (2005) firstly analyzed the drawback of Szmidt and Kacprzyk's distance measures and then presented a new definition of intuitionistic fuzzy distance measure. Later, Xu and Chen (2008) reviewed the existing intuitionistic fuzzy distance measures and then gave a new definition of distance measure for IFSs.

Definition 1.17 (Xu & Chen, 2008). Let $M, N \in IFSs(U)$. A real-valued function $d: IFSs(U) \times IFSs(U) \to [0, 1]$ is said to be a distance measure for IFSs if it satisfies the following axioms:

(d1) $0 \le d(M,N) \le 1,$

(d2) $d(M,N) = 0$ iff $M = N,$

(d3) $d(M,N) = d(N,M),$

(d4) If $M \subseteq N \subseteq O,$ then $d(M,O) \ge d(M,N)$ and $d(M,O) \ge d(N,O), \forall O \in IFS(U).$

1.5. Multiple Criteria Decision Making (MCDM)

Multiple criteria decision making (MCDM) is the process that combines alternative's performance across several contradicting, qualitative and/or quantitative criteria and determines the best feasible solution. For instance, in a personal context, the job one desires may depend upon its status, locality, remuneration, growth opportunities, working environments etc. The laptop one purchases may be characterized in terms of storage, price, style, comfort, battery and so forth. Few terms such as objectives, goals, alternatives, criteria, decision makers (DMs) and weights are frequently used in the MCDM process. Based on existing studies, the meanings of these terms are presented as follows:

Objective: An objective is something to be pursued to its fullest. For example, a car manufacturer may want to maximize gas mileage or minimize production cost or minimize its level of air pollution. An objective generally indicates the direction of change desired.

Goals: Goals are levels of aspiration. Often the goals are referred as constraints because they are designed to limit and restrict the alternative set. For example, the standard gas mileage, say

20 miles/gallon, set up by the federal government for 1980 models, is a constraint, whereas 30 miles/gallon may serve as a goal for the car manufacturer.

Alternatives: In MCDM, alternatives are "different possible courses of action". Each alternative can be characterized by a number of criteria, i.e., gas mileage, purchasing cost, horsepower, etc of a car.

Criterion: A criterion is a measurable characteristic of an alternative. Performance parameters, components, factors, characteristics, attributes and properties are synonyms for criteria. A criterion should provide a means of evaluating the levels of an objective.

Decision makers: The person or group of individuals who is responsible for making strategically important decisions based on a number of given evaluation criteria.

Weight: This term represents the relative importance of criteria and DMs.

1.5.1. Basic Concepts and Approaches for MCDM

Linguistic Variable: A linguistic variable is a variable the values of which are linguistic terms. Linguistic terms have been found intuitively easy to use in expressing the subjectiveness and/or qualitative imprecision of a DM's assessments.

Decision Matrix: A decision matrix consists of rows and columns that allow the evaluation of alternatives relative to various decision criteria. A MCDM problem consisting of 's' alternatives $\{P_1, P_2, ..., P_s\}$ and 't' criteria $\{Q_1, Q_2, ..., Q_t\}$ can be represented using a decision matrix $\Omega = (\theta_{ij})_{s\times t}$, shown in Eq. (1.8). Here, θ_{ij} determines the performance of i^{th} alternative P_i with respect to j^{th} criterion Q_j.

$$\Omega=\left(\theta_{ij}\right)_{s\times t}=\begin{array}{c} \\ P_1 \\ P_2 \\ \vdots \\ P_s \end{array}\begin{array}{c} \begin{array}{cccc} Q_1 & Q_2 & \cdots & Q_t \end{array} \\ \begin{bmatrix} \theta_{11} & \theta_{12} & \cdots & \theta_{1t} \\ \theta_{21} & \theta_{22} & \cdots & \theta_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{s1} & \theta_{s1} & \cdots & \theta_{st} \end{bmatrix} \end{array}. \tag{1.8}$$

Optimal Solution: An optimal solution refers to the best solution for a MCDM problem. In other words, it is defined as a solution which satisfies all the constraints with maximum or minimum objective function value.

Positive Ideal Solution (PIS): PIS is the solution that maximizes the benefit criteria and minimizes the cost criteria. It is also defined as a hypothetical solution for which all the criteria values correspond to the maximum criteria values in the database comprising the satisfying solutions.

Negative Ideal Solution (NIS): NIS is the solution that maximizes the cost criteria and minimizes the benefit criteria. It is also defined as a hypothetical solution for which all the criteria values correspond to the minimum criteria values in the database.

Efficient solution: This solution is named differently by different disciplines: non-dominated solution, non-inferior solution and efficient solution in MCDM, a set of admissible alternatives in statistical decision theory and Pareto-optimal solution in economics. A feasible solution in MCDM is non-dominated if there exists no other feasible solution that will yield an improvement in one objective/attribute without causing a degradation in at least one other objective/attribute.

Several approaches have been introduced to solve the MCDM problems, such as TOPSIS (Technique for Order Preference by Similarity to Ideal Solution), VIKOR (Vlsekriterijumska Optimizacija I Kompromisno Resenje), COPRAS (Complex Proportional Assessment), CoCoSo (Combined Compromise Solution), MARCOS (Measurement Alternatives and Ranking according to Compromise Solution) etc.

TOPSIS: The TOPSIS approach is based on the concept that the chosen alternative should have the shortest distance from the positive-ideal solution and the longest distance from the negative-ideal solution.

VIKOR: The VIKOR approach is a compromise ranking approach for MCDM problems. It determines a compromise solution, providing a maximum utility for the majority and a minimum regret for the opponent.

COPRAS: The COPRAS method considers the ratios to the ideal solution and the worst solution simultaneously.

CoCoSo: The CoCoSo approach is based on an integrated simple additive weighting and exponentially weighted product model. It gives a compendium of combined solution based on compromise attitudes and aggregation strategies

MARCOS: The MARCOS approach is based on the measurement of alternatives and their ranking in relation to a compromise solution. The compromise solution includes determination of utility functions according to the distance from PIS and NIS, and their aggregations.

1.6. Organizations of the Thesis

This thesis is divided into seven chapters which present entropy and divergence measures with novel multi-criteria decision making methods and their applications in various fields.

In Chapter 1, the background of the proposed work, the fundamental concepts of intuitionistic fuzzy set, Pythagorean fuzzy set, interval-valued Pythagorean fuzzy set, information measures and multiple criteria decision making are presented.

In Chapter 2, the comprehensive literature related to the digital transformation in the financial sectors, interval-valued Pythagorean fuzzy set, the multi-objective optimization based on ratio analysis with the full multiplicative form (MULTIMOORA) method is presented.

In Chapter 3, a novel entropy measure is proposed to describe the degree of uncertainty of an interval-valued Pythagorean fuzzy set. Comparison with existing entropy measures are presented to show the effectiveness of the proposed entropy measure in the context of interval-valued Pythagorean fuzzy set. Further, new divergence measure is proposed to quantify the degree of discrimination between interval-valued Pythagorean fuzzy sets. Several properties of the proposed divergence measure are discussed in detail. Comparative study is presented to overcome the limitations of the existing interval-valued Pythagorean fuzzy divergence measures.

In Chapter 4, a hybrid decision support system is proposed based on the combination of interval-valued Pythagorean fuzzy entropy measure, the ranking sum model and the MULTIMOORA methods. The combination of entropy measure-based procedure and the ranking sum model is used to take the benefits of both the objective and subjective weight of criteria under interval-valued Pythagorean fuzzy environment, while the collective MULTIMOORA method is presented to evaluate and rank the alternatives from multiple criteria and interval-valued Pythagorean fuzzy information perspective.

In Chapter 5, the key challenges are identified based on the questionnaire and literature review, which may affect the digital transformation in the sustainable financial service systems. Further, the IVPF-Entropy-RS-MULTIMOORA method is implemented to evaluate the key challenges of the digital transformation in the sustainable financial service systems.

In Chapter 6, sensitivity analysis is discussed to prove the stability of the obtained results by IVPF-Entropy-RS-MULTIMOORA method. In addition, comparison with existing MCDM methods is presented to illustrate the robustness of the proposed methods under the context of IVPFSs.

In Chapter 7, the concluding remarks of the proposed work and future scope are presented.

CHAPTER 2

LITERATURE REVIEW

2.1. The Financial Services Industry and Sustainable Development

It is not easy to achieve commercial success within the current extremely competitive setting of the financial services sector (Asif and Sargeant, 2000). The last bastion of competitive advantage is effectively meeting the consumers' requirements. A growing realization indicates that the most important point in the development of sustainable competitive advantage is becoming customer-driven (Bennett, 1992). Those companies that succeed in the provision of quality services for their customers are considered the companies with a great opportunity for gaining success in such a competitive market. In modern societies, banks and insurance firms typically provide financial services (Dehnert, 2020). Both above-noted sectors share a number of functional similarities, which involves the risk transformation function. To start with, banks resolve various risk proclivities of debtors and investors in the credit and investment function, although the insurance business model involves identifying and calculating risk and balancing in underwriting procedures. Next, due to the maturity transformation function, banks are capable of reconciling various maturity interests of creditors/debtors, while specific insurance firms carry out savings and deposit businesses as well, e.g., life insurers. Finally, the customer service function distributes complex financial products through customer advisory services.

Within the financial sector, the first activities are mostly focused on managing the internal environment (Jeucken and Bouma, 1999). The undertakings to decrease the direct environmental effects lead to a number of positive public reputations (Babiak and Trendafilova, 2011); although the most important connection between sustainable development and the financial division is indirect; it is through lending or investing in insurance, or project finance (Scholtens, 2008). As a result, as a subsequent step, this sector gets focused on the management of environmental risks in their business through the integration of the assessment of sustainability risks into their credit risk management procedures (Evangelinos and Nikolaou, 2009).

Environmental risks and regulations considerably affect the risk of a credit portfolio of a bank; thus, they need to be perfectly managed (Weber et al., 2008). This is also the case for insurance since this sector also requires managing environmental risks. After the advent of "socially responsible investment (SRI)" products and services, financial institutions started to affect sustainable development (SD) over their core business (Cerin and Scholtens, 2011). While on the one hand, environmental insurance is necessary (Weber et al., 2014); on the other hand, the market for environmental liability insurance is comparatively risky since environmental regulations suffer typically from uncertainty (Abraham, 1988). As it is widely acknowledged, in comparison with banks, insurance is more susceptible to environmental risks, although the influence of the insurance sector on SD is still rather indirect. It delivers insurance for its clients' projects; however, banks directly affect SD (Scholtens, 2011). Insurance has a limited opportunity to affect its clients' sustainability performance.

Recently, digital transformation has attracted much attention from scholars working in different domains such as marketing, business, management, ITs, and ISs. The growth in “information and communication technologies (ICTs)” have greatly affected companies and organizations. The changes that occurred to conventional business ecosystems have led to the creation of novel business environments termed generally as ‘digital business ecosystems’. Changes that occurred to business ecosystems have influenced the strategic decisions of organizations, which are generally made in relation to both external and internal environments. The frequency and size of such changes have caused the concept of change to be of higher meaningfulness (Emre and Ayberk, 2020). The quick growth of technology and various changes to current global markets have magnified a novel cooperative adaptation process. Such DT and the use of innovative technologies have raised many questions in regard to the changes that conventional firms, management practices, and strategies required to use to give suitable responses to them (Hess et al., 2016). Such response includes the formation of novel business models and/or enhancements to the currently-used business models using digital technologies (Stjepi et al., 2020).

Financial service providers must focus on several daily challenges during their business activities. In recent years, several studies have been carried out with a focus on digital transformation and the subsequent risks from both practical and theoretical perspectives (Bohnert et al., 2019). There is a robust connection between the respective firms and their customers on the basis of trust and long-term connections; this is the case for both banking and insurance sectors (Boot, 2000; Csiszar and Heidrich, 2006). Moreover, the financial market has encountered fundamental innovations mostly due to the quick enhancements in technical possibilities and delivery networks (Bömer and Maxin, 2018). Several innovative technologies such as “big data analytics (BDA)”, Robo-advisory, and “artificial intelligence (AI)” provide novel opportunities for firms to give support to their current business processes and customers. Such technologies can create/enhance novel financial services to meet the changes occurring in consumers’ needs (Keller et al., 2018). However, digital advances have fostered sales of financial services via digital channels (for instance, online platforms); they have provided challenges and opportunities to both firms and consumers. Nowadays, new competitors are moving continuously into the market. Through innovative business models, InsurTechs, FinTechs, or global Internet giants such as Facebook, Google, and Amazon (also termed BigTechs) have threatened incumbent firms (Arner et al., 2016), which show different reactions towards such new competitors.

In comparison to other industrial sectors, the financial sector is less exposed to stakeholders’ pressures (Qian et al., 2020) or the regulations about community, environment, or labor issues (Ertugrul and Hegde, 2009). However, numerous companies in the financial sector are on the radar of nongovernment institutions over financing projects or borrowers with businesses that can damage SD and the natural environment (Noor and Syumanda, 2006).

The financial sector primarily consists of insurance, banking, and asset management; banking and asset management are typically offered jointly. Various conventional products offered by insurances and banks have complementary and substitutive characteristics (Liu and Lee, 2019). In spite of the obvious converging impacts between insurances and banks, Beltratti and Corvino (2008) highlighted the fundamental differences that exist between the two business models. These differences are mainly connected to demography (for instance, sales channels, regulations, and accounting), the scale of operations, and the liability structure. Such differences form the relevance and speed of their digital transformations in the financial sector.

Currently, the literature consists of a few studies systematically analyzing the impacts on the processes and the influence of leveraging digital technologies in this sector. The study of (Cziesla 2014) is one of the few integrative studies on this issue, which performed a meta-analysis of the extant literature that is relevant to digital transformation.

The financial market exerts a governing impact upon the whole economy, society, and sustainable development through its activities (Choi and Wang, 2009). The financial market plays an intermediate role in distributing capital into various sectors, markets, projects, and regions; this market might prefer to be concentrated on financial derivatives. The consequence of the robust impacts of the financial industry upon society and the economy was observed in the last financial crisis. During this crisis (at least one of the key reasons), academic studies were focused on the influences of the financial sector on societies and sustainability practices. In this context, 'sustainability' is generally described as a development that satisfies the present generation's requirements without compromising the future generations' requirements (Brundtland, 1987). Sustainability is characterized not only by this intergenerational aspect but also by intra-generational equity between north and south (Barkemeyer et al., 2014) and by taking into account both societal and environmental aspects of development (Vifell and Soneryd, 2012).

2.2. Digital Transformation in the Sustainable Financial Services Sector

The sector of financial services, as mentioned earlier, comprises different main components, i.e., banking asset management. DT greatly impacts financial services since digital technologies have changed businesses in three dimensions: value creation, value proposition, and customer interaction (Pousttchi and Gleiss, 2019). The "value creation model (VCM)" shows how DT affects the way the products/services in the financial services sector are created (Pousttchi et al., 2019). It necessitates some essential processes for the execution of various business functions, e.g., maturity, risk, or information transformation. Scientific studies have classified the empirical elements of the business models influenced by DT from a variety of perspectives, such as customer orientation (Gimpel et al., 2018).

The all-inclusive concept of DT encompasses technologies as well as organizational and strategic changes (Diener and Špaček, 2021). Such transformation is due to the growth of technology, the advent of innovative business models, and alterations to customers' expectations (Omar, 2016). Nowadays, a number of definitions of 'digitalization' are generally acknowledged. This term is a complex issue encompassing a number of areas such as shifts in thinking, alterations to leadership, technology adoption, resources digitalization, and innovation acceptance (Francis et al., 2018). Note that there is a difference between 'digitalization' and 'digitization'. While the former addresses the influences of digital technologies on the organization, the latter signifies the shift from analogous solutions to digital solutions. In fact, digitalization refers to organizational revitalization by means of ICT (Hensmans, 2020).

A set of barriers seems to block the DT, hindering or destroying the whole process. Nevertheless, DT can resolve the challenges and problems the banks presently face. The principal practices of DT, e.g., digital trends, leadership, digital strategies, DT skills, "digital technologies (DTLs) adoption, and a "customer-centric approach (CCA), are the influences that are noted to endure on digital maturity levels (Lotriet and Kokotwane Dltshego, 2020). The term "digital transformation (DT)" is often misunderstood as simply deploying state-of-the-art ICT. Technological investment entails some risks and requires understanding the relationships between technological and organizational culture and institutional modifications within some definite boundaries of regulatory procedures. DT should not be understood as a definite, simple, and/or predictable process. Rather, this process could be disruptive or transformative, with irreversible effects on associated organizational outcomes in relation to technical capabilities and behaviors (Krasonikolakis et al., 2020).

Digitalization has made important contributions to the sustainable development objectives of the United Nations. Only the transformation of prevailing businesses could help to solve the economic and environmental challenges of the future in a sustainable way (Bican and Brem, 2020). Digital transformation generates novel social groups—partly human, semi-human, or non-human. Some of these groups already exist, and some could be predicted to exist soon because of the latest advances in fields such as software engineering, brain wearables, and robotics. The growth of our dependency on digital services and tools could bring about a number of challenges to both organizations and human beings (Fekete and Rhyner, 2020). According to Forcadell et al. (2020), digitalization may cause some challenges that can hamper its potential benefits and compromise its survival. This is why corporate sustainability plays a noteworthy role in putting digitalization into effect. This can compensate for the disadvantages of digitalization. Particularly, integrating digitalization with corporate sustainability could help to transform the organizational nature of banks by simultaneously narrowing their boundaries and expanding their scope. El Hilali et al. (2020) called for further attention to imaginable ways to achieve sustainability in the course of DT. Based on their findings, firms can achieve sustainability by effectively considering the customers, data processing, and innovation. However, they did not prove the considerable role of competition in the enhancement of the firms' commitment to sustainability. It was also endorsed partly in the study of Ordieres-Meré et al. (2020), where they showed the positive impacts of knowledge creation that could be facilitated by directly or indirectly applying digitalization. Technology has been found a factor capable of disrupting the financial industry, solving friction points for customers and businesses, and injecting more resilience and sustainability into the overall business. In addition, sustainable financial technology could have a great contribution to the stability of the financial system (Moro-Visconti et al., 2020).

Climate change has many economic implications that could result in a great transformation in financial services (Bopp, 2020). Indeed, this phenomenon is at the top of the business agenda among numerous issues that are critical to the financial industry. By considering the association between the financial sector and "sustainable development (SD)", at least three significant aspects must be taken into account. First, the financial sector can affect the environmental and sustainability aspects of the clients, e.g., projects, debtors, and investees (Thompson and Cowton, 2004; Weber, 2014), which is considered the indirect influence of this sector on sustainable development. Evidently, the indirect influences of finance are of high importance as access to capital is generally a key premise for being successful in business. Then, in a lot of ways, regulations related to environmental issues have influenced and still affect the financial sector (Weber et al., 2010). For example, environmental regulations in regard to the pollution of water, soil, and air have affected the ways environmental risks were managed in credit risk management during the 1990s (Boyer and Laffont, 1997). The numerous opportunities and risks associated with sustainability (for instance, alleviation of poverty and climate change) have increased and are still arising, which need to be addressed well by the financial sector (Richardson, 2009). On the other hand, it should be noted that the financial sector has often been reactive rather than proactive regarding such sustainability challenges.

Third, the pressures exerted by stakeholders with a focus on SD affect the reputational risks of financial institutions (Evangelinos and Nikolaou, 2009) and their financial performance (Scholtens and Zhou, 2008).

The field of "information systems (ISs)" has created the analytical/theoretical frameworks addressing the transformational impacts that may be exerted due to diffusing, integrating, and implementing the "information technologies (ITs)" on intricate business ecosystems (Werth et al., 2020). From the primary computer systems to the latest "digital technologies (DTLs)", e.g., "social, mobile, analytics, cloud and internet of things (SMACIT)" technologies (Sebastian et al., 2020), the promptness and degree of technological innovations have set the pace of "digital transformation (DT)" in services and industrial sectors. They have led to scientific discourse in the domain of business and IS research (Kutzner et al., 2018). Digital transformation was described by Vial (2019) as a process through which DTLs cause some disruptions triggering strategic responses from the companies seeking to adjust their paths for value creation and, at the same time, managing the structural changes and organizational obstacles influencing both positive and negative concerns of such transformational process. Accordingly, the disruptive impacts resulting from digital transformation could be taken into account as the direct consequence of the second-order technological disruptions induced by the aggregated effects of multiple digital innovations upon economic, social, and political norms (Schuelke-Leech, 2018). Technologically, digital innovation could appear in the form of novel digitalized products or service innovations, business or technical process enhancements, innovative digitally-driven business models, and digital business strategies that are based on the latest paradigms of value creation (Hess et al., 2020). Due to the high significance of the influence of digital transformation on companies, societies, and industries, in the past decade, it has attracted lots of attention from scholars working in different fields. Some prominent examples in this regard are sustainability (Feroz et al., 2021; Pamucar et al., 2022), and healthcare (Kraus et al., 2021), supply chain finance (Kamaci and Petchimuthu, 2022; Mahmoudi et al., 2022), strategy and organizational change (Hanelt et al., 2021; Khan et al., 2021), and IT (Papagiannidis et al., 2020).

Various researchers in the management domain have investigated the challenges the industry incumbents face when innovating their businesses (Eklund and Kapoor, 2019). In particular, the industry of financial services is currently experiencing a fundamental transformation. The formerly-stable market reveals extraordinary competitive dynamics, regulatory changes, and non-/near- banks as asymmetric competitors in the digital technologies era. Practitioners mention a disruptive change that is able to lower the significance of conventional financial service providers. This has been recently exemplified by (Dehnert, 2020), who focused on future industry perspectives in regard to profitability measures. The study of (Chiorazzo et al., 2018) was concentrated on the digital transformation of incumbent financial service providers that had three criteria: high market power, revenue streams from conventional services, and physical branches (Chiorazzo et al., 2018). Insurance firms and incumbent banks have significant roles in the sustainable development of society. Such importance is due to the fact that there are numerous significant economic functions, e.g., the promotion of saving and wealth formation and the credit supply to the economy.

Financial service providers are traditionally concerned about B2C retail businesses with four main product types, i.e., payment, financing, investment, and insurance (Alt and Reinhold, 2012), with two key subsectors, i.e., banking and insurance. The former involves transferring, accumulating, increasing savings, and providing capital, while the latter primarily implies transferring and managing the risks. From the traditional perspective, financial services are the least interesting products, making it difficult to differentiate among them. On the other hand, digitalization makes customer orientation a key aspect of the competition (Bons et al., 2012). New digitally-enabled competitors position themselves with various digital products that are standardized and easy to handle. With a decrease in switching costs, consumers will be able to select from amongst the offers given by both conventional and new financial service providers for their accounts, loans, payments, investments, mortgages, or insurance products, which questions their former robust, trust-based relationship with their financial service providers (Pousttchi and Dehnert, 2018).

The ICT has enabled the "digital transformation (DT)" of financial services as well as the development of Fintech, which have resulted in not only the enhancement of the resources available in service systems (for instance, resource "density") but also their transferability. This has led to novel technology-empowered value co-creation processes (Breidbach and Maglio, 2016). Owing to the recent developments in technology, the prevalent effects of ICT on financial services are no more surprising, and the investigation of the ICT implications in services represents more broadly a key service research priority (Ostrom et al., 2015). On the other hand, the current contributions in the service research discipline have been questioned due to the fact that they could not offer deep insight into the emergent digital service innovations (Lusch and Nambisan, 2015) in general, and also they could not offer a deep understanding about Fintech as an evolving service context, in particular (Breidbach and Ranjan, 2017). For example, Bharadwaj (2000) stated that companies with high levels of IT capabilities could perform better than the control sample of companies regarding various profit- and cost-based performance measures. For example, Aral and Weill (2007) maintained that the total IT investment of a company is not associated with PERF; though, investments in definite IT assets consistent with their strategic purposes are able to explain the performance differences. It should be noted that the findings of more recently-conducted studies have suggested that digital transformation mediates the relationships between PERF and IT investments (Nwankpa and Roumani, 2016). Particularly, findings of some studies showed that DT largely and positively affects different factors such as FSP PERF in the long term (Scott et al., 2017), organizational agility (Ravichandran, 2018), and productivity (Bertoni and Croce, 2011). In another study, DeYoung et al. (2007) showed that DT has a positive association with community bank profitability. A recurrent finding indicates that specific configurations are vital to PERF (Ray et al., 2005). Such a gap in the knowledge is partially due to the fact that ICT and service research have been carried out conventionally in disciplinary silos (Brust et al., 2017), which is further amplified due to the high speed of the emergence of innovative innovation, disruptive technologies (Christensen, 2006).

2.3. Interval-Valued Pythagorean Fuzzy Set

Due to the lack of complete knowledge, subjectivity of human mind and vague phenomenon, the notion of fuzzy set (FS) (Zadeh, 1965) has successfully been applied for various purposes in different areas and proven its effectiveness for dealing with imprecise and fuzzy information. The theory of FS is characterized by the membership degree, which takes all the values between 0 and 1, and the non-membership degree is defined as one minus the membership degree. However, it may not always be true that the degree of non-membership of an element is equal to one minus the membership degree because there may be some hesitation degree. To overcome the limitation of FS, Atanassov (1986) introduced the concept of intuitionistic fuzzy set (IFS), which is characterized by the degrees of member-ship, non-membership and hesitancy. In IFS, the degree of hesitancy is defined as one minus the sum of membership and non-membership degrees. The theory of IFS is more powerful as compared to FS as it deals with membership, non-membership, and hesitant degrees. In IFS, an element is expressed by the degrees of membership and non-membership with their sum is bounded to unity. In the literature, several real-life applications have been presented in the context of IFSs (Rani et al., 2019; Mishra et al., 2020; Pandey et al., 2023; Tripathi et al., 2023; Alrasheedi et al., 2023).

However, in many situations, the sum of degrees of membership and non-membership may be greater than one, while their quadratic sum is less than or equal to 1. To treat these situations, Yager (2014) presented the idea of Pythagorean fuzzy set (PFS), in which each element is expressed by the membership and non-membership degrees with their quadratic sum is less than or equal to 1. For example, there may be a situation where the experts may give the degrees to which an option U_i satisfies the criterion T_j is 0.6 and dissatisfies the criterion is 0.5. Here, it can be easily seen that $0.6+0.5>1$, therefore, IFS fails to express such types of situations, while it can be effectively handled by the PFS theory. Accordingly, the PFS theory has proven as a more suitable and flexible tool to manage the uncertainty of MCDM problems. For instance, Biswas and Deb (2021) presented a combined MCDM method based on the Pythagorean fuzzy aggregation operators using the power operations and Schweizer and Sklar t-norm and t-conorm and applied to develop a MCGDM approach under PFS environment. Li et al. (2022) presented a comprehensive evaluation model and evaluation method based on the Pythagorean fuzzy Technique for Order Preference by Similarity to Ideal Solution (PF-TOPSIS) method for evaluating the dispatching results of power system with high penetration of renewable energy. Although, Liu et al. (2022) studied a hybrid Pythagorean fuzzy evaluation based on distance from the average solution (PF-EDAS) method and its application in the assessment and prioritization of sustainable circular suppliers. To measure the degree of distance between PFSs, Mahanta and Panda (2021) suggested a new Pythagorean fuzzy distance measure. Further, their application has been presented in the context of decision making, pattern recognition and medical diagnosis, where the information is presented in terms of PFNs. Chaurasiya and Jain (2022) proposed an integrated Pythagorean fuzzy entropy measure based-complex proportional assessment has developed to evaluate the multi-criteria healthcare waste treatment methods. Hajiaghaei-Keshteli et al. (2023) developed a hybrid Pythagorean fuzzy TOPSIS method and its application in green suppliers evaluation. Based on the ordered weighted averaging and the probabilistic ordered weighted averaging operators, Verma and Mittal (2023) presented some Pythagorean fuzzy cosine similarity aggregation operators. Chaurasiya and Jain (2023) proposed a hybrid complex proportional assessment framework using Pythagorean fuzzy numbers and presented its application in the identification of best management software from multiple criteria perspective. Habib et al. (2023) presented a new Pythagorean fuzzy cognitive map and its application in the treatment of pregnant women with heart disease. Mishra et al. (2023) extended the intuitionistic fuzzy similarity measure from Pythagorean fuzzy perspectives and presented a hybrid additive ratio assessment framework to deal with sustainable biomass crop selection problem with Pythagorean fuzzy information.

In the PFS theory, the degrees of membership and non-membership are exact numbers, which is hard for the experts to define their exact value in several decision-making problems. To conquer this issue, Peng and Yang (2016) pioneered the idea of interval-valued Pythagorean fuzzy set (IVIFS), which deals with uncertainty in practical decision-making problems. Its basic feature is that both membership and non-membership functions of an element to a given set are considered and taken as interval values rather than exact numbers. As an extended version of PFS, the IVPFS theory provides a more effective and reasonable way to cope with imprecise and uncertain information. Due to its higher flexibility in dealing with uncertain data, the IVPFS doctrine has been broadly explored from different perspectives. Fu et al. (2020) proposed an innovative product ranking method that incorporates the feature–opinion pairs mining and interval-valued Pythagorean fuzzy information. Additionally, the authors have presented an IVPF Heronian mean operator to consider the interrelationship between product attributes. With the use of IVPFSs, Mohagheghi et al. (2020) proposed a hybrid multi-objective model to evaluate and prioritize the high technology project portfolios by considering the multiple experts and criteria. By means of IVPFSs, Ayyildiz and Gumus (2021) proposed the supply chain operations reference model with new metrics related to Industry 4.0. For this purpose, they extended the Analytic Hierarchy Process under IVPF environment and applied to evaluate the supply chain performance. Tang and Yang (2021) firstly presented the concept of IVPF fuzzy preference relation and further proposed a novel IVPF multi-attribute group decision-making model for assessing and ranking the sustainable e-bike sharing recycling suppliers in the context of IVPFSs. Yin et al. (2022) proposed a multi-objective programming method for the assessment of rail transit photovoltaic power station sites with multiple aspects of sustainability. Mishra et al. (2022) proposed a similarity measure to compute the similarity between IVPFSs. Moreover, they introduced an integrated IVPF similarity measure-based complex proportional assessment framework and its application in the assessment of waste-to-energy technologies. Al-Barakati et al. (2022) integrated the concept of similarity between

IVPFSs, the score function and the weighted aggregated sum product assessment method with IVPF information, and applied to evaluate the renewable energy resource selection problem. Senapati et al. (2022) defined the Aczel-Alsina operations on interval-valued Pythagorean fuzzy numbers (IVPFNs) and then developed a series of Aczel-Alsina aggregation operators for IVPFNs. Further, they proposed an algorithm to solve the multi-criteria emerging information technology software selection problem under the context of IVPFSs. Ren et al. (2023) integrated the power average and Maclaurin symmetric mean operators to construct new IVPF power Maclaurin symmetric mean operators. Using these operators, the authors have proposed a soft computing multi-criteria group decision-making method based on TOPSIS approach and IVPF information. They presented the significance of their proposed approach and the exploration of feature extraction methods. Mishra et al. (2023) proposed fairly aggregation operators to combine the individuals' decision information. Moreover, the authors have proposed a hybrid multi-attribute ideal-real comparative analysis (MAIRCA) method based on the combination of entropy and and pivot pairwise relative criteria importance assessment (PIPRECIA) model and applied to evaluate the blockchain platforms for healthcare supply chain management. To consider the interrelationships between arguments, Rani et al. (2023a) proposed some interaction aggregation operators for IVPFNs with their characteristics. To compare the IVPFNs, they proposed a score function and presented their properties. Furthermore, the authors have introduced a hybrid operational competitiveness rating approach for evaluating and prioritizing the metaverse integration alternatives of sharing economy in transportation sector. Rani et al. (2023b) proposed a collective weighted integrated sum product approach based on the Dombi aggregation operators, entropy and PIPRECIA model with IVPF information. They applied their method for industry 4.0 technology assessment and digital transformation. To promote the sustainable development of offshore wind-solar-seawater pumped storage power project, Mao et al. (2023) proposed a hybridized MCDM approach, and extended it to the IVPF environment with its application in investment project selection.

2.4. IVPF-Entropy and Divergence Measures

Atanassov (1986) extended the idea of fuzzy set to "intuitionistic fuzzy set (IFS)", which is characterized by the "membership degree (MD)" and "non-membership degree (NMD)" with their sum is less than or equal to one. However, in many situations, the sum of MD and NMD may be greater than one, while their quadratic sum is less than or equal to one. To deal with these situations, Yager (2013) gave the theory of "Pythagorean fuzzy sets (PFSs)", which is also characterized by the MD and NMD with their quadratic sum is restricted to unity. Thus, the PFS theory has been proven as a more effective and flexible tool to deal with MCDM problems with uncertain information (Yager, 2014). In order to overcome the drawbacks of PFS theory, Peng and Yang (2016) extended the concept of PFS to "interval-valued Pythagorean fuzzy set (IVPFS)". In IVPFS, both the MD and the NMD of an element are taken and presented in terms of interval values instead of real numbers. Many theories and applications have been developed under the context of IVPFSs (Mohagheghi et al., 2020; Fu et al., 2020; Ibrahim et al., 2023; Mishra et al., 2023; Luo et al., 2023; Ren et al., 2023).

To measure the uncertain information in decision making problems, De Luca and Termini (1972) introduced the axiomatic definition of an entropy measure for fuzzy sets. Thereafter, a lot of work has been done on the fuzzy entropy measure in different fields (Mishra et al., 2014; Hooda and Mishra, 2015; Mishra et al., 2016a,b). The concept of entropy measure for IFSs has been defined by Burillo and Bustince (1996). After that, Szmidt and Kacprzyk (2001) proposed an entropy measure for IFSs which is the generalization of De Luca and Termini's (1972) fuzzy entropy. Hung and Yang (2006) introduced the axiomatic ideas of entropy measure for IFSs by using the theory of probability. For PFSs, Peng et al. (2017) presented different measures (similarity, inclusion, entropy and distance measures) for PFSs and discussed their corresponding relationships. Yang and Hussain (2018) introduced PF-entropy based on probabilistic-type, distance, and min-max operators. Xue et al. (2018) defined a new entropy measure for PFSs by considering the similarity part and hesitant part, which show the fuzziness and uncertainty, respectively. Thao and Smarandache (2019) extended the concept of fuzzy entropy for PFSs and presented various numerical illustrations to evaluate the applicability of their developed measure. In addition, they have also applied to propose a COPRAS method under Pythagorean fuzzy environment. Further, Rani, et al. (2019) defined novel entropy and implemented for the determination of criteria's weights in the VIKOR technique. Rani, et al. (2020) introduced PF-entropy and divergence measures and further, utilized to find criteria weights in the Pythagorean fuzzy WASPAS approach. In context of IVPFSs, only few authors have developed the concept of entropy measure (Peng and Li, 2019; Mishra et al., 2023).

2.5. MULTMOORA Method

An MCDM, one of the important branches of decision science, is a systematic way to find the best possible solution according to the given set of criteria. In today's life, the practical decision making problems are more and trickier to anticipate owing to the intricacy and uncertainty of the real-world problems. In recent years, lots of methods have been commenced for solving real MCDM applications, wherein each of them has their own advantages and restrictions. In

2006, Brauers & Zavadskas (2006) commenced the MOORA (Multi-Objective Optimization based on the Ratio Analysis) model, which combines the ratio system (RS) and reference point (RP) model. Further, Brauers & Zavadskas (2010) enhanced the robustness of MOORA technique by adding the full multiplicative form (FMF), and the extended technique was called MULTIMOORA model. Compared with AHP (Analytic Hierarchy Process), TOPSIS (Technique for Order of Preference by Similarity to Ideal Solution) (Chen & Hwang, 1992), VIKOR (VIseKriterijumska Optimizacija I Kompromisno Resenje in Serbian) (Opricovic, 1998) and TODIM (TOmada de Decisão Interativa e Multicritério), the MULTIMOORA method has more superiority, easy mathematical expressions, less computation time, good stability and strong robustness (Brauers & Zavadskas, 2012). Over the past few years, the MULTIMOORA method has been expanded under diverse fuzzy settings. For instance, Li (2014) proposed a novel hesitant fuzzy MULTIMOORA approach for assessing MCDM problems with hesitant fuzzy sets (HFSs). Hafezalkotob et al. (2016) studied new generalization of MULTIMOORA technique using interval numbers and further, implemented on a material selection of power gears. In a study, Stanujkic et al. (2017) gave a neutrosophic MULTIMOORA approach and proved its suitability through a realistic example. Hafezalkotob et al. (2019) presented all-inclusive analysis of MULTIMOORA framework based on theoretical and practical aspects. Xian et al. (2020) introduced a novel MULTIMOORA technique with interval 2-tuple Pythagorean fuzzy linguistic set and employed for solving group decision making applications. Wang et al. (2020) set up a collective decision support system with the combination of triangular fuzzy entropy with MULTIMOORA approach, and applied for assessing the sustainable battery suppliers. Liu et al. (2021) introduced a revised version of MULTIMOORA approach for evaluating sustainable suppliers with intuitionistic linguistic rough numbers. Recently, Sarabi & Darestani (2021) designed a hybrid fuzzy decision support system by integrating MULTIMOORA and BWM (Best Worst Method) for logistic provider selection with fuzzy information. Rani and Mishra (2021) gave Einstein aggregation operators-based MULTIMOORA method for electric vehicle charging station selection. Baidya et al. (2021) developed BCF-Archimedean power weighted opearoter-based CRITIC-MULTIMOORA method to solve third-party reverse logistics providers' (3PRLP) selection problem. Saraji et al. (2021) presented an integrated hesitant-SWARA-MULTIMOORA method to the analysis and assesses the challenges to adapt the online education during the COVID-19 outbreak. He et al. (2021) proposed an integrated decision-making method using SWARA and MULTIMOORA under Interval-Valued Pythagorean Fuzzy Sets (IVPFSs) to examine the current status of sustainable community-based tourism (CBT) in the Indian Himalayan region context.

Multi-criteria decision making (MCDM) can be categorized into the outranking approaches and the utility value-based approaches. The outranking approaches are inadequate in dealing with large number of criteria and alternatives due to complex calculations. The utility-based approaches only use a single normalization process to non-dimensionalize assessment degrees over diverse criteria, which would bias the outcomes because of the fault normalized values for aggregation. To overcome these limitations, Liao and Wu (2020) proposed the idea of DNMA method for solving MCDM problems with conflicting criteria. As a utility-based approach, the DNMA method takes the merits of different normalization procedures and aggregation functions which are integrated in a suitable way. It considers both the utility values and the subordinate ranks of alternatives to determine the collective values of alternatives by a weighted Euclidean distance formula. Wu and Liao (2019) presented the main advantages of classical DNMA method by comparing it with some of the existing utility-based methods from different points of view. Liao et al. (2019) proposed a novel extension of DNMA method from hesitant fuzzy set perspective. In addition, the proposed hesitant fuzzy information-based DNMA method has implemented to deal with the lung cancer screening problem. Nie et al. (2019) developed a novel multi-expert MCDM methodology based on the DNMA method with a cardinal consensus reaching process, in which the assessment ratings of alternatives by means of multiple criteria are articulated as hesitant fuzzy linguistic term sets. Lai et al. (2020) proposed an improved Z-number based DNMA method to manage the MCDM problems with conflicting criteria. Further, they applied to evaluate and prioritize the sustainable cloud service providers under Z-number environment. Zhang et al. (2020) proposed an extended version of DNMA method from Pythagorean fuzzy information perspective. They applied to evaluate the multi-criteria internet financial products assessment problem and compared with existing TOPSIS and VIKOR methods under Pythagorean fuzzy environment. Wang and Rani (2021) extended the DNMA method to the intuitionistic fuzzy environment to solve the uncertain MCDM problems. Further, the proposed method has applied to evaluate and rank the sustainable risk factors in supply chain management. Based on the full consistency method (FUCOM) weighting model, Saha et al. (2022) put forward a hybridized DNMA method with q-rung orthopair fuzzy information perspective and presented its application in the assessment of healthcare waste treatment technologies with multiple tangible and intangible criteria. Hezam et al. (2022) proposed an integrated MCDM framework based on the ranking sum model, the Method based on the Removal Effects of Criteria (MEREC) model and the DNMA method, and applied to deal with the multi-criteria alternatives fuel vehicles assessment problem. Al-Barakati and Rani (2023) proposed an innovative interval-valued intuitionistic fuzzy DNMA method which involves two target-based normalizations and three subordinate utility models. Further, they used their method for assessing the healthcare waste disposal technologies assessment problem under interval-valued intuitionistic fuzzy environment. Mishra et al. (2023) presented a novel single-valued neutrosophic DNMA method based on the Archimedean-Dombi operators, the MEREC and the Stepwise Weight Assessment Ratio Analysis (SWARA) models. They implemented their hybrid DNMA method for evaluating and prioritizing the sustainable location for the construction of lithium-ion batteries' manufacturing plant. Hezam et al. (2023) introduced an integrated Fermatean fuzzy DNMA method using the criteria Importance through inter-criteria correlation and SWARA weighting models. In addition, they evaluated their DNMA method on a case study of digital technologies assessment for disabled persons and presented its sensitivity and comparative analyses.

2.6. Research Problems for the Study

This study answers the following research questions:

- What are the key significant challenges/drivers to the assessment of digital transformation?
- What are the significant values/weights of the challenges/drivers that may affect the process of digital transformation in the sustainable financial service systems?
- Which one is the most suitable method to evaluate the key challenges of digital transformation in the sustainable financial service systems from uncertainty perspective?

2.7. Objective of the Study

The aim and objective of the thesis entitled **"Interval-valued Pythagorean Fuzzy Decision-Making Models for Evaluating Challenges of Digital Transformation"** are as follows:

- To develop new entropy and divergence measures to handle the uncertainty under interval-valued Pythagorean fuzzy environment to determine the significance/weight values DT challenges in sustainable financial service systems of the banking sector.
- To develop a hybrid decision-making models to rank and evaluate the DT challenges in sustainable financial service systems of the banking sector from interval-valued Pythagorean fuzzy perspective.
- The proposed comprehensive framework is applied to rank evaluate digital transformation challenges in sustainable financial service systems of the banking sector. Also, with the consideration of these digital transformation challenges, we determine the suitable sustainable financial service systems of the banking sector.

CHAPTER 3

INTERVAL-VALUED PYTHAGOREAN FUZZY ENTROPY AND DIVERGENCE MEASURES

3.1. Introduction

In fuzzy set theory and its extensions, the notion of divergence measure (DM) is utilized to enumerate the discrimination between two sets. Due to its advantages, the development of different DMs and their applications in different areas is always a hot topic amongst the researchers (Kadian and Kumar, 2020; Rani and Jain, 2020; Verma, 2021). Khan et al. (2022) studied several DMs and their properties under circular intuitionistic fuzzy sets. For the first time, Arthi and Mohana (2021) introduced the idea of DM for IVPFSs and presented its application in decision-making problems. Thus, the developments of entropy and divergence measures for IVPFSs are open issues for the researchers.

The rest part of this chapter is arranged as follows: Section 3.2 proposes a new entropy measure to present the uncertainty of an IVPFS. Section 3.3 discusses the comparison between introduced and existing entropy measures under IVPFS background. Section 3.4 proposes a novel divergence measure to compute the discrimination between IVPFSs and further discusses its properties. Section 3.5 presents the comparison of proposed and existing divergence measures, which proves the effectiveness of the proposed IVPF-divergence measure over the existing ones.

3.2. Proposed New IVPF-Entropy

As a relative degree of randomness, entropy has been used to quantify the degree of fuzziness of a FS or system. In other words, the entropy of fuzzy set offers the average amount of fuzziness/ambiguity present in the fuzzy set. In the process of MCDM, it is employed to estimate the criteria weights. Based on Rani and Jain (2019) entropy measure for IVIFS, we develop new entropy measure for IVPFS, which as

Theorem 3.1. For an IVPFS $E \in IVPFS(\Omega)$, where $\Omega = \{\varepsilon_1, \varepsilon_2, ..., \varepsilon_n\}$ is universe of discourse, the IVPF-entropy measure is defined as

$$\wp(E) = \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{i=1}^{n} \left(\left(\frac{\left((\mu_E^-(\varepsilon_i))^2 + (\mu_E^+(\varepsilon_i))^2\right) + 2 - \left((\nu_E^-(\varepsilon_i))^2 + (\nu_E^+(\varepsilon_i))^2\right)}{4} \right) \times e^{\left(\frac{\left((\nu_E^-(\varepsilon_i))^2 + (\nu_E^+(\varepsilon_i))^2\right) + 2 - \left((\mu_E^-(\varepsilon_i))^2 + (\mu_E^+(\varepsilon_i))^2\right)}{4} \right)} \right.$$

$$\left. + \left(\left(\frac{\left((\nu_E^-(\varepsilon_i))^2 + (\nu_E^+(\varepsilon_i))^2\right) + 2 - \left((\mu_E^-(\varepsilon_i))^2 + (\mu_E^+(\varepsilon_i))^2\right)}{4} \right) \right) \times e^{\left(\frac{\left((\mu_E^-(\varepsilon_i))^2 + (\mu_E^+(\varepsilon_i))^2\right) + 2 - \left((\nu_E^-(\varepsilon_i))^2 + (\nu_E^+(\varepsilon_i))^2\right)}{4} \right)} - 1 \right]. \tag{3.1}$$

Proof. To prove this theorem, the given Eq. (3.1) must satisfy the axioms (s1)-(s4) of Definition 1.15.

(s1). Let E be a crisp set, i.e., $\mu_E^-(\varepsilon_i) = 1 = \mu_E^+(\varepsilon_i)$, $\nu_E^-(\varepsilon_i) = 0 = \nu_E^+(\varepsilon_i)$ or $\mu_E^-(\varepsilon_i) = 0 = \mu_E^+(\varepsilon_i)$, $\nu_E^-(\varepsilon_i) = 1 = \nu_E^+(\varepsilon_i)$, $\forall \varepsilon_i \in \Omega$. Then, Eq. (1) implies that $\wp(E) = 0$.

Conversely, let $\wp(E) = 0$. Also, assume that

$$\frac{\left((\mu_E^-(\varepsilon_i))^2+(\mu_E^+(\varepsilon_i))^2\right)+2-\left((\nu_E^-(\varepsilon_i))^2+(\nu_E^+(\varepsilon_i))^2\right)}{4}=f_E(\varepsilon_i). \tag{3.2}$$

From Eq. (3.1) and Eq. (3.2) becomes

$$\wp(E)=\frac{1}{n\left(\sqrt{e}-1\right)}\sum_{i=1}^{n}\left(f_E(\varepsilon_i)\times e^{(1-f_E(\varepsilon_i))}+\left(1-f_E(\varepsilon_i)\right)\times e^{f_E(\varepsilon_i)}-1\right)=0. \tag{3.3}$$

Eq. (3.3) becomes zero if and only if either $f_E(\varepsilon_i)=0$ or $1-f_E(\varepsilon_i)=0 \Rightarrow f_E(\varepsilon_i)=1,\ \forall\ \varepsilon_i\in\Omega.$ It means that

$$\frac{\left((\mu_E^-(\varepsilon_i))^2+(\mu_E^+(\varepsilon_i))^2\right)+2-\left((\nu_E^-(\varepsilon_i))^2+(\nu_E^+(\varepsilon_i))^2\right)}{4}=0 \tag{3.4}$$

or

$$\frac{\left((\mu_E^-(\varepsilon_i))^2+(\mu_E^+(\varepsilon_i))^2\right)+2-\left((\nu_E^-(\varepsilon_i))^2+(\nu_E^+(\varepsilon_i))^2\right)}{4}=1. \tag{3.5}$$

Eq. (3.4) and Eq. (3.5) imply that $\mu_E^-\left(\varepsilon_i\right)=\mu_E^+\left(\varepsilon_i\right)=1,\ \nu_E^-\left(\varepsilon_i\right)=\nu_E^+\left(\varepsilon_i\right)=0$ or $\mu_E^-\left(\varepsilon_i\right)=\mu_E^+\left(\varepsilon_i\right)=0,\ \nu_E^-\left(\varepsilon_i\right)=\nu_E^+\left(\varepsilon_i\right)=1.$ Therefore, E is a crisp set.

(s2). Let $\mu_E^-\left(\varepsilon_i\right)=\nu_E^-\left(\varepsilon_i\right)$ and $\mu_E^+\left(\varepsilon_i\right)=\nu_E^+\left(\varepsilon_i\right)$. Then, from Eq. (1), we obtain $\wp\left(E\right)=1.$

Let us assume that $\wp\left(E\right)=\frac{1}{n}\sum_{i=1}^{n}\varphi(f_E(\varepsilon_i)),$ where

$$\varphi(f_E(\varepsilon_i))=\frac{1}{\left(\sqrt{e}-1\right)}\left(f_E(\varepsilon_i)\times e^{(1-f_E(\varepsilon_i))}+\left(1-f_E(\varepsilon_i)\right)\times e^{f_E(\varepsilon_i)}-1\right),\ \forall\,\varepsilon_i\in\Omega. \tag{3.6}$$

Let $\wp\left(E\right)=1\Rightarrow\frac{1}{n}\sum_{i=1}^{n}\varphi(f_E(\varepsilon_i))=1.$ Therefore

$$\varphi(f_E(\varepsilon_i))=1,\ \forall\ \varepsilon_i\in\Omega. \tag{3.7}$$

Differentiating (3.7) with respect to $f_E\left(\varepsilon_i\right)$, we get

$$\frac{\partial \varphi(f_E(\varepsilon_i))}{\partial f_E(\varepsilon_i)} = \frac{1}{\left(\sqrt{e}-1\right)}\left(e^{(1-f_E(\varepsilon_i))} - f_E(\varepsilon_i)\times e^{(1-f_E(\varepsilon_i))} - e^{f_E(\varepsilon_i)} + \left(1-f_E(\varepsilon_i)\right)\times e^{f_E(\varepsilon_i)}\right) = 0,\ \forall\ \varepsilon_i \in \Omega,$$

$$\Rightarrow \left(1-f_E(\varepsilon_i)\right)\times e^{(1-f_E(\varepsilon_i))} = f_E(\varepsilon_i)\times e^{f_E(\varepsilon_i)}, \forall\, \varepsilon_i \in \Omega. \tag{3.8}$$

Using the fact that $\varphi(\varepsilon) = \varepsilon\, e^{\varepsilon}$ is a bijection function, so that we can write $\left(1-f_E(\varepsilon_i)\right) = f_E(\varepsilon_i)$, therefore $f_E(\varepsilon_i) = 0.5,\ \forall\ \varepsilon_i \in \Omega.$

Now,

$$\left(\frac{\partial^2 \varphi(f_E(\varepsilon_i))}{\partial\left(f_E(\varepsilon_i)\right)^2}\right)_{f_E(\varepsilon_i)=0.5} = -2\left(e^{0.5} + 0.5\times e^{0.5}\right) < 0,\ \forall\, \varepsilon_i \in \Omega. \tag{3.9}$$

Hence, the function $f_E(\varepsilon_i)$ is a concave function and has a global maximum at $f_E(\varepsilon_i) = 0.5$. As $\wp\left(E\right) = \frac{1}{n}\sum_{i=1}^{n}\varphi(f_E(\varepsilon_i)),$ therefore $\wp\left(E\right)$ attains the maximum value at $f_E(\varepsilon_i) = 0.5$. Thus, $f_E(\varepsilon_i) = 0.5$ implies that $\frac{\left((\mu_E^-(\varepsilon_i))^2 + (\mu_E^+(\varepsilon_i))^2\right) + 2 - \left((\nu_E^-(\varepsilon_i))^2 + (\nu_E^+(\varepsilon_i))^2\right)}{4} = \frac{1}{2},\ \forall\ \varepsilon_i \in \Omega.$ It is possible if and only if $\mu_B^-\left(\varepsilon_i\right) = \nu_B^-\left(\varepsilon_i\right)$ and $\mu_B^+\left(\varepsilon_i\right) = \nu_B^+\left(\varepsilon_i\right)$.

(s3) It is obvious by Definition 1.15.

(s4) Let

$$g(x, y) = \left[\left(\frac{x+2-y}{4}\right)e^{\left(\frac{y+2-x}{4}\right)} + \left(\frac{y+2-x}{4}\right)e^{\left(\frac{x+2-y}{4}\right)} - 1\right], \tag{3.10}$$

Taking the partial derivatives of g with respect to *x* and *y*, respectively, we have

$$\frac{\partial g}{\partial x} = \frac{1}{4}\left[\left(\frac{y+2-x}{4}\right)e^{\left(\frac{y+2-x}{4}\right)} - \left(\frac{x+2-y}{4}\right)e^{\left(\frac{x+2-y}{4}\right)}\right], \tag{3.11}$$

$$\frac{\partial g}{\partial y}=\frac{1}{4}\left[\left(\frac{x+2-y}{4}\right)e^{\left(\frac{x+2-y}{4}\right)}-\left(\frac{y+2-x}{4}\right)e^{\left(\frac{y+2-x}{4}\right)}\right]. \tag{3.12}$$

In order to find critical point of g, we set $\frac{\partial g}{\partial x}=0$ and $\frac{\partial g}{\partial y}=0.$ This gives

$$x=y. \tag{3.13}$$

From (3.12) and (3.13), we obtain $\frac{\partial g}{\partial x}\geq 0,$ when $x\leq y$ and $\frac{\partial g}{\partial x}\leq 0$ when $x\geq y,$ for any $x,\, y\in[0,1].$ Thus, $g(x,y)$ is increasing with respect to x for $x\leq y$ and decreasing when $x\geq y.$ Similarly, we obtain that $\frac{\partial g}{\partial y}\leq 0,$ when $x\leq y$ and $\frac{\partial g}{\partial y}\geq 0$ when $x\geq y.$ Consider any two IVPFSs E and F with $E\subseteq F,$ i.e.,

$$\mu_E^-(\varepsilon_i)\leq\mu_F^-(\varepsilon_i)\leq\nu_F^-(\varepsilon_i)\leq\nu_E^-(\varepsilon_i),\;\; \mu_E^+(\varepsilon_i)\leq\mu_F^+(\varepsilon_i)\leq\nu_F^+(\varepsilon_i)\leq\nu_E^+(\varepsilon_i)$$

or

$$\mu_E^-(\varepsilon_i)\geq\mu_F^-(\varepsilon_i)\geq\nu_F^-(\varepsilon_i)\geq\nu_E^-(\varepsilon_i),\;\; \mu_E^+(\varepsilon_i)\geq\mu_F^+(\varepsilon_i)\geq\nu_F^+(\varepsilon_i)\geq\nu_E^+(\varepsilon_i),$$

for each $\varepsilon_i\in\Omega.$ Then from the monotonicity of $g(x,y)$ and Eq. (3.1), we obtain that $\wp(E)\leq\wp(F)$ when $E\subseteq F.$ [Proved]

Theorem 3.2. Let $\wp(.)$ be an IVPF-entropy on $\Omega.$ For any $E,F\in IVPFS(\Omega),$ we have

$$\wp(E\cup F)+\wp(E\cap F)=\wp(E)+\wp(F),$$

Proof: Since

$$\wp(E)+\wp(F)=\frac{1}{n\left(\sqrt{e}-1\right)}\sum_{i=1}^{n}\left(\left(\frac{\left((\mu_E^{-}(\varepsilon_i))^2+(\mu_E^{+}(\varepsilon_i))^2\right)+2-\left((\nu_E^{-}(\varepsilon_i))^2+(\nu_E^{+}(\varepsilon_i))^2\right)}{4}\right)\times e^{\left(\frac{\left((\nu_E^{-}(\varepsilon_i))^2+(\nu_E^{+}(\varepsilon_i))^2\right)+2-\left((\mu_E^{-}(\varepsilon_i))^2+(\mu_E^{+}(\varepsilon_i))^2\right)}{4}\right)}\right.$$

$$\left.+\left(\left(\frac{\left((\nu_E^{-}(\varepsilon_i))^2+(\nu_E^{+}(\varepsilon_i))^2\right)+2-\left((\mu_E^{-}(\varepsilon_i))^2+(\mu_E^{+}(\varepsilon_i))^2\right)}{4}\right)\right)\times e^{\left(\frac{\left((\mu_E^{-}(\varepsilon_i))^2+(\mu_E^{+}(\varepsilon_i))^2\right)+2-\left((\nu_E^{-}(\varepsilon_i))^2+(\nu_E^{+}(\varepsilon_i))^2\right)}{4}\right)}-1\right)+$$

$$\frac{1}{n\left(\sqrt{e}-1\right)}\sum_{i=1}^{n}\left(\left(\frac{\left((\mu_F^{-}(\varepsilon_i))^2+(\mu_F^{+}(\varepsilon_i))^2\right)+2-\left((\nu_F^{-}(\varepsilon_i))^2+(\nu_F^{+}(\varepsilon_i))^2\right)}{4}\right)\times e^{\left(\frac{\left((\nu_F^{-}(\varepsilon_i))^2+(\nu_F^{+}(\varepsilon_i))^2\right)+2-\left((\mu_F^{-}(\varepsilon_i))^2+(\mu_F^{+}(\varepsilon_i))^2\right)}{4}\right)}\right.$$

$$\left.+\left(\left(\frac{\left((\nu_F^{-}(\varepsilon_i))^2+(\nu_F^{+}(\varepsilon_i))^2\right)+2-\left((\mu_F^{-}(\varepsilon_i))^2+(\mu_F^{+}(\varepsilon_i))^2\right)}{4}\right)\right)\times e^{\left(\frac{\left((\mu_F^{-}(\varepsilon_i))^2+(\mu_F^{+}(\varepsilon_i))^2\right)+2-\left((\nu_F^{-}(\varepsilon_i))^2+(\nu_F^{+}(\varepsilon_i))^2\right)}{4}\right)}-1\right).$$

$$\wp(E\cup F)=\frac{1}{n\left(\sqrt{e}-1\right)}\sum_{i=1}^{n}\left(\left(\frac{\left((\mu_{E\cup F}^{-}(\varepsilon_i))^2+(\mu_{E\cup F}^{+}(\varepsilon_i))^2\right)+2-\left((\nu_{E\cup F}^{-}(\varepsilon_i))^2+(\nu_{E\cup F}^{+}(\varepsilon_i))^2\right)}{4}\right)\times e^{\left(\frac{\left((\nu_{E\cup F}^{-}(\varepsilon_i))^2+(\nu_{E\cup F}^{+}(\varepsilon_i))^2\right)+2-\left((\mu_{E\cup F}^{-}(\varepsilon_i))^2+(\mu_{E\cup F}^{+}(\varepsilon_i))^2\right)}{4}\right)}\right.$$

$$+\left(\left(\frac{\left((\nu_{E\cup F}^{-}(\varepsilon_i))^2+(\nu_{E\cup F}^{+}(\varepsilon_i))^2\right)+2-\left((\mu_{E\cup F}^{-}(\varepsilon_i))^2+(\mu_{E\cup F}^{+}(\varepsilon_i))^2\right)}{4}\right)\times e^{\left(\frac{\left((\mu_{E\cup F}^{-}(\varepsilon_i))^2+(\mu_{E\cup F}^{+}(\varepsilon_i))^2\right)+2-\left((\nu_{E\cup F}^{-}(\varepsilon_i))^2+(\nu_{E\cup F}^{+}(\varepsilon_i))^2\right)}{4}\right)}-1\right),$$

$$\wp\left(E\cap F\right)=\frac{1}{n\left(\sqrt{e}-1\right)}\sum_{i=1}^{n}\left(\left(\frac{\left((\mu_{E\cap F}^{-}(\varepsilon_i))^2+(\mu_{E\cap F}^{+}(\varepsilon_i))^2\right)+2-\left((\nu_{E\cap F}^{-}(\varepsilon_i))^2+(\nu_{E\cap F}^{+}(\varepsilon_i))^2\right)}{4}\right)\times e^{\left(\frac{\left((\nu_{E\cap F}^{-}(\varepsilon_i))^2+(\nu_{E\cap F}^{+}(\varepsilon_i))^2\right)+2-\left((\mu_{E\cap F}^{-}(\varepsilon_i))^2+(\mu_{E\cap F}^{+}(\varepsilon_i))^2\right)}{4}\right)}\right.$$

$$\left.+\left(\left(\frac{\left((\nu_{E\cap F}^{-}(\varepsilon_i))^2+(\nu_{E\cap F}^{+}(\varepsilon_i))^2\right)+2-\left((\mu_{E\cap F}^{-}(\varepsilon_i))^2+(\mu_{E\cap F}^{+}(\varepsilon_i))^2\right)}{4}\right)\times e^{\left(\frac{\left((\mu_{E\cap F}^{-}(\varepsilon_i))^2+(\mu_{E\cap F}^{+}(\varepsilon_i))^2\right)+2-\left((\nu_{E\cap F}^{-}(\varepsilon_i))^2+(\nu_{E\cap F}^{+}(\varepsilon_i))^2\right)}{4}\right)}-1\right)\right).$$

If $E\subseteq F$, then $\mu_E^{-}(\varepsilon_i)\le\mu_F^{-}(\varepsilon_i)$, $\mu_E^{+}(\varepsilon_i)\le\mu_F^{+}(\varepsilon_i)$, $\nu_E^{-}(\varepsilon_i)\ge\nu_F^{-}(\varepsilon_i)$ and $\nu_E^{+}(\varepsilon_i)\ge\nu_F^{+}(\varepsilon_i)$, it implies that

$$\wp\left(E\cup F\right)=\frac{1}{n\left(\sqrt{e}-1\right)}\sum_{i=1}^{n}\left(\left(\frac{\left((\mu_F^{-}(\varepsilon_i))^2+(\mu_F^{+}(\varepsilon_i))^2\right)+2-\left((\nu_F^{-}(\varepsilon_i))^2+(\nu_F^{+}(\varepsilon_i))^2\right)}{4}\right)\times e^{\left(\frac{\left((\nu_F^{-}(\varepsilon_i))^2+(\nu_F^{+}(\varepsilon_i))^2\right)+2-\left((\mu_F^{-}(\varepsilon_i))^2+(\mu_F^{+}(\varepsilon_i))^2\right)}{4}\right)}\right.$$

$$\left.+\left(\left(\frac{\left((\nu_F^{-}(\varepsilon_i))^2+(\nu_F^{+}(\varepsilon_i))^2\right)+2-\left((\mu_F^{-}(\varepsilon_i))^2+(\mu_F^{+}(\varepsilon_i))^2\right)}{4}\right)\times e^{\left(\frac{\left((\mu_F^{-}(\varepsilon_i))^2+(\mu_F^{+}(\varepsilon_i))^2\right)+2-\left((\nu_E^{-}(\varepsilon_i))^2+(\nu_F^{+}(\varepsilon_i))^2\right)}{4}\right)}-1\right)\right),$$

$$\wp\left(E\cap F\right)=\frac{1}{n\left(\sqrt{e}-1\right)}\sum_{i=1}^{n}\left(\left(\frac{\left(\left(\mu_E^-(\varepsilon_i)\right)^2+\left(\mu_E^+(\varepsilon_i)\right)^2\right)+2-\left(\left(\nu_E^-(\varepsilon_i)\right)^2+\left(\nu_E^+(\varepsilon_i)\right)^2\right)}{4}\right)\times e^{\left(\frac{\left((\nu_E^-(\varepsilon_i))^2+(\nu_E^+(\varepsilon_i))^2\right)+2-\left((\mu_E^-(\varepsilon_i))^2+(\mu_E^+(\varepsilon_i))^2\right)}{4}\right)}\right.$$

$$\left.+\left(\left(\frac{\left(\left(\nu_E^-(\varepsilon_i)\right)^2+\left(\nu_E^+(\varepsilon_i)\right)^2\right)+2-\left(\left(\mu_E^-(\varepsilon_i)\right)^2+\left(\mu_E^+(\varepsilon_i)\right)^2\right)}{4}\right)\right)\times e^{\left(\frac{\left((\mu_E^-(\varepsilon_i))^2+(\mu_E^+(\varepsilon_i))^2\right)+2-\left((\nu_E^-(\varepsilon_i))^2+(\nu_E^+(\varepsilon_i))^2\right)}{4}\right)}-1\right).$$

So, we have

$$\wp\left(E\cup F\right)+\wp\left(E\cap F\right)=\wp\left(E\right)+\wp\left(F\right).$$

Similarly, if $E\supseteq F,$ then $\mu_E^-(\varepsilon_i)\geq\mu_F^-(\varepsilon_i),$ $\mu_E^+(\varepsilon_i)\geq\mu_F^+(\varepsilon_i),$ $\nu_E^-(\varepsilon_i)\leq\nu_F^-(\varepsilon_i)$ and $\nu_E^+(\varepsilon_i)\leq\nu_F^+(\varepsilon_i),$ we will obtain $\wp\left(E\cup F\right)+\wp\left(E\cap F\right)=\wp\left(E\right)+\wp\left(F\right).$

3.3. Comparison with Existing IVPF-Entropies

This section discusses the comparison between proposed and existing entropy measures (Peng and Li, 2019; Rani et al., 2023).

Peng and Li (2019):

$$\wp_P\left(E\right)=1-\sqrt[p]{\frac{1}{2n}\sum_{i=1}^{n}\left(\left|(\mu_E^-(\varepsilon_i))^2-(\nu_E^-(\varepsilon_i))^2\right|^p+\left|(\mu_E^+(\varepsilon_i))^2-(\nu_E^+(\varepsilon_i))^2\right|^p\right)}. \tag{3.14}$$

Rani et al. (2023):

$$\wp_{R1}\left(E\right)=1-\frac{1}{n}\sum_{i=1}^{n}\left[\left\{\left(\frac{\left(\mu_E^-\left(\varepsilon_i\right)\right)^2+\left(\mu_E^+\left(\varepsilon_i\right)\right)^2}{2}\right)-\left(\frac{\left(\nu_E^-\left(\varepsilon_i\right)\right)^2+\left(\nu_E^+\left(\varepsilon_i\right)\right)^2}{2}\right)\right\}I_{\left[\substack{\left(\mu_E^-(\varepsilon_i)\right)^2+\left(\mu_E^+(\varepsilon_i)\right)^2\\ \geq\left(\nu_E^-(\varepsilon_i)\right)^2+\left(\nu_E^+(\varepsilon_i)\right)^2}\right]}\right.$$

$$+\left\{\left(\frac{\left(\nu_E^-(\varepsilon_i)\right)^2+\left(\nu_E^+(\varepsilon_i)\right)^2}{2}\right)-\left(\frac{\left(\mu_E^-(\varepsilon_i)\right)^2+\left(\mu_E^+(\varepsilon_i)\right)^2}{2}\right)\right\} I_{\left[\substack{\left(\mu_E^-(\varepsilon_i)\right)^2+\left(\mu_E^+(\varepsilon_i)\right)^2 \\ <\left(\nu_E^-(\varepsilon_i)\right)^2+\left(\nu_E^+(\varepsilon_i)\right)^2}\right]}\Bigg]. \tag{3.15}$$

$$\wp_{R2}(E)=\frac{1}{n\left(1-e^{(-0.5)}\right)}\sum_{i=1}^{n}\left[\left\{1-e^{\left(-\left(\frac{\left(\nu_F^-(\varepsilon_i)\right)^2+\left(\nu_F^+(\varepsilon_i)\right)^2+2-\left(\mu_F^-(\varepsilon_i)\right)^2-\left(\mu_F^+(\varepsilon_i)\right)^2}{4}\right)\right)}\right\} I_{\left[\substack{\left(\mu_E^-(\varepsilon_i)\right)^2+\left(\mu_E^+(\varepsilon_i)\right)^2 \\ \geq\left(\nu_E^-(\varepsilon_i)\right)^2+\left(\nu_E^+(\varepsilon_i)\right)^2}\right]}\right.$$

$$\left.+\left\{1-e^{\left(-\left(\frac{\left(\mu_F^-(\varepsilon_i)\right)^2+\left(\mu_F^+(\varepsilon_i)\right)^2+2-\left(\nu_F^-(\varepsilon_i)\right)^2-\left(\nu_F^+(\varepsilon_i)\right)^2}{4}\right)\right)}\right\} I_{\left[\substack{\left(\mu_E^-(\varepsilon_i)\right)^2+\left(\mu_E^+(\varepsilon_i)\right)^2 \\ <\left(\nu_E^-(\varepsilon_i)\right)^2+\left(\nu_E^+(\varepsilon_i)\right)^2}\right]}\right]. \tag{3.16}$$

Example 3.1 (adopted from Hung & Yang, 2006). Suppose that $E=\left\{\left\langle \varepsilon_i,\left(\left[\mu^-(\varepsilon_i),\mu^+(\varepsilon_i)\right],\left[\nu^-(\varepsilon_i),\nu^+(\varepsilon_i)\right]\right)\right\rangle:\varepsilon_i\in\Omega\right\}$. For any $n\in\mathbb{N}^+$, define the IVPFS E^n as follows:

$$E^n=\left\{\left\langle \varepsilon_i,\left(\left[\left(\mu_F^-(\varepsilon_i)\right)^n,\left(\mu_F^+(\varepsilon_i)\right)^n\right],\left[\left(\nu_F^-(\varepsilon_i)\right)^n,\left(\nu_F^+(\varepsilon_i)\right)^n\right]\right)\right\rangle:\varepsilon_i\in\Omega\right\}. \tag{3.17}$$

The IVPFS E was defined on the universal set $\Omega=\{6,7,8,9,10\}$ as follows:

$$E=\left\{\begin{array}{l}\langle 6,[0.1,0.2],[0.6,0.7]\rangle,\langle 7,[0.3,0.5],[0.4,0.5]\rangle, \\ \langle 8,[0.6,0.7],[0.1,0.2]\rangle,\langle 9,[0.8,0.9],[0,0.1]\rangle,\langle 10,[1,1],[0,0]\rangle\end{array}\right\}.$$

Similarly, we can define the following IVPFSs:

$$E^2=\left\{\begin{array}{l}\langle 6,[0.01,0.04],[0.84,0.91]\rangle,\langle 7,[0.09,0.25],[0.64,0.75]\rangle, \\ \langle 8,[0.36,0.49],[0.19,0.36]\rangle,\langle 9,[0.64,0.81],[0,0.19]\rangle,\langle 10,[1,1],[0,0]\rangle\end{array}\right\},$$

$$E^3=\left\{\begin{array}{l}\langle 6,[0.0010,0.0080],[0.9360,0.9730]\rangle,\langle 7,[0.0270,0.1250],[0.7840,0.8750]\rangle, \\ \langle 8,[0.2160,0.3430],[0.2710,0.4880]\rangle,\langle 9,[0.5120,0.7290],[0,0.2710]\rangle,\langle 10,[1,1],[0,0]\rangle\end{array}\right\},$$

$$E^4 = \left\{ \begin{array}{l} \langle 6,[0.0001,0.0016],[0.9744,0.9919]\rangle, \langle 7,[0.0081,0.0625],[0.8704,0.9375]\rangle, \\ \langle 8,[0.1296,0.2401],[0.3439,0.5904]\rangle, \langle 9,[0.4096,0.6561],[0,0.3439]\rangle, \langle 10,[1,1],[0,0]\rangle \end{array} \right\}.$$

Given the characterization of the linguistic concept, the IVPFS is utilized to describe the effectiveness of structural linguistic variable. De et al. (2000) measured as "LARGE" on for the classification of linguistic variables. Now, we consider the following operators: considered as "More or less LARGE"; considered as "Very LARGE"; considered as "Quite Very LARGE"; considered as "Very Very LARGE".

E considered as "More or less LARGE";

E^2 considered as "Very LARGE";

E^3 considered as "Quite Very LARGE";

E^4 considered as "Very Very LARGE".

Based on the mathematical operations, the entropy measures of IVIFSs as discussed above should fulfill the condition as mentioned in Hung & Yang (2006) (see Table 3.1), i.e., $\wp(E) > \wp(E^2) > \wp(E^3) > \wp(E^4)$. It can be easily seen that the purposed entropy measure also looks quite reasonable as compared with existing entropy measures of IVPFSs.

Table 3.1: Comparison of proposed and existing entropy measures for IVPFSs

	$\wp_P(.)$	$\wp_{R1}(.)$	$\wp_{R2}(.)$	$\wp(.)$
E	0.524	0.893	0.489	0.637
E^2	0.472	0.719	0.433	0.593
E^3	0.424	0.625	0.394	0.519
E^4	0.382	0.570	0.356	0.464

Example 3.2. Let $E, F, G, H \in IVPFSs(\Omega)$ such that

$$E=\{\langle \varepsilon_1,[0.2,0.2],[0.2,0.3]\rangle : \varepsilon_1 \in \Omega\},$$

$$F=\{\langle \varepsilon_1,[0.2,0.3],[0.4,0.6]\rangle : \varepsilon_1 \in \Omega\},$$

$$G=\{\langle \varepsilon_1,[0.6,0.6],[0.2,0.2]\rangle : \varepsilon_1 \in \Omega\}$$

and

$$H=\{\langle \varepsilon_1,[0.7,0.7],[0.3,0.3]\rangle : \varepsilon_1 \in \Omega\}.$$

The proposed and existing entropy measures are computed for given IVPFSs, which are given in Table 3.2. Here, we can see that proposed and existing entropy measures follow the pattern $\wp(E)>\wp(F)>\wp(G)>\wp(H)$.

Table 3.2: Computational results for proposed and existing entropy measures

IVPFSs	$([0.2,0.2],[0.2,0.3])$	$([0.2,0.3],[0.4,0.6])$	$([0.6,0.6],[0.2,0.2])$	$([0.7,0.7],[0.3,0.3])$
Proposed	0.999	0.964	0.902	0.846
$\wp_P(E)$	0.958	0.771	0.680	0.600
$\wp_{R1}(E)$	0.975	0.805	0.680	0.600
$\wp_{R2}(E)$	0.981	0.842	0.733	0.659

3.3. Proposed New IVPF-Divergence Measures

Definition 3.1. Suppose $E=([\mu_1^-,\mu_1^+],[\nu_1^-,\nu_1^+])$ and $F=([\mu_2^-,\mu_2^+],[\nu_2^-,\nu_2^+])$ be two IVPFSs. Now, the divergence measure for IVPFSs presented as

$$\Im(E,F)=\frac{1}{4n(e^2-1)}\times\sum_{i=1}^{n}\left[\left((\mu_1^-(\varepsilon_i))^2-(\mu_2^-(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\mu_1^-(\varepsilon_i))^2}{1+(\mu_1^-(\varepsilon_i))^2+(\mu_2^-(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\mu_2^-(\varepsilon_i))^2}{1+(\mu_1^-(\varepsilon_i))^2+(\mu_2^-(\varepsilon_i))^2}\right)}\right)\right.$$

$$+\left((\mu_1^+(\varepsilon_i))^2-(\mu_2^+(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\mu_1^+(\varepsilon_i))^2}{1+(\mu_1^+(\varepsilon_i))^2+(\mu_2^+(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\mu_2^+(\varepsilon_i))^2}{1+(\mu_1^+(\varepsilon_i))^2+(\mu_2^+(\varepsilon_i))^2}\right)}\right)$$

$$+\left((\nu_1^-(\varepsilon_i))^2-(\nu_2^-(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\nu_1^-(\varepsilon_i))^2}{1+(\nu_1^-(\varepsilon_i))^2+(\nu_2^-(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\nu_2^-(\varepsilon_i))^2}{1+(\nu_1^-(\varepsilon_i))^2+(\nu_2^-(\varepsilon_i))^2}\right)}\right)$$

$$+\left((\nu_1^+(\varepsilon_i))^2-(\nu_2^+(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\nu_1^+(\varepsilon_i))^2}{1+(\nu_1^+(\varepsilon_i))^2+(\nu_2^+(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\nu_2^+(\varepsilon_i))^2}{1+(\nu_1^+(\varepsilon_i))^2+(\nu_2^+(\varepsilon_i))^2}\right)}\right)\Bigg]. \quad (3.18)$$

Theorem 3.3. The function $\Im(E,F)$ given in Eq. (3.18), is a valid IVPF-DM.

Proof. To proof this theorem, the function $\Im(E,F)$ must have to fulfill the axioms of Definition 1.16.

(A1) It is obvious that $\Im(E,F)=\Im(F,E)$.

(A2) For $E,F\in IVPFSs(\Omega)$, if $\Im(E,F)=0$, then

$$\Bigg[\left((\mu_1^-(\varepsilon_i))^2-(\mu_2^-(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\mu_1^-(\varepsilon_i))^2}{1+(\mu_1^-(\varepsilon_i))^2+(\mu_2^-(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\mu_2^-(\varepsilon_i))^2}{1+(\mu_1^-(\varepsilon_i))^2+(\mu_2^-(\varepsilon_i))^2}\right)}\right)$$

$$+\left((\mu_1^+(\varepsilon_i))^2-(\mu_2^+(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\mu_1^+(\varepsilon_i))^2}{1+(\mu_1^+(\varepsilon_i))^2+(\mu_2^+(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\mu_2^+(\varepsilon_i))^2}{1+(\mu_1^+(\varepsilon_i))^2+(\mu_2^+(\varepsilon_i))^2}\right)}\right)$$

$$+\left((\nu_1^-(\varepsilon_i))^2-(\nu_2^-(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\nu_1^-(\varepsilon_i))^2}{1+(\nu_1^-(\varepsilon_i))^2+(\nu_2^-(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\nu_2^-(\varepsilon_i))^2}{1+(\nu_1^-(\varepsilon_i))^2+(\nu_2^-(\varepsilon_i))^2}\right)}\right)$$

$$+\left((\nu_1^+(\varepsilon_i))^2-(\nu_2^+(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\nu_1^+(\varepsilon_i))^2}{1+(\nu_1^+(\varepsilon_i))^2+(\nu_2^+(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\nu_2^+(\varepsilon_i))^2}{1+(\nu_1^+(\varepsilon_i))^2+(\nu_2^+(\varepsilon_i))^2}\right)}\right)\Bigg]=0.$$

Since all four terms are negative for input values, therefore, we have

$$\left((\mu_1^-(\varepsilon_i))^2-(\mu_2^-(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\mu_1^-(\varepsilon_i))^2}{1+(\mu_1^-(\varepsilon_i))^2+(\mu_2^-(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\mu_2^-(\varepsilon_i))^2}{1+(\mu_1^-(\varepsilon_i))^2+(\mu_2^-(\varepsilon_i))^2}\right)}\right)=0,$$

$$\left((\mu_1^+(\varepsilon_i))^2-(\mu_2^+(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\mu_1^+(\varepsilon_i))^2}{1+(\mu_1^+(\varepsilon_i))^2+(\mu_2^+(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\mu_2^+(\varepsilon_i))^2}{1+(\mu_1^+(\varepsilon_i))^2+(\mu_2^+(\varepsilon_i))^2}\right)}\right)=0,$$

$$\left((\nu_1^-(\varepsilon_i))^2-(\nu_2^-(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\nu_1^-(\varepsilon_i))^2}{1+(\nu_1^-(\varepsilon_i))^2+(\nu_2^-(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\nu_2^-(\varepsilon_i))^2}{1+(\nu_1^-(\varepsilon_i))^2+(\nu_2^-(\varepsilon_i))^2}\right)}\right)=0,$$

$$\left((\nu_1^+(\varepsilon_i))^2-(\nu_2^+(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\nu_1^+(\varepsilon_i))^2}{1+(\nu_1^+(\varepsilon_i))^2+(\nu_2^+(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\nu_2^+(\varepsilon_i))^2}{1+(\nu_1^+(\varepsilon_i))^2+(\nu_2^+(\varepsilon_i))^2}\right)}\right)=0.$$

This implies that

$$\left((\mu_1^-(\varepsilon_i))^2-(\mu_2^-(\varepsilon_i))^2\right)=0 \text{ and } \left(e^{\left(\frac{4(\mu_1^-(\varepsilon_i))^2}{1+(\mu_1^-(\varepsilon_i))^2+(\mu_2^-(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\mu_2^-(\varepsilon_i))^2}{1+(\mu_1^-(\varepsilon_i))^2+(\mu_2^-(\varepsilon_i))^2}\right)}\right)=0,$$

$$\left((\mu_1^+(\varepsilon_i))^2-(\mu_2^+(\varepsilon_i))^2\right)=0 \text{ and } \left(e^{\left(\frac{4(\mu_1^+(\varepsilon_i))^2}{1+(\mu_1^+(\varepsilon_i))^2+(\mu_2^+(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\mu_2^+(\varepsilon_i))^2}{1+(\mu_1^+(\varepsilon_i))^2+(\mu_2^+(\varepsilon_i))^2}\right)}\right)=0,$$

$$\left((\nu_1^-(\varepsilon_i))^2-(\nu_2^-(\varepsilon_i))^2\right)=0 \text{ and } \left(e^{\left(\frac{4(\nu_1^-(\varepsilon_i))^2}{1+(\nu_1^-(\varepsilon_i))^2+(\nu_2^-(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\nu_2^-(\varepsilon_i))^2}{1+(\nu_1^-(\varepsilon_i))^2+(\nu_2^-(\varepsilon_i))^2}\right)}\right)=0,$$

$$\left((\nu_1^+(\varepsilon_i))^2-(\nu_2^+(\varepsilon_i))^2\right)=0 \text{ and } \left(e^{\left(\frac{4(\nu_1^+(\varepsilon_i))^2}{1+(\nu_1^+(\varepsilon_i))^2+(\nu_2^+(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\nu_2^+(\varepsilon_i))^2}{1+(\nu_1^+(\varepsilon_i))^2+(\nu_2^+(\varepsilon_i))^2}\right)}\right)=0.$$

From all these cases, we get $\mu_1^-(\varepsilon_i)=\mu_2^-(\varepsilon_i)$, $\mu_1^+(\varepsilon_i)=\mu_2^+(\varepsilon_i)$, $\nu_1^-(\varepsilon_i)=\nu_2^-(\varepsilon_i)$ and $\nu_1^+(\varepsilon_i)=\nu_2^+(\varepsilon_i)$.

Thus, $E=F$. Similarly, we can prove that if $E=F$, then $\Im(E,F)=0$.

(A3) $\Im(E\cap G, F\cap G)=\dfrac{1}{4n\left(e^2-1\right)}\times$

$$\sum_{i=1}^{n}\left[\left(\begin{array}{l}\left(\min\left\{\mu_1^-(\varepsilon_i),\mu_3^-(\varepsilon_i)\right\}\right)^2\\ -\left(\min\left\{\mu_2^-(\varepsilon_i),\mu_3^-(\varepsilon_i)\right\}\right)^2\end{array}\right)\left(\begin{array}{l} e^{\left(\frac{4\left(\min\left\{\mu_1^-(\varepsilon_i),\mu_3^-(\varepsilon_i)\right\}\right)^2}{1+\left(\min\left\{\mu_1^-(\varepsilon_i),\mu_3^-(\varepsilon_i)\right\}\right)^2+\left(\min\left\{\mu_2^-(\varepsilon_i),\mu_3^-(\varepsilon_i)\right\}\right)^2}\right)}\\ -e^{\left(\frac{4\left(\min\left\{\mu_2^-(\varepsilon_i),\mu_3^-(\varepsilon_i)\right\}\right)^2}{1+\left(\min\left\{\mu_1^-(\varepsilon_i),\mu_3^-(\varepsilon_i)\right\}\right)^2+\left(\min\left\{\mu_2^-(\varepsilon_i),\mu_3^-(\varepsilon_i)\right\}\right)^2}\right)}\end{array}\right)\right.$$

$$+\left(\begin{array}{l}\left(\min\left\{\mu_1^+(\varepsilon_i),\mu_3^+(\varepsilon_i)\right\}\right)^2-\\ \left(\min\left\{\mu_2^+(\varepsilon_i),\mu_3^+(\varepsilon_i)\right\}\right)^2\end{array}\right)\left(\begin{array}{l} e^{\left(\frac{4\left(\min\left\{\mu_1^+(\varepsilon_i),\mu_3^+(\varepsilon_i)\right\}\right)^2}{1+\left(\min\left\{\mu_1^+(\varepsilon_i),\mu_3^+(\varepsilon_i)\right\}\right)^2+\left(\min\left\{\mu_2^+(\varepsilon_i),\mu_3^+(\varepsilon_i)\right\}\right)^2}\right)}\\ -e^{\left(\frac{4\min\left\{\left(\mu_2^+(\varepsilon_i)\right)^2,\left(\mu_3^+(\varepsilon_i)\right)^2\right\}}{1+\left(\min\left\{\mu_1^+(\varepsilon_i),\mu_3^+(\varepsilon_i)\right\}\right)^2+\left(\min\left\{\mu_2^+(\varepsilon_i),\mu_3^+(\varepsilon_i)\right\}\right)^2}\right)}\end{array}\right)$$

$$+\left(\begin{array}{l}\left(\max\left\{\nu_1^-(\varepsilon_i),\nu_3^-(\varepsilon_i)\right\}\right)^2\\ -\left(\max\left\{\nu_2^-(\varepsilon_i),\nu_3^-(\varepsilon_i)\right\}\right)^2\end{array}\right)\left(\begin{array}{l} e^{\left(\frac{4\left(\max\left\{\nu_1^-(\varepsilon_i),\nu_3^-(\varepsilon_i)\right\}\right)^2}{1+\left(\max\left\{\nu_1^-(\varepsilon_i),\nu_3^-(\varepsilon_i)\right\}\right)^2+\left(\max\left\{\nu_2^-(\varepsilon_i),\nu_3^-(\varepsilon_i)\right\}\right)^2}\right)}\\ -e^{\left(\frac{4\max\left\{\left(\nu_2^-(\varepsilon_i)\right)^2,\left(\nu_3^-(\varepsilon_i)\right)^2\right\}}{1+\left(\max\left\{\nu_1^-(\varepsilon_i),\nu_3^-(\varepsilon_i)\right\}\right)^2+\left(\max\left\{\nu_2^-(\varepsilon_i),\nu_3^-(\varepsilon_i)\right\}\right)^2}\right)}\end{array}\right)$$

$$\left.+\left(\begin{array}{l}\left(\max\left\{\nu_1^+(\varepsilon_i),\nu_3^+(\varepsilon_i)\right\}\right)^2\\ -\max\left\{\left(\nu_2^+(\varepsilon_i)\right)^2,n_3^+(\varepsilon_i)\right\}\end{array}\right)\left(\begin{array}{l} e^{\left(\frac{4\left(\max\left\{\nu_1^+(\varepsilon_i),\nu_3^+(\varepsilon_i)\right\}\right)^2}{1+\left(\max\left\{\nu_1^+(\varepsilon_i),\nu_3^+(\varepsilon_i)\right\}\right)^2+\left(\max\left\{\nu_2^+(\varepsilon_i),\nu_3^+(\varepsilon_i)\right\}\right)^2}\right)}\\ -e^{\left(\frac{4\left(\max\left\{\nu_2^+(\varepsilon_i),\nu_3^+(\varepsilon_i)\right\}\right)^2}{1+\left(\max\left\{\nu_1^+(\varepsilon_i),\nu_3^+(\varepsilon_i)\right\}\right)^2+\left(\max\left\{\nu_2^+(\varepsilon_i),\nu_3^+(\varepsilon_i)\right\}\right)^2}\right)}\end{array}\right)\right]. \tag{3.19}$$

From $\min\left\{\mu_1^-(\varepsilon_i),\mu_3^-(\varepsilon_i)\right\}$, $\min\left\{\mu_2^-(\varepsilon_i),\mu_3^-(\varepsilon_i)\right\}$, $\min\left\{\mu_1^+(\varepsilon_i),\mu_3^+(\varepsilon_i)\right\}$, $\min\left\{\mu_2^+(\varepsilon_i),\mu_3^+(\varepsilon_i)\right\}$, $\max\left\{\nu_1^-(\varepsilon_i),\nu_3^-(\varepsilon_i)\right\}$, $\max\left\{\nu_2^-(\varepsilon_i),\nu_3^-(\varepsilon_i)\right\}$, $\max\left\{\nu_1^+(\varepsilon_i),\nu_3^+(\varepsilon_i)\right\}$, and $\max\left\{\nu_2^+(\varepsilon_i),\nu_3^+(\varepsilon_i)\right\}$, we deduce the following results:

$$\mu_1^-(\varepsilon_i)\le\mu_3^-(\varepsilon_i)\le\mu_2^-(\varepsilon_i) \text{ or } \mu_2^-(\varepsilon_i)\le\mu_3^-(\varepsilon_i)\le\mu_1^-(\varepsilon_i) \text{ or } \mu_1^+(\varepsilon_i)\le\mu_3^+(\varepsilon_i)\le\mu_2^+(\varepsilon_i) \text{ or } \mu_2^+(\varepsilon_i)\le\mu_3^+(\varepsilon_i)\le\mu_1^+(\varepsilon_i) \tag{3.20}$$

$$\mu_3^-(\varepsilon_i)\le\left\{\mu_1^-(\varepsilon_i)\ \&\ \mu_2^-(\varepsilon_i)\right\} \text{ or } \mu_3^-(\varepsilon_i)\ge\left\{\mu_1^-(\varepsilon_i)\ \&\ \mu_2^-(\varepsilon_i)\right\} \text{ or } \mu_3^+(\varepsilon_i)\le\left\{\mu_1^+(\varepsilon_i)\ \&\ \mu_2^+(\varepsilon_i)\right\} \text{ or } \mu_3^+(\varepsilon_i)\ge\left\{\mu_1^+(\varepsilon_i)\ \&\ \mu_2^+(\varepsilon_i)\right\}, \tag{3.21}$$

$$\nu_1^-(\varepsilon_i)\le\nu_3^-(\varepsilon_i)\le\nu_2^-(\varepsilon_i) \text{ or } \nu_2^-(\varepsilon_i)\le\nu_3^-(\varepsilon_i)\le\nu_1^-(\varepsilon_i) \text{ or } \nu_1^+(\varepsilon_i)\le\nu_3^+(\varepsilon_i)\le\nu_2^+(\varepsilon_i) \text{ or } \nu_2^+(\varepsilon_i)\le\nu_3^+(\varepsilon_i)\le\nu_1^+(\varepsilon_i) \tag{3.22}$$

$$\nu_3^-(\varepsilon_i)\le\left\{\nu_1^-(\varepsilon_i)\ \&\nu_2^-(\varepsilon_i)\right\} \text{ or } \nu_3^-(\varepsilon_i)\ge\left\{\nu_1^-(\varepsilon_i)\ \&\nu_2^-(\varepsilon_i)\right\} \text{ or } \nu_3^+(\varepsilon_i)\le\left\{\nu_1^+(\varepsilon_i)\ \&\nu_2^+(\varepsilon_i)\right\} \text{ or } \nu_3^+(\varepsilon_i)\ge\left\{\nu_1^+(\varepsilon_i)\ \&\nu_2^+(\varepsilon_i)\right\}. \tag{3.23}$$

The proof is easy for the situations in (3.21) and (3.23). Now, we prove the results for (3.20) and (3.22), then Eq. (3.18) becomes

$$\Im\left(E\cap G,F\cap G\right)=\frac{1}{4n\left(e^2-1\right)}\times\sum_{i=1}^{n}\Bigg[\left(\left(\mu_3^-(\varepsilon_i)\right)^2-\left(\mu_1^-(\varepsilon_i)\right)^2\right)\left(e^{\left(\frac{4\left(\mu_3^-(\varepsilon_i)\right)^2}{1+\left(\mu_1^-(\varepsilon_i)\right)^2+\left(\mu_3^-(\varepsilon_i)\right)^2}\right)}-e^{\left(\frac{4\left(\mu_1^-(\varepsilon_i)\right)^2}{1+\left(\mu_1^-(\varepsilon_i)\right)^2+\left(\mu_3^-(\varepsilon_i)\right)^2}\right)}\right)+\left(\left(\mu_3^+(\varepsilon_i)\right)^2-\left(\mu_1^+(\varepsilon_i)\right)^2\right)\left(e^{\left(\frac{4\left(\mu_3^+(\varepsilon_i)\right)^2}{1+\left(\mu_1^+(\varepsilon_i)\right)^2+\left(\mu_3^+(\varepsilon_i)\right)^2}\right)}-e^{\left(\frac{4\left(\mu_1^+(\varepsilon_i)\right)^2}{1+\left(\mu_1^+(\varepsilon_i)\right)^2+\left(\mu_3^+(\varepsilon_i)\right)^2}\right)}\right)$$

$$+\left(\left(\nu_2^-(\varepsilon_i)\right)^2-\left(\nu_3^-(\varepsilon_i)\right)^2\right)\left(e^{\left(\frac{4\left(\nu_2^-(\varepsilon_i)\right)^2}{1+\left(\nu_2^-(\varepsilon_i)\right)^2+\left(\nu_3^-(\varepsilon_i)\right)^2}\right)}-e^{\left(\frac{4\left(\nu_3^-(\varepsilon_i)\right)^2}{1+\left(\nu_2^-(\varepsilon_i)\right)^2+\left(\nu_3^-(\varepsilon_i)\right)^2}\right)}\right)+\left(\left(\nu_2^+(\varepsilon_i)\right)^2-\left(\nu_3^+(\varepsilon_i)\right)^2\right)\left(e^{\left(\frac{4\left(\nu_2^+(\varepsilon_i)\right)^2}{1+\left(\nu_2^+(\varepsilon_i)\right)^2+\left(\nu_3^+(\varepsilon_i)\right)^2}\right)}-e^{\left(\frac{4\left(\nu_3^+(\varepsilon_i)\right)^2}{1+\left(\nu_2^+(\varepsilon_i)\right)^2+\left(\nu_3^+(\varepsilon_i)\right)^2}\right)}\right)\Bigg].$$

$$\Im\left(E,F\right)=\frac{1}{4n\left(e^2-1\right)}\times\sum_{i=1}^{n}\Bigg[\left(\left(\mu_2^-(\varepsilon_i)\right)^2-\left(\mu_1^-(\varepsilon_i)\right)^2\right)\left(e^{\left(\frac{4\left(\mu_2^-(\varepsilon_i)\right)^2}{1+\left(\mu_1^-(\varepsilon_i)\right)^2+\left(\mu_2^-(\varepsilon_i)\right)^2}\right)}-e^{\left(\frac{4\left(\mu_1^-(\varepsilon_i)\right)^2}{1+\left(\mu_1^-(\varepsilon_i)\right)^2+\left(\mu_2^-(\varepsilon_i)\right)^2}\right)}\right)+\left(\left(\mu_2^+(\varepsilon_i)\right)^2-\left(\mu_1^+(\varepsilon_i)\right)^2\right)\left(e^{\left(\frac{4\left(\mu_2^+(\varepsilon_i)\right)^2}{1+\left(\mu_1^+(\varepsilon_i)\right)^2+\left(\mu_2^+(\varepsilon_i)\right)^2}\right)}-e^{\left(\frac{4\left(\mu_1^+(\varepsilon_i)\right)^2}{1+\left(\mu_1^+(\varepsilon_i)\right)^2+\left(\mu_2^+(\varepsilon_i)\right)^2}\right)}\right)$$

$$+\begin{pmatrix}\left(\nu_1^-(\varepsilon_i)\right)^2-\\\left(\nu_2^-(\varepsilon_i)\right)^2\end{pmatrix}\begin{pmatrix}e^{\left(\frac{4\left(\nu_1^-(\varepsilon_i)\right)^2}{1+\left(\nu_1^-(\varepsilon_i)\right)^2+\left(\nu_2^-(\varepsilon_i)\right)^2}\right)}-\\e^{\left(\frac{4\left(\nu_2^-(\varepsilon_i)\right)^2}{1+\left(\nu_1^-(\varepsilon_i)\right)^2+\left(\nu_2^-(\varepsilon_i)\right)^2}\right)}\end{pmatrix}+\begin{pmatrix}\left(\nu_1^+(\varepsilon_i)\right)^2-\\\left(\nu_2^+(\varepsilon_i)\right)^2\end{pmatrix}\begin{pmatrix}e^{\left(\frac{4\left(\nu_1^+(\varepsilon_i)\right)^2}{1+\left(\nu_1^+(\varepsilon_i)\right)^2+\left(\nu_2^+(\varepsilon_i)\right)^2}\right)}-\\e^{\left(\frac{4\left(\nu_2^+(\varepsilon_i)\right)^2}{1+\left(\nu_1^+(\varepsilon_i)\right)^2+\left(\nu_2^+(\varepsilon_i)\right)^2}\right)}\end{pmatrix}.$$

From cases (3.20) and (3.22), we have

$$\left(\mu_3^-(\varepsilon_i)\right)^2-\left(\mu_1^-(\varepsilon_i)\right)^2\le\left(\mu_2^-(\varepsilon_i)\right)^2-\left(\mu_1^-(\varepsilon_i)\right)^2,$$

$$\left(e^{\left(\frac{4\left(\mu_3^-(\varepsilon_i)\right)^2}{1+\left(\mu_1^-(\varepsilon_i)\right)^2+\left(\mu_3^-(\varepsilon_i)\right)^2}\right)}-e^{\left(\frac{4\left(\mu_1^-(\varepsilon_i)\right)^2}{1+\left(\mu_1^-(\varepsilon_i)\right)^2+\left(\mu_3^-(\varepsilon_i)\right)^2}\right)}\right)\le\left(e^{\left(\frac{4\left(\mu_2^-(\varepsilon_i)\right)^2}{1+\left(\mu_1^-(\varepsilon_i)\right)^2+\left(\mu_2^-(\varepsilon_i)\right)^2}\right)}-e^{\left(\frac{4\left(\mu_1^-(\varepsilon_i)\right)^2}{1+\left(\mu_1^-(\varepsilon_i)\right)^2+\left(\mu_2^-(\varepsilon_i)\right)^2}\right)}\right),$$

$$\left(\mu_3^+(\varepsilon_i)\right)^2-\left(\mu_1^+(\varepsilon_i)\right)^2\le\left(\mu_2^+(\varepsilon_i)\right)^2-\left(\mu_1^+(\varepsilon_i)\right)^2,$$

$$\left(e^{\left(\frac{4\left(\mu_3^+(\varepsilon_i)\right)^2}{1+\left(\mu_1^+(\varepsilon_i)\right)^2+\left(\mu_3^+(\varepsilon_i)\right)^2}\right)}-e^{\left(\frac{4\left(\mu_1^+(\varepsilon_i)\right)^2}{1+\left(\mu_1^+(\varepsilon_i)\right)^2+\left(\mu_3^+(\varepsilon_i)\right)^2}\right)}\right)\le\left(e^{\left(\frac{4\left(\mu_2^+(\varepsilon_i)\right)^2}{1+\left(\mu_1^+(\varepsilon_i)\right)^2+\left(\mu_2^+(\varepsilon_i)\right)^2}\right)}-e^{\left(\frac{4\left(\mu_1^+(\varepsilon_i)\right)^2}{1+\left(\mu_1^+(\varepsilon_i)\right)^2+\left(\mu_2^+(\varepsilon_i)\right)^2}\right)}\right),$$

$$\left(\nu_3^-(\varepsilon_i)\right)^2-\left(\nu_1^-(\varepsilon_i)\right)^2\le\left(\nu_2^-(\varepsilon_i)\right)^2-\left(\nu_1^-(\varepsilon_i)\right)^2,$$

$$\left(e^{\left(\frac{4\left(\nu_3^-(\varepsilon_i)\right)^2}{1+\left(\nu_1^-(\varepsilon_i)\right)^2+\left(\nu_3^-(\varepsilon_i)\right)^2}\right)}-e^{\left(\frac{4\left(\nu_1^-(\varepsilon_i)\right)^2}{1+\left(\nu_1^-(\varepsilon_i)\right)^2+\left(\nu_3^-(\varepsilon_i)\right)^2}\right)}\right)\le\left(e^{\left(\frac{4\left(\nu_2^-(\varepsilon_i)\right)^2}{1+\left(\nu_1^-(\varepsilon_i)\right)^2+\left(\nu_2^-(\varepsilon_i)\right)^2}\right)}-e^{\left(\frac{4\left(\nu_1^-(\varepsilon_i)\right)^2}{1+\left(\nu_1^-(\varepsilon_i)\right)^2+\left(\nu_2^-(\varepsilon_i)\right)^2}\right)}\right),$$

$$\left(\nu_3^+(\varepsilon_i)\right)^2-\left(\nu_1^+(\varepsilon_i)\right)^2\le\left(\nu_2^+(\varepsilon_i)\right)^2-\left(\nu_1^+(\varepsilon_i)\right)^2,$$

$$\left(e^{\left(\frac{4\left(\nu_3^+(\varepsilon_i)\right)^2}{1+\left(\nu_1^+(\varepsilon_i)\right)^2+\left(\nu_3^+(\varepsilon_i)\right)^2}\right)}-e^{\left(\frac{4\left(\nu_1^+(\varepsilon_i)\right)^2}{1+\left(\nu_1^+(\varepsilon_i)\right)^2+\left(\nu_3^+(\varepsilon_i)\right)^2}\right)}\right)\le\left(e^{\left(\frac{4\left(\nu_2^+(\varepsilon_i)\right)^2}{1+\left(\nu_1^+(\varepsilon_i)\right)^2+\left(\nu_2^+(\varepsilon_i)\right)^2}\right)}-e^{\left(\frac{4\left(\nu_1^+(\varepsilon_i)\right)^2}{1+\left(\nu_1^+(\varepsilon_i)\right)^2+\left(\nu_2^+(\varepsilon_i)\right)^2}\right)}\right).$$

This implies that $\Im(E \cap G, F \cap G) \leq \Im(E, F)$.

(A4) Similarly, we can prove that $\Im(E \cup G, F \cup G) \leq \Im(E, F)$.

Thus, the measure Eq. (3.18) holds all the axioms of divergence measure on IVPFSs. Hence, it is a valid IVPF-DM.

Theorem 3.4. Let $E, F, G \in IVPFSs(\Omega)$. Then the proposed IVPF-DM $\Im(E, F)$ given by Eq. (3.18) satisfies the following properties:

(i) $\Im(E, F) = \Im(E^c, F^c)$;

(ii) $\Im(E, F^c) = \Im(E^c, F)$;

(iii) $\Im(E, E^c) = 1$ if E is a crisp set;

(iv) $\Im(E, E^c) = 0$ if and only if $\mu_E^-(\varepsilon_i) = \nu_E^-(\varepsilon_i)$ and $\mu_E^+(\varepsilon_i) = \nu_E^+(\varepsilon_i)$, $\forall\ \varepsilon_i \in \Omega$;

(v) $\Im(E, E \cup F) = \Im(E \cap F, F) = \Im(E, F)$ for $E \subseteq F$ and $F \subseteq E$;

(vi) $\Im(E \cap F, E \cup F) = \Im(E, F)$ for $E \subseteq F$;

(vii) $\Im(E, F) \leq \Im(E, G)$ for $E \subseteq F \subseteq G$;

(viii) $\Im(F, G) \leq \Im(E, G)$ for $E \subseteq F \subseteq G$.

Proof: (i) Since $E^c = \left(\left[\nu_1^-, \nu_1^+\right], \left[\mu_1^-, \mu_1^+\right]\right)$ and $F^c = \left(\left[\nu_2^-, \nu_2^+\right], \left[\mu_2^-, \mu_2^+\right]\right)$, therefore, we have

$$\Im(E^c, F^c) = \frac{1}{4n\left(e^2 - 1\right)} \times$$

$$\sum_{i=1}^{n}\left[\left((\nu_1^-(\varepsilon_i))^2 - (\nu_2^-(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\nu_1^-(\varepsilon_i))^2}{1+(\nu_1^-(\varepsilon_i))^2+(\nu_2^-(\varepsilon_i))^2}\right)} - e^{\left(\frac{4(\nu_2^-(\varepsilon_i))^2}{1+(\nu_1^-(\varepsilon_i))^2+(\nu_2^-(\varepsilon_i))^2}\right)}\right)\right.$$

$$+\left((\nu_1^+(\varepsilon_i))^2-(\nu_2^+(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\nu_1^+(\varepsilon_i))^2}{1+(\nu_1^+(\varepsilon_i))^2+(\nu_2^+(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\nu_2^+(\varepsilon_i))^2}{1+(\nu_1^+(\varepsilon_i))^2+(\nu_2^+(\varepsilon_i))^2}\right)}\right)$$

$$+\left((\mu_1^-(\varepsilon_i))^2-(\mu_2^-(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\mu_1^-(\varepsilon_i))^2}{1+(\mu_1^-(\varepsilon_i))^2+(\mu_2^-(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\mu_2^-(\varepsilon_i))^2}{1+(\mu_1^-(\varepsilon_i))^2+(\mu_2^-(\varepsilon_i))^2}\right)}\right)$$

$$+\left((\mu_1^+(\varepsilon_i))^2-(\mu_2^+(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\mu_1^+(\varepsilon_i))^2}{1+(\mu_1^+(\varepsilon_i))^2+(\mu_2^+(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\mu_2^+(\varepsilon_i))^2}{1+(\mu_1^+(\varepsilon_i))^2+(\mu_2^+(\varepsilon_i))^2}\right)}\right)\Bigg].$$

Thus, we have $\Im(E,F)=\Im(E^c,F^c)$.

(ii) From Eq. (3.18), we have

$$\Im(E,F^c)=\frac{1}{4n(e^2-1)}\times$$

$$\sum_{i=1}^{n}\Bigg[\left((\mu_1^-(\varepsilon_i))^2-(\nu_2^-(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\mu_1^-(\varepsilon_i))^2}{1+(\mu_1^-(\varepsilon_i))^2+(\nu_2^-(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\nu_2^-(\varepsilon_i))^2}{1+(\mu_1^-(\varepsilon_i))^2+(\nu_2^-(\varepsilon_i))^2}\right)}\right)$$

$$+\left((\mu_1^+(\varepsilon_i))^2-(\nu_2^+(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\mu_1^+(\varepsilon_i))^2}{1+(\mu_1^+(\varepsilon_i))^2+(\nu_2^+(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\nu_2^+(\varepsilon_i))^2}{1+(\mu_1^+(\varepsilon_i))^2+(\nu_2^+(\varepsilon_i))^2}\right)}\right)$$

$$+\left((\nu_1^-(\varepsilon_i))^2-(\mu_2^-(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\nu_1^-(\varepsilon_i))^2}{1+(\nu_1^-(\varepsilon_i))^2+(\mu_2^-(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\mu_2^-(\varepsilon_i))^2}{1+(\nu_1^-(\varepsilon_i))^2+(\mu_2^-(\varepsilon_i))^2}\right)}\right)$$

$$+\left((\nu_1^+(\varepsilon_i))^2-(\mu_2^+(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\nu_1^+(\varepsilon_i))^2}{1+(\nu_1^+(\varepsilon_i))^2+(\mu_2^+(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\mu_2^+(\varepsilon_i))^2}{1+(\nu_1^+(\varepsilon_i))^2+(\mu_2^+(\varepsilon_i))^2}\right)}\right)\Bigg]$$

$$=\frac{1}{4n(e^2-1)}\times\sum_{i=1}^{n}\Bigg[\left((\nu_1^-(\varepsilon_i))^2-(\mu_2^-(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\nu_1^-(\varepsilon_i))^2}{1+(\nu_1^-(\varepsilon_i))^2+(\mu_2^-(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\mu_2^-(\varepsilon_i))^2}{1+(\nu_1^-(\varepsilon_i))^2+(\mu_2^-(\varepsilon_i))^2}\right)}\right)$$

$$+\left((\nu_1^+(\varepsilon_i))^2-(\mu_2^+(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\nu_1^+(\varepsilon_i))^2}{1+(\nu_1^+(\varepsilon_i))^2+(\mu_2^+(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\mu_2^+(\varepsilon_i))^2}{1+(\nu_1^+(\varepsilon_i))^2+(\mu_2^+(\varepsilon_i))^2}\right)}\right)$$

$$+\left((\mu_1^-(\varepsilon_i))^2-(\nu_2^-(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\mu_1^-(\varepsilon_i))^2}{1+(\mu_1^-(\varepsilon_i))^2+(\nu_2^-(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\nu_2^-(\varepsilon_i))^2}{1+(\mu_1^-(\varepsilon_i))^2+(\nu_2^-(\varepsilon_i))^2}\right)}\right)$$

$$+\left((\mu_1^+(\varepsilon_i))^2-(\nu_2^+(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\mu_1^+(\varepsilon_i))^2}{1+(\mu_1^+(\varepsilon_i))^2+(\nu_2^+(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\nu_2^+(\varepsilon_i))^2}{1+(\mu_1^+(\varepsilon_i))^2+(\nu_2^+(\varepsilon_i))^2}\right)}\right)\Bigg].$$

Therefore, we have $\Im\left(E,F^c\right)=\Im\left(E^c,F\right)$.

(iii) We have

$$\Im\left(E,E^c\right)=\frac{1}{4n\left(e^2-1\right)}\times\sum_{i=1}^{n}\Bigg[\left((\mu_1^-(\varepsilon_i))^2-(\nu_1^-(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\mu_1^-(\varepsilon_i))^2}{1+(\mu_1^-(\varepsilon_i))^2+(\nu_1^-(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\nu_1^-(\varepsilon_i))^2}{1+(\mu_1^-(\varepsilon_i))^2+(\nu_1^-(\varepsilon_i))^2}\right)}\right)$$

$$+\left((\mu_1^+(\varepsilon_i))^2-(\nu_1^+(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\mu_1^+(\varepsilon_i))^2}{1+(\mu_1^+(\varepsilon_i))^2+(\nu_1^+(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\nu_1^+(\varepsilon_i))^2}{1+(\mu_1^+(\varepsilon_i))^2+(\nu_1^+(\varepsilon_i))^2}\right)}\right)$$

$$+\left((\nu_1^-(\varepsilon_i))^2-(\mu_1^-(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\nu_1^-(\varepsilon_i))^2}{1+(\nu_1^-(\varepsilon_i))^2+(\mu_1^-(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\mu_1^-(\varepsilon_i))^2}{1+(\nu_1^-(\varepsilon_i))^2+(\mu_1^-(\varepsilon_i))^2}\right)}\right)$$

$$+\left((\nu_1^+(\varepsilon_i))^2-(\mu_1^+(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\nu_1^+(\varepsilon_i))^2}{1+(\nu_1^+(\varepsilon_i))^2+(\mu_1^+(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\mu_1^+(\varepsilon_i))^2}{1+(\nu_1^+(\varepsilon_i))^2+(\mu_1^+(\varepsilon_i))^2}\right)}\right)\Bigg]. \tag{3.24}$$

Let E be a crisp set, i.e., $\mu_E^-(\varepsilon_i)=1=\mu_E^+(\varepsilon_i)$, $\nu_E^-(\varepsilon_i)=0=\nu_E^+(\varepsilon_i)$ or $\mu_E^-(\varepsilon_i)=0=\mu_E^+(\varepsilon_i)$, $\nu_E^-(\varepsilon_i)=1=\nu_E^+(\varepsilon_i)$, $\forall\,\varepsilon_i\in\Omega$. Then Eq. (3.24) becomes $\Im\left(E,E^c\right)=1$.

(iv) If $\Im\left(E,E^c\right)=0$, then

$$\Im\left(E,E^{c}\right)=\frac{1}{4n\left(e^{2}-1\right)}\times\sum_{i=1}^{n}\left[\left((\mu_1^{-}(\varepsilon_i))^2-(\nu_1^{-}(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\mu_1^{-}(\varepsilon_i))^2}{1+(\mu_1^{-}(\varepsilon_i))^2+(\nu_1^{-}(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\nu_1^{-}(\varepsilon_i))^2}{1+(\mu_1^{-}(\varepsilon_i))^2+(\nu_1^{-}(\varepsilon_i))^2}\right)}\right)\right.$$

$$+\left((\mu_1^{+}(\varepsilon_i))^2-(\nu_1^{+}(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\mu_1^{+}(\varepsilon_i))^2}{1+(\mu_1^{+}(\varepsilon_i))^2+(\nu_1^{+}(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\nu_1^{+}(\varepsilon_i))^2}{1+(\mu_1^{+}(\varepsilon_i))^2+(\nu_1^{+}(\varepsilon_i))^2}\right)}\right)$$

$$+\left((\nu_1^{-}(\varepsilon_i))^2-(\mu_1^{-}(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\nu_1^{-}(\varepsilon_i))^2}{1+(\nu_1^{-}(\varepsilon_i))^2+(\mu_1^{-}(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\mu_1^{-}(\varepsilon_i))^2}{1+(\nu_1^{-}(\varepsilon_i))^2+(\mu_1^{-}(\varepsilon_i))^2}\right)}\right)$$

$$\left.+\left((\nu_1^{+}(\varepsilon_i))^2-(\mu_1^{+}(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\nu_1^{+}(\varepsilon_i))^2}{1+(\nu_1^{+}(\varepsilon_i))^2+(\mu_1^{+}(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\mu_1^{+}(\varepsilon_i))^2}{1+(\nu_1^{+}(\varepsilon_i))^2+(\mu_1^{+}(\varepsilon_i))^2}\right)}\right)\right]=0,$$

it implies that

$$\left((\mu_1^{-}(\varepsilon_i))^2-(\nu_1^{-}(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\mu_1^{-}(\varepsilon_i))^2}{1+(\mu_1^{-}(\varepsilon_i))^2+(\nu_1^{-}(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\nu_1^{-}(\varepsilon_i))^2}{1+(\mu_1^{-}(\varepsilon_i))^2+(\nu_1^{-}(\varepsilon_i))^2}\right)}\right)$$

$$+\left((\mu_1^{+}(\varepsilon_i))^2-(\nu_1^{+}(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\mu_1^{+}(\varepsilon_i))^2}{1+(\mu_1^{+}(\varepsilon_i))^2+(\nu_1^{+}(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\nu_1^{+}(\varepsilon_i))^2}{1+(\mu_1^{+}(\varepsilon_i))^2+(\nu_1^{+}(\varepsilon_i))^2}\right)}\right)$$

$$+\left((\nu_1^{-}(\varepsilon_i))^2-(\mu_1^{-}(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\nu_1^{-}(\varepsilon_i))^2}{1+(\nu_1^{-}(\varepsilon_i))^2+(\mu_1^{-}(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\mu_1^{-}(\varepsilon_i))^2}{1+(\nu_1^{-}(\varepsilon_i))^2+(\mu_1^{-}(\varepsilon_i))^2}\right)}\right)$$

$$\left.+\left((\nu_1^{+}(\varepsilon_i))^2-(\mu_1^{+}(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\nu_1^{+}(\varepsilon_i))^2}{1+(\nu_1^{+}(\varepsilon_i))^2+(\mu_1^{+}(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\mu_1^{+}(\varepsilon_i))^2}{1+(\nu_1^{+}(\varepsilon_i))^2+(\mu_1^{+}(\varepsilon_i))^2}\right)}\right)\right]=0,$$

$$\Leftrightarrow\left((\mu_1^{-}(\varepsilon_i))^2-(\nu_1^{-}(\varepsilon_i))^2\right)=0\ \text{ or }\left(e^{\left(\frac{4(\mu_1^{-}(\varepsilon_i))^2}{1+(\mu_1^{-}(\varepsilon_i))^2+(\nu_1^{-}(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\nu_1^{-}(\varepsilon_i))^2}{1+(\mu_1^{-}(\varepsilon_i))^2+(\nu_1^{-}(\varepsilon_i))^2}\right)}\right)=0,$$

$$\left((\mu_1^+(\varepsilon_i))^2-(\nu_1^+(\varepsilon_i))^2\right)=0 \text{ or } \left(e^{\left(\frac{4(\mu_1^+(\varepsilon_i))^2}{1+(\mu_1^+(\varepsilon_i))^2+(\nu_1^+(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\nu_1^+(\varepsilon_i))^2}{1+(\mu_1^+(\varepsilon_i))^2+(\nu_1^+(\varepsilon_i))^2}\right)}\right)=0,$$

$$\left((\nu_1^-(\varepsilon_i))^2-(\mu_1^-(\varepsilon_i))^2\right)=0 \text{ or } \left(e^{\left(\frac{4(\nu_1^-(\varepsilon_i))^2}{1+(\nu_1^-(\varepsilon_i))^2+(\mu_1^-(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\mu_1^-(\varepsilon_i))^2}{1+(\nu_1^-(\varepsilon_i))^2+(\mu_1^-(\varepsilon_i))^2}\right)}\right)=0,$$

$$\left((\nu_1^+(\varepsilon_i))^2-(\mu_1^+(\varepsilon_i))^2\right)=0 \text{ or } \left.\left(e^{\left(\frac{4(\nu_1^+(\varepsilon_i))^2}{1+(\nu_1^+(\varepsilon_i))^2+(\mu_1^+(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\mu_1^+(\varepsilon_i))^2}{1+(\nu_1^+(\varepsilon_i))^2+(\mu_1^+(\varepsilon_i))^2}\right)}\right)\right]=0,$$

$$\Leftrightarrow \mu_E^-(\varepsilon_i)=\nu_E^-(\varepsilon_i) \text{ and } \mu_E^-(\varepsilon_i)=\nu_E^-(\varepsilon_i),\ \forall\ \varepsilon_i\in\Omega.$$

Conversely, if $\mu_E^-(\varepsilon_i)=\nu_E^-(\varepsilon_i)$ and $\mu_E^+(\varepsilon_i)=\nu_E^+(\varepsilon_i)$, then

$$\Im\left(E,E^c\right)=\frac{1}{4n\left(e^2-1\right)}\times\sum_{i=1}^{n}\left[\left((\mu_1^-(\varepsilon_i))^2-(\mu_1^-(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\mu_1^-(\varepsilon_i))^2}{1+(\mu_1^-(\varepsilon_i))^2+(\mu_1^-(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\mu_1^-(\varepsilon_i))^2}{1+(\mu_1^-(\varepsilon_i))^2+(\mu_1^-(\varepsilon_i))^2}\right)}\right)\right.$$

$$+\left((\mu_1^+(\varepsilon_i))^2-(\mu_1^+(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\mu_1^+(\varepsilon_i))^2}{1+(\mu_1^+(\varepsilon_i))^2+(\mu_1^+(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\mu_1^+(\varepsilon_i))^2}{1+(\mu_1^+(\varepsilon_i))^2+(\mu_1^+(\varepsilon_i))^2}\right)}\right)$$

$$+\left((\nu_1^-(\varepsilon_i))^2-(\nu_1^-(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\nu_1^-(\varepsilon_i))^2}{1+(\nu_1^-(\varepsilon_i))^2+(\nu_1^-(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\nu_1^-(\varepsilon_i))^2}{1+(\nu_1^-(\varepsilon_i))^2+(\nu_1^-(\varepsilon_i))^2}\right)}\right)$$

$$\left.+\left((\nu_1^+(\varepsilon_i))^2-(\nu_1^+(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\nu_1^+(\varepsilon_i))^2}{1+(\nu_1^+(\varepsilon_i))^2+(\nu_1^+(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\nu_1^+(\varepsilon_i))^2}{1+(\nu_1^+(\varepsilon_i))^2+(\nu_1^+(\varepsilon_i))^2}\right)}\right)\right],$$

it implies that $\Im\left(E,E^c\right)=0.$

(v) We have

$$\Im\left(E,E\cup F\right)=\frac{1}{4n\left(e^2-1\right)}\times\sum_{i=1}^{n}\left[\left((\mu_1^-(\varepsilon_i))^2-\max\left((\mu_1^-(\varepsilon_i))^2,(\mu_2^-(\varepsilon_i))^2\right)\right)\right.$$

$$\times\left(e^{\left(\frac{4(\mu_1^-(\varepsilon_i))^2}{1+(\mu_1^-(\varepsilon_i))^2+\max\left((\mu_1^-(\varepsilon_i))^2,(\mu_2^-(\varepsilon_i))^2\right)}\right)}-e^{\left(\frac{4\max\left((\mu_1^-(\varepsilon_i))^2,(\mu_2^-(\varepsilon_i))^2\right)}{1+(\mu_1^-(\varepsilon_i))^2+\max\left((\mu_1^-(\varepsilon_i))^2,(\mu_2^-(\varepsilon_i))^2\right)}\right)}\right)+\left((\mu_1^+(\varepsilon_i))^2-\max\begin{pmatrix}(\mu_1^+(\varepsilon_i))^2,\\(\mu_2^+(\varepsilon_i))^2\end{pmatrix}\right)$$

$$\times\left(e^{\left(\frac{4(\mu_1^+(\varepsilon_i))^2}{1+(\mu_1^+(\varepsilon_i))^2+\max\left((\mu_1^+(\varepsilon_i))^2,(\mu_2^+(\varepsilon_i))^2\right)}\right)}-e^{\left(\frac{4\max\left((\mu_1^+(\varepsilon_i))^2,(\mu_2^+(\varepsilon_i))^2\right)}{1+(\mu_1^+(\varepsilon_i))^2+\max\left((\mu_1^+(\varepsilon_i))^2,(\mu_2^+(\varepsilon_i))^2\right)}\right)}\right)+\left((\nu_1^-(\varepsilon_i))^2-\min\begin{pmatrix}(\nu_1^-(\varepsilon_i))^2,\\(\nu_2^-(\varepsilon_i))^2\end{pmatrix}\right)$$

$$\left(e^{\left(\frac{4(\nu_1^-(\varepsilon_i))^2}{1+(\nu_1^-(\varepsilon_i))^2+\min\left((\nu_1^-(\varepsilon_i))^2,(\nu_2^-(\varepsilon_i))^2\right)}\right)}-e^{\left(\frac{4\min\left((\nu_1^-(\varepsilon_i))^2,(\nu_2^-(\varepsilon_i))^2\right)}{1+(\nu_1^-(\varepsilon_i))^2+\min\left((\nu_1^-(\varepsilon_i))^2,(\nu_2^-(\varepsilon_i))^2\right)}\right)}\right)+\left((\nu_1^+(\varepsilon_i))^2-\min\begin{pmatrix}(\nu_1^+(\varepsilon_i))^2,\\(\nu_2^+(\varepsilon_i))^2\end{pmatrix}\right)$$

$$\left.\times\left(e^{\left(\frac{4(\nu_1^+(\varepsilon_i))^2}{1+(\nu_1^+(\varepsilon_i))^2+\min\left((\nu_1^+(\varepsilon_i))^2,(\nu_2^+(\varepsilon_i))^2\right)}\right)}-e^{\left(\frac{4\min\left((\nu_1^+(\varepsilon_i))^2,(\nu_2^+(\varepsilon_i))^2\right)}{1+(\nu_1^+(\varepsilon_i))^2+\min\left((\nu_1^+(\varepsilon_i))^2,(\nu_2^+(\varepsilon_i))^2\right)}\right)}\right)\right].$$

$$\Im\left(E\cap F,F\right)=\frac{1}{4n\left(e^2-1\right)}\times\sum_{i=1}^{n}\left[\left(\min\left((\mu_1^-(\varepsilon_i))^2,(\mu_2^-(\varepsilon_i))^2\right)-(\mu_2^-(\varepsilon_i))^2\right)\right.$$

$$\times\left(e^{\left(\frac{4\min\left((\mu_1^-(\varepsilon_i))^2,(\mu_2^-(\varepsilon_i))^2\right)}{1+\min\left((\mu_1^-(\varepsilon_i))^2,(\mu_2^-(\varepsilon_i))^2\right)+(\mu_2^-(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\mu_2^-(\varepsilon_i))^2}{1+\min\left((\mu_1^-(\varepsilon_i))^2,(\mu_2^-(\varepsilon_i))^2\right)+(\mu_2^-(\varepsilon_i))^2}\right)}\right)+\left(\min\begin{pmatrix}(\mu_1^+(\varepsilon_i))^2,\\(\mu_2^+(\varepsilon_i))^2\end{pmatrix}-(\mu_2^+(\varepsilon_i))^2\right)$$

$$\times\left(e^{\left(\frac{4\min\left((\mu_1^+(\varepsilon_i))^2,(\mu_2^+(\varepsilon_i))^2\right)}{1+\min\left((\mu_1^+(\varepsilon_i))^2,(\mu_2^+(\varepsilon_i))^2\right)+(\mu_2^+(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\mu_2^+(\varepsilon_i))^2}{1+\min\left((\mu_1^+(\varepsilon_i))^2,(\mu_2^+(\varepsilon_i))^2\right)+(\mu_2^+(\varepsilon_i))^2}\right)}\right)+\left(\max\begin{pmatrix}(\nu_1^-(\varepsilon_i))^2,\\(\nu_2^-(\varepsilon_i))^2\end{pmatrix}-(\nu_2^-(\varepsilon_i))^2\right)$$

$$\left(e^{\left(\frac{4\max\left((\nu_1^-(\varepsilon_i))^2,(\nu_2^-(\varepsilon_i))^2\right)}{1+\max\left((\nu_1^-(\varepsilon_i))^2,(\nu_2^-(\varepsilon_i))^2\right)+(\nu_2^-(\varepsilon_i))^2}\right)} - e^{\left(\frac{4(\nu_2^-(\varepsilon_i))^2}{1+\max\left((\nu_1^-(\varepsilon_i))^2,(\nu_2^-(\varepsilon_i))^2\right)+(\nu_2^-(\varepsilon_i))^2}\right)} \right) + \left(\max\begin{pmatrix}(\nu_1^+(\varepsilon_i))^2, \\ (\nu_2^+(\varepsilon_i))^2\end{pmatrix} - (\nu_2^+(\varepsilon_i))^2 \right)$$

$$\times \left(e^{\left(\frac{4\max\left((\nu_1^+(\varepsilon_i))^2,(\nu_2^+(\varepsilon_i))^2\right)}{1+\max\left((\nu_1^+(\varepsilon_i))^2,(\nu_2^+(\varepsilon_i))^2\right)+(\nu_2^+(\varepsilon_i))^2}\right)} - e^{\left(\frac{4(\nu_2^+(\varepsilon_i))^2}{1+\max\left((\nu_1^+(\varepsilon_i))^2,(\nu_2^+(\varepsilon_i))^2\right)+(\nu_2^+(\varepsilon_i))^2}\right)} \right)\Bigg]\Bigg].$$

If $E \subseteq F$, then $\mu_E^-(\varepsilon_i) \le \mu_F^-(\varepsilon_i)$, $\mu_E^+(\varepsilon_i) \le \mu_F^+(\varepsilon_i)$, $\nu_E^-(\varepsilon_i) \ge \nu_F^-(\varepsilon_i)$ and $\nu_E^+(\varepsilon_i) \ge \nu_F^+(\varepsilon_i)$, it implies that

$$\Im\left(E, E \cup F\right) = \frac{1}{4n\left(e^2-1\right)} \times$$

$$\sum_{i=1}^{n} \Bigg[\left((\mu_1^-(\varepsilon_i))^2 - (\mu_2^-(\varepsilon_i))^2\right) \times \left(e^{\left(\frac{4(\mu_1^-(\varepsilon_i))^2}{1+(\mu_1^-(\varepsilon_i))^2+(\mu_2^-(\varepsilon_i))^2}\right)} - e^{\left(\frac{4(\mu_2^-(\varepsilon_i))^2}{1+(\mu_1^-(\varepsilon_i))^2+(\mu_2^-(\varepsilon_i))^2}\right)} \right)$$

$$+ \left((\mu_1^+(\varepsilon_i))^2 - (\mu_2^+(\varepsilon_i))^2\right) \times \left(e^{\left(\frac{4(\mu_1^+(\varepsilon_i))^2}{1+(\mu_1^+(\varepsilon_i))^2+(\mu_2^+(\varepsilon_i))^2}\right)} - e^{\left(\frac{4(\mu_2^+(\varepsilon_i))^2}{1+(\mu_1^+(\varepsilon_i))^2+(\mu_2^+(\varepsilon_i))^2}\right)} \right)$$

$$+ \left((\nu_1^-(\varepsilon_i))^2 - (\nu_2^-(\varepsilon_i))^2\right) \times \left(e^{\left(\frac{4(\nu_1^-(\varepsilon_i))^2}{1+(\nu_1^-(\varepsilon_i))^2+(\nu_2^-(\varepsilon_i))^2}\right)} - e^{\left(\frac{4(\nu_2^-(\varepsilon_i))^2}{1+(\nu_1^-(\varepsilon_i))^2+(\nu_2^-(\varepsilon_i))^2}\right)} \right)$$

$$+ \left((\nu_1^+(\varepsilon_i))^2 - (\nu_2^+(\varepsilon_i))^2\right) \times \left(e^{\left(\frac{4(\nu_1^+(\varepsilon_i))^2}{1+(\nu_1^+(\varepsilon_i))^2+(\nu_2^+(\varepsilon_i))^2}\right)} - e^{\left(\frac{4(\nu_2^+(\varepsilon_i))^2}{1+(\nu_1^+(\varepsilon_i))^2+(\nu_2^+(\varepsilon_i))^2}\right)} \right) \Bigg]. \qquad (3.25)$$

$$\Im\left(E \cap F, F\right) = \frac{1}{4n\left(e^2-1\right)} \times$$

$$\sum_{i=1}^{n} \Big[\left((\mu_1^-(\varepsilon_i))^2 - (\mu_2^-(\varepsilon_i))^2\right) \times \left(e^{\left(\frac{4(\mu_1^-(\varepsilon_i))^2}{1+(\mu_1^-(\varepsilon_i))^2+(\mu_2^-(\varepsilon_i))^2}\right)} - e^{\left(\frac{4(\mu_2^-(\varepsilon_i))^2}{1+(\mu_1^-(\varepsilon_i))^2+(\mu_2^-(\varepsilon_i))^2}\right)} \right)$$

$$+\left((\mu_1^+(\varepsilon_i))^2-(\mu_2^+(\varepsilon_i))^2\right)\times\left(e^{\left(\frac{4(\mu_1^+(\varepsilon_i))^2}{1+(\mu_1^+(\varepsilon_i))^2+(\mu_2^+(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\mu_2^+(\varepsilon_i))^2}{1+(\mu_1^+(\varepsilon_i))^2+(\mu_2^+(\varepsilon_i))^2}\right)}\right)$$

$$+\left((\nu_1^-(\varepsilon_i))^2-(\nu_2^-(\varepsilon_i))^2\right)\times\left(e^{\left(\frac{4(\nu_1^-(\varepsilon_i))^2}{1+(\nu_1^-(\varepsilon_i))^2+(\nu_2^-(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\nu_2^-(\varepsilon_i))^2}{1+(\nu_1^-(\varepsilon_i))^2+(\nu_2^-(\varepsilon_i))^2}\right)}\right)$$

$$+\left((\nu_1^+(\varepsilon_i))^2-(\nu_2^+(\varepsilon_i))^2\right)\times\left(e^{\left(\frac{4(\nu_1^+(\varepsilon_i))^2}{1+(\nu_1^+(\varepsilon_i))^2+(\nu_2^+(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\nu_2^+(\varepsilon_i))^2}{1+(\nu_1^+(\varepsilon_i))^2+(\nu_2^+(\varepsilon_i))^2}\right)}\right)\Bigg]. \qquad (3.26)$$

From Eq. (3.25) and Eq. (3.26), we get $\Im(E, E\cup F)=\Im(E\cap F, F)=\Im(E,F)$.

Similarly, if $F\subseteq E$, then $\mu_F^-(\varepsilon_i)\le\mu_E^-(\varepsilon_i)$, $\mu_F^+(\varepsilon_i)\le\mu_E^+(\varepsilon_i)$, $\nu_F^-(\varepsilon_i)\ge\nu_E^-(\varepsilon_i)$ and $\nu_F^+(\varepsilon_i)\ge\nu_E^+(\varepsilon_i)$, it implies that $\Im(E, E\cup F)=\Im(E\cap F, F)=\Im(E,F)$.

(vi) $\Im(E\cap F, E\cup F)=$

$$\frac{1}{4n\left(e^2-1\right)}\times\sum_{i=1}^{n}\Bigg[\left(\min\left((\mu_1^-(\varepsilon_i))^2,(\mu_2^-(\varepsilon_i))^2\right)-\max\left((\mu_1^-(\varepsilon_i))^2,(\mu_2^-(\varepsilon_i))^2\right)\right)$$

$$\times\left(e^{\left(\frac{4\min\left((\mu_1^-(\varepsilon_i))^2,(\mu_2^-(\varepsilon_i))^2\right)}{1+\min\left((\mu_1^-(\varepsilon_i))^2,(\mu_2^-(\varepsilon_i))^2\right)+\max\left((\mu_1^-(\varepsilon_i))^2,(\mu_2^-(\varepsilon_i))^2\right)}\right)}-e^{\left(\frac{4\max\left((\mu_1^-(\varepsilon_i))^2,(\mu_2^-(\varepsilon_i))^2\right)}{1+\min\left((\mu_1^-(\varepsilon_i))^2,(\mu_2^-(\varepsilon_i))^2\right)+\max\left((\mu_1^-(\varepsilon_i))^2,(\mu_2^-(\varepsilon_i))^2\right)}\right)}\right)$$

$$+\left(\min\left((\mu_1^+(\varepsilon_i))^2,(\mu_2^+(\varepsilon_i))^2\right)-\max\left((\mu_1^+(\varepsilon_i))^2,(\mu_2^+(\varepsilon_i))^2\right)\right)$$

$$\times\left(e^{\left(\frac{4\min\left((\mu_1^+(\varepsilon_i))^2,(\mu_2^+(\varepsilon_i))^2\right)}{1+\min\left((\mu_1^+(\varepsilon_i))^2,(\mu_2^+(\varepsilon_i))^2\right)+\max\left((\mu_1^+(\varepsilon_i))^2,(\mu_2^+(\varepsilon_i))^2\right)}\right)}-e^{\left(\frac{4\max\left((\mu_1^+(\varepsilon_i))^2,(\mu_2^+(\varepsilon_i))^2\right)}{1+\min\left((\mu_1^+(\varepsilon_i))^2,(\mu_2^+(\varepsilon_i))^2\right)+\max\left((\mu_1^+(\varepsilon_i))^2,(\mu_2^+(\varepsilon_i))^2\right)}\right)}\right)$$

$$+\left(\max\left((\nu_1^-(\varepsilon_i))^2,(\nu_2^-(\varepsilon_i))^2\right)-\min\left((\nu_1^-(\varepsilon_i))^2,(\nu_2^-(\varepsilon_i))^2\right)\right)$$

$$\times\left(e^{\left(\frac{4\max\left((\nu_1^-(\varepsilon_i))^2,(\nu_2^-(\varepsilon_i))^2\right)}{1+\max\left((\nu_1^-(\varepsilon_i))^2,(\nu_2^-(\varepsilon_i))^2\right)+\min\left((\nu_1^-(\varepsilon_i))^2,(\nu_2^-(\varepsilon_i))^2\right)}\right)}-e^{\left(\frac{4\min\left((\nu_1^-(\varepsilon_i))^2,(\nu_2^-(\varepsilon_i))^2\right)}{1+\max\left((\nu_1^-(\varepsilon_i))^2,(\nu_2^-(\varepsilon_i))^2\right)+\min\left((\nu_1^-(\varepsilon_i))^2,(\nu_2^-(\varepsilon_i))^2\right)}\right)}\right)$$

$$+\left(\max\left((\nu_1^+(\varepsilon_i))^2,(\nu_2^+(\varepsilon_i))^2\right)-\min\left((\nu_1^+(\varepsilon_i))^2,(\nu_2^+(\varepsilon_i))^2\right)\right)$$

$$\times\left(e^{\left(\frac{4\max\left((\nu_1^+(\varepsilon_i))^2,(\nu_2^+(\varepsilon_i))^2\right)}{1+\max\left((\nu_1^+(\varepsilon_i))^2,(\nu_2^+(\varepsilon_i))^2\right)+\min\left((\nu_1^+(\varepsilon_i))^2,(\nu_2^+(\varepsilon_i))^2\right)}\right)}-e^{\left(\frac{4\min\left((\nu_1^+(\varepsilon_i))^2,(\nu_2^+(\varepsilon_i))^2\right)}{1+\max\left((\nu_1^+(\varepsilon_i))^2,(\nu_2^+(\varepsilon_i))^2\right)+\min\left((\nu_1^+(\varepsilon_i))^2,(\nu_2^+(\varepsilon_i))^2\right)}\right)}\right)\Bigg].$$

If $E\subseteq F$, then $\mu_E^-(\varepsilon_i)\leq\mu_F^-(\varepsilon_i)$, $\mu_E^+(\varepsilon_i)\leq\mu_F^+(\varepsilon_i)$, $\nu_E^-(\varepsilon_i)\geq\nu_F^-(\varepsilon_i)$ and $\nu_E^+(\varepsilon_i)\geq\nu_F^+(\varepsilon_i)$, it implies that

$$\Im\left(E\cap F,E\cup F\right)=\frac{1}{4n\left(e^2-1\right)}\times\sum_{i=1}^{n}\Bigg[\left((\mu_1^-(\varepsilon_i))^2-(\mu_2^-(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\mu_1^-(\varepsilon_i))^2}{1+(\mu_1^-(\varepsilon_i))^2+(\mu_2^-(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\mu_2^-(\varepsilon_i))^2}{1+(\mu_1^-(\varepsilon_i))^2+(\mu_2^-(\varepsilon_i))^2}\right)}\right)$$

$$+\left((\mu_1^+(\varepsilon_i))^2-(\mu_2^+(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\mu_1^+(\varepsilon_i))^2}{1+(\mu_1^+(\varepsilon_i))^2+(\mu_2^+(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\mu_2^+(\varepsilon_i))^2}{1+(\mu_1^+(\varepsilon_i))^2+(\mu_2^+(\varepsilon_i))^2}\right)}\right)$$

$$+\left((\nu_1^-(\varepsilon_i))^2-(\nu_2^-(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\nu_1^-(\varepsilon_i))^2}{1+(\nu_1^-(\varepsilon_i))^2+(\nu_2^-(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\nu_2^-(\varepsilon_i))^2}{1+(\nu_1^-(\varepsilon_i))^2+(\nu_2^-(\varepsilon_i))^2}\right)}\right)$$

$$+\left((\nu_1^+(\varepsilon_i))^2-(\nu_2^+(\varepsilon_i))^2\right)\left(e^{\left(\frac{4(\nu_1^+(\varepsilon_i))^2}{1+(\nu_1^+(\varepsilon_i))^2+(\nu_2^+(\varepsilon_i))^2}\right)}-e^{\left(\frac{4(\nu_2^+(\varepsilon_i))^2}{1+(\nu_1^+(\varepsilon_i))^2+(\nu_2^+(\varepsilon_i))^2}\right)}\right)\Bigg].$$

Thus, $\Im\left(E\cap F,E\cup F\right)=\Im\left(E,F\right)$.

(vii)-(viii) If $E\subseteq F\subseteq G$, then obviously $\Im\left(E,F\right)\leq\Im\left(E,G\right)$ and $\Im\left(F,G\right)\leq\Im\left(E,G\right)$.

3.5. Comparison with Existing IVPF-Divergence Measures

This section compares the proposed divergence measure with some of the existing divergence measures, presented by (Kumar et al., 2020; Mishra et al., 2022; Al-Barakati et al., 2022).

Kumar et al. (2020):

$$d_K\left(\psi_1,\psi_2\right)=\sqrt[p]{\frac{1}{4n}\sum_{i=1}^{n}\begin{pmatrix}\left|(\mu_1^-(\varepsilon_i))^2-(\mu_2^-(\varepsilon_i))^2\right|^p+\left|(\mu_1^+(\varepsilon_i))^2-(\mu_2^+(\varepsilon_i))^2\right|^p+\\ \left|(\nu_1^-(\varepsilon_i))^2-(\nu_2^-(\varepsilon_i))^2\right|^p+\left|(\nu_1^+(\varepsilon_i))^2-(\nu_2^+(\varepsilon_i))^2\right|^p+\\ \left|(\pi_1^-(\varepsilon_i))^2-(\pi_2^-(\varepsilon_i))^2\right|^p+\left|(\pi_1^+(\varepsilon_i))^2-(\pi_2^+(\varepsilon_i))^2\right|^p\end{pmatrix}}.$$

Mishra et al. (2022):

$$S_M\left(\psi_1,\psi_2\right)=\frac{1}{6n}\sum\Big[2\left(\mu_1^-\left(\varepsilon_i\right)\mu_2^-\left(\varepsilon_i\right)+\mu_1^+\left(\varepsilon_i\right)\mu_2^+\left(\varepsilon_i\right)+\nu_1^-\left(\varepsilon_i\right)\nu_2^-\left(\varepsilon_i\right)+\nu_1^+\left(\varepsilon_i\right)\nu_2^+\left(\varepsilon_i\right)\right)$$

$$+\pi_1^-\left(\varepsilon_i\right)\pi_2^-\left(\varepsilon_i\right)+\pi_1^+\left(\varepsilon_i\right)\pi_2^+\left(\varepsilon_i\right)+\sqrt{\left(1-\mu_1^{-2}\left(\varepsilon_i\right)\right)\left(1-\mu_2^{-2}\left(\varepsilon_i\right)\right)}+\sqrt{\left(1-(\mu_1^+(\varepsilon_i))^2\right)\left(1-(\mu_2^{+2}(\varepsilon_i))^2\right)}$$

$$+\sqrt{\left(1-(\nu_1^-(\varepsilon_i))^2\right)\left(1-(\nu_2^-(\varepsilon_i))^2\right)}+\sqrt{\left(1-(\nu_1^+(\varepsilon_i))^2\right)\left(1-(\nu_2^+(\varepsilon_i))^2\right)}\Big],$$

$$d_M\left(\psi_1,\psi_2\right)=1-S_M\left(\psi_1,\psi_2\right)$$

Al-Barakati et al. (2022):

$$S_1\left(A,B\right)=\frac{1}{4n}\sum\Big[\mu_A^-\left(z_i\right)\mu_B^-\left(z_i\right)+\mu_A^+\left(z_i\right)\mu_B^+\left(z_i\right)+2\left(\nu_A^-\left(z_i\right)\nu_B^-\left(z_i\right)+\nu_A^+\left(z_i\right)\nu_B^+\left(z_i\right)\right)$$

$$+\pi_A^-\left(z_i\right)\pi_B^-\left(z_i\right)+\pi_A^+\left(z_i\right)\pi_B^+\left(z_i\right)+\sqrt{\left(1-\nu_A^{-2}\left(z_i\right)\right)\left(1-\nu_B^{-2}\left(z_i\right)\right)}+\sqrt{\left(1-\nu_A^{+2}\left(z_i\right)\right)\left(1-\nu_B^{+2}\left(z_i\right)\right)}\Big],$$

$$d_1\left(E,F\right)=1-S_1\left(E,F\right).$$

Here, Table 3.3 demonstrates a comparison of divergence measures for IVPFSs with different counter-intuitive examples.

Table 3.3: Comparison results between proposed and existing divergence measures

	A	$d_k(\mathrm{A},\mathrm{B})$	$d_M(\mathrm{A},\mathrm{B})$	$d_1(\mathrm{A},\mathrm{B})$	Proposed

	B				
Case-I	$([0.26,0.36],[0.26,0.36])$ $([0.36,0.46],[0.36,0.46])$	0.0288	0.0485	0.2085	0.0113
Case-II	$([0.26,0.36],[0.36,0.46])$ $([0.36,0.46],[0.26,0.36])$	0.0192	0.0480	0.2042	0.0074
Case-III	$([1.00,1.00],[0.00,0.00])$ $([0.00,0.00],[1.00,1.00])$	0.5067	0.4872	0.5067	0.4259
Case-IV	$([1.00,1.00],[0.00,0.00])$ $([0.00,0.00],[0.00,0.00])$	**0.0358**	**0.0654**	**0.1991**	0.0022
Case-V	$([0.50,0.0],[0.50,0.50])$ $([0.00,0.00],[0.00,0.00])$	**0.0358**	**0.0654**	**0.1991**	0.0020
Case-VI	$([0.36,0.46],[0.16,0.26])$ $([0.46,0.56],[0.26,0.36])$	0.0406	**0.0933**	0.2323	0.0210
Case-VII	$([0.36,0.46],[0.16,0.26])$ $([0.46,0.56],[0.16,0.26])$	0.0384	**0.0933**	0.2336	0.0207

The divergence and distance measures that are evaluated by different cases are depicted in Table 3.3. By comparing Table 3.3, it can be seen that the proposed divergence measure between the IVPFNs can overcome the drawbacks of the existing divergence measures (Kumar et al., 2020; Mishra et al., 2022; Al-Barakati et al., 2022) between the IVPFNs. By means of the obtained results, we get some interesting outcomes. Finally, it is worth mentioned that the developed divergence measure provides reasonable results under considered sets, whilst extant measures generate some counter-intuitive cases.

CHAPTER 4

PROPOSED INTERVAL-VALUED PYTHAGOREAN FUZZY DECISION-MAKING MODEL

4.1. Introduction

The MULTIMOORA (MULTIplicative Multi-Objective Optimization by Ratio Analysis) is an effective model in handling MCDM problems, which combines the ratio system (RS) and reference point (RP) model with the full multiplicative form (FMF). It has more superiority, easy mathematical expressions, less computation time, good stability and strong robustness over the other existing approaches.

The rest sections of this chapter are organized as follows: Section 4.2 defines the decision-making model from multiple criteria and IVPF information perspective. Section 4.3 proposes an integrated weighting model by combining the IVPF-entropy measure based formula for objective weight and IVPF-SWARA based model for subjective weight of criteria under the context of IVPFSs. Based on these two sections, Section 4.4 develops a hybrid MULTIMOORA method under IVPFS environment.

4.2. Defining the IVPF-Decision Matrix

Step 1: Obtain the "IVPF-decision matrix (IVPF-DM)."

For the MCDM process, consider a set of alternatives $R=\{R_1,R_2,...,R_m\}$ over the set of criteria $C=\{C_1,C_2,...,C_n\}$.The "decision expert (DE)" provides his/her assessments λ_{ij} of options $R_i\left(i=1(1)m\right)$ over attribute $C_j\left(j=1(1)n\right)$ in term of "linguistic decision-matrix (LDM)".

Step 2: Estimate the DEs' weights.

Consider a set of DEs $E=\{E_1,E_2,...,E_l\}$ with important weight $\psi=\left(\psi_1,\psi_2,...,\psi_\ell\right)^T$. The DE's weight is obtained as "linguistic values (LVs)" and presented by IVPFNs. Let $E_k=\left(\left[\mu_k^-,\ \mu_k^+\right],\left[\nu_k^-,\nu_k^+\right]\right), k=1,2,...,l$ be the IVPFN, represents the significance of k^{th} DE. Therefore, the weight of k^{th} DE is obtained as (Rahimi et al., 2022)

$$\psi_k=\frac{\left(\left(\mu_k^-\right)^2+\left(\mu_k^+\right)^2\right)\left(2+\left(\pi_k^-\right)^2+\left(\pi_k^+\right)^2\right)}{\sum_{k=1}^{\ell}\left(\left(\left(\mu_k^-\right)^2+\left(\mu_k^+\right)^2\right)\left(2+\left(\pi_k^-\right)^2+\left(\pi_k^+\right)^2\right)\right)}, \tag{4.1}$$

where $\psi_k\geq 0$ and $\sum_{k=1}^{l}\psi_k=1$.

Step 3: Obtain the "aggregated-IVPF-DM (AIVPF-DM)".

Let $Z=\left(z_{ij}^{(k)}\right)$ be the "linguistic decision-matrix (LDM)" of k^{th} expert. To combine all the distinct LDMs, we use the "IVPF-averaging aggregation operator" and create an AIVPF-DM□ $=\left[z_{ij}\right]_{m\times n}$,

where $z_{ij}=\left(\left[\mu_{ij}^-,\ \mu_{ij}^+\right],\left[\nu_{ij}^-,\ \nu_{ij}^+\right]\right)$ such that

$$z_{ij} = \sum_{k=1}^{\ell} \psi_k z_{ij}^{(k)} = \left(\left[\sqrt{1 - \prod_{k=1}^{\ell} \left(1 - \left(\mu_{ijk}^{-}\right)^2\right)^{\psi_k}}, \sqrt{1 - \prod_{k=1}^{\ell} \left(1 - \left(\mu_{ijk}^{+}\right)^2\right)^{\psi_k}} \right], \right. \tag{4.2}$$

$$\left. \left[\prod_{k=1}^{\ell} \left(\nu_{ijk}^{-}\right)^{\psi_k}, \prod_{k=1}^{\ell} \left(\nu_{ijk}^{+}\right)^{\psi_k} \right] \right).$$

On similar line, we use the "IVPF-geometric aggregation operator" and create an AIVPF-DM $= \left[z_{ij} \right]_{m \times n}$, where $z_{ij} = \left(\left[\mu_{ij}^{-}, \mu_{ij}^{+} \right], \left[\nu_{ij}^{-}, \nu_{ij}^{+} \right] \right)$ such that

$$z_{ij} = \sum_{k=1}^{\ell} \psi_k z_{ij}^{(k)} = \left(\left[\prod_{k=1}^{\ell} \left(\mu_{ijk}^{-}\right)^{\psi_k}, \prod_{k=1}^{\ell} \left(\mu_{ijk}^{+}\right)^{\psi_k} \right], \right. \tag{4.3}$$

$$\left. \left[\sqrt{1 - \prod_{k=1}^{\ell} \left(1 - \left(\nu_{ijk}^{-}\right)^2\right)^{\psi_k}}, \sqrt{1 - \prod_{k=1}^{\ell} \left(1 - \left(\nu_{ijk}^{+}\right)^2\right)^{\psi_k}} \right] \right).$$

4.3. Estimation of Criteria Weights

During the process of multi-attribute decision analysis (MADA), one of the most challenging issues for decision makers (DMs) is the assessment of criteria importance degrees. Lots of procedures have been suggested by different scholars for the computation of the criteria weights. In a general classification, criteria weights are determined with two different approaches: objective and subjective.

For subjective weights of criteria, Stillwell et al. (1981) pioneered an effective RS model, which can successfully help the DEs in the ranking of criteria importance degrees. Narayanamoorthy et al. (2020) proposed a collective weighting procedure with CRITIC and RS models for assessing the significant indicators in bio-medical waste disposal methods. Based on the proposed weight-determining model, they suggested a hybrid hesitant fuzzy-information based approach for assessing the bio-medical waste disposal technologies. Recently, Hezam et al. (2022) discussed an

IF-information based RS model with the purpose of evaluating the sustainability criteria in the assessment of alternative fuel vehicles.

Step 1: Determine the criteria weights by integrated weighting model.

To compute the importance degree of each criterion, we develop a procedure by using the proposed IVPF-entropy and divergence measures. Consider that the importance degree of each criterion is different. Suppose $w=\left(w_1, w_2, \ldots, w_n\right)^T$ with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$ is a weight vector of the criterion set. To calculate the value of w, we present the following steps:

Step 2: Compute the objective weight of each criterion.

To compute the objective weights of criteria, an approach based on the entropy and divergence measures is presented, which is given as follows:

$$w_j^o = \frac{\sum_{i=1}^{m}\left(1-\bar{E}\left(z_{ij}\right)\right)}{\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\left(1-\bar{E}\left(z_{ij}\right)\right)\right)}, \tag{4.4}$$

where $\bar{E}\left(z_{ij}\right)=E\left(z_{ij}\right)/\max_{i=1,\ldots,m} E\left(z_{ij}\right),\ j=1,2,\ldots,n,$

signifies the proposed IVPF-entropy $E\left(z_{ij}\right)$ given in Eq. (3.1).

Step 3: Compute the subjective weight of each criterion.

First, we estimate the AIVPF-DM based on the linguistic assessment degrees provided by the DEs through the IVPFWA operator and obtained given as

$$N=\left(z_j\right)_{1\times n}=IVPFWA_{\vartheta_k}\left(z_j^{(1)}, z_j^{(2)}, \ldots, z_j^{(l)}\right),$$

or

$$N=\left(z_j\right)_{1\times n}=IVPFWG_{\vartheta_k}\left(z_j^{(1)}, z_j^{(2)}, \ldots, z_j^{(l)}\right).$$

Next, compute the IVPF-score value of AIVPF-DM for DT challenges using (1.15) and given as follows:

$$\bar{\eta}_j=\frac{1}{2}\left(\mathrm{S}\left(z_j\right)+1\right).$$

The subjective weighting procedure allows showing the opinions and assessment ratings of DEs. The procedure of MCDM and the DEs' opinion of each option over different criteria play important roles in choosing the best option for the given problem. In this regard, the DE provides their subjective rating (Stillwell et al., 1981; Hezam et al., 2022). Now, based on the IVPF-score value of AIVPF-DM for DT challenges, we determine the rank of DT challenges and develop the rank sum method (RSM) for obtaining the subjective weight of DT challenges is

$$w_j^s = \frac{n - r_j + 1}{\sum_{j=1}^{n} \left(n - r_j + 1\right)}, \tag{4.5}$$

where r_j means the rank of each attribute, j=1, 2, 3, ..., n.

Step 4: Find the combined weight of each criterion.

With the use of Eq. (4.4) and Eq. (4.5), the combined weighting formula for each criterion is given as follows:

$$w_j = \gamma\, w_j^s + \left(1 - \gamma\right) w_j^o, \tag{4.6}$$

where γ is the weight-determining coefficient and $\gamma \in [0, 1]$.

4.4. Proposed IVPF-MULTIMOORA Ranking Model

During the last decade, numerous new methodologies have been proposed to deal with real MCDM issues, where each of them has its own benefits and limitations (He et al., 2021). Brauers and Zavadskas (2010) pioneered the Multi-Objective Optimization based on Ratio Analysis plus the full multiplicative form (MULTIMOORA) procedure, which integrates three aggregation models, namely ratio system (RS), reference point (RP), and the Full Multiplicative Form (FMF). In comparison with numerous extant models, the MULTIMOORA method has some advantages such as easier mathematical terminologies, less complexity and higher robustness (Brauers and

Zavadskas, 2011, 2012). Due to its unique advantages over other MCDM methods, the classical MULTIMOORA method has been employed to solve different realistic problems (Stankevičienė et al., 2019; Rani and Mishra, 2021). However, there is no research regarding examining the digital transformation challenges in sustainable financial service systems of the banking sector by utilizing the MULTIMOORA approach under IVPFSs settings.

The steps for the IVPF-Entropy-RS-MULTIMOORA (see Fig. 4.1) approach are discussed by

Step 1: Assess the preferences of alternatives using the RS model.

The following sub-steps show the evaluation of an optimal option using the RS model:

Step 1.1: Compute Y_i^+ and Y_i^- by the IVPFWA operator as follows:

$$Y_i^+ = \left(\left[\sqrt{1 - \prod_{j \in C_b} \left(1 - \left(\mu_{ij}^-\right)^2\right)^{w_j}}, \sqrt{1 - \prod_{j \in C_b} \left(1 - \left(\mu_{ij}^+\right)^2\right)^{w_j}} \right], \left[\prod_{j \in C_b} \left(\nu_{ij}^-\right)^{w_j}, \prod_{j \in C_b} \left(\nu_{ij}^+\right)^{w_j} \right] \right), \tag{4.7}$$

$$Y_i^- = \left(\left[\sqrt{1 - \prod_{j \in C_n} \left(1 - \left(\mu_{ij}^-\right)^2\right)^{w_j}}, \sqrt{1 - \prod_{j \in C_n} \left(1 - \left(\mu_{ij}^+\right)^2\right)^{w_j}} \right], \left[\prod_{j \in C_n} \left(\nu_{ij}^-\right)^{w_j}, \prod_{j \in C_n} \left(\nu_{ij}^+\right)^{w_j} \right] \right), \tag{4.8}$$

where Y_i^+ and Y_i^- signify the significance values of the option with the benefit and cost attributes, and C_b and C_n represent the benefit and cost type challenges.

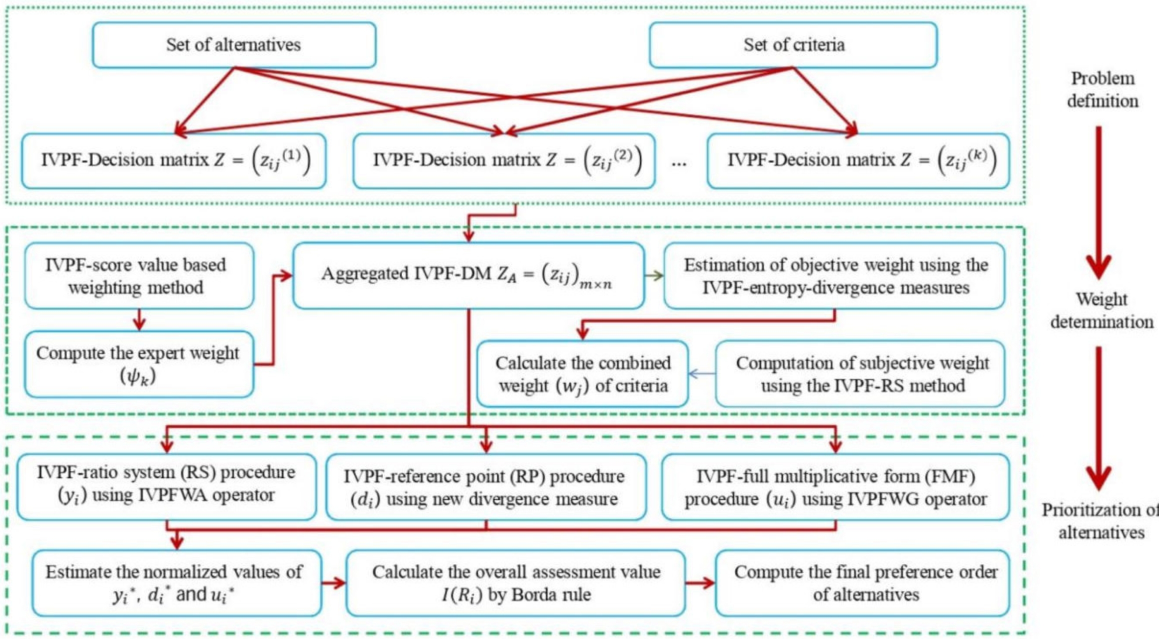

Fig. 4.1: Flowchart of developed IVPF-entropy-divergence-rank sum-MULTIMOORA model

Step 1.2: Compute the y_i^+ and y_i^- by the IVPF-score degrees using Eq. (1.15) and obtained as follows:

$$y_i^+ = S\left(Y_i^+\right) \text{ and } y_i^- = S\left(Y_i^-\right) \tag{4.9}$$

Step 1.3: Estimate the final utility degree for each alternative as

$$y_i = y_i^+ - y_i^-. \tag{4.10}$$

Step 2: Assess the preference of alternative using the RP model.

The following steps include the ranking of the options to find the optimal one using the RP procedure:

Step 2.1: Compute the reference point. The coordinate value of the RP $r_j^* = \{r_1^*, r_2^*, ..., r_n^*\}$ is an IVPFN r_j^* is defined as 1 and 0, and is computed by using:

$$r_j^* = \langle[1,1],[0,0]\rangle_{1\times n}. \tag{4.11}$$

Step 2.2: Estimate the distances from the alternatives to all the coordinates of the RP as

$$D_{ij} = w_j\left(D\left(z_{ij}, r_j^*\right)\right) \tag{4.12}$$

where D signifies the proposed IVPF-divergence measure obtained by Eq. (3.18).

Step 2.3: Evaluate the highest distance of each alternative as follows:

$$d_i = \max_j D_{ij}, i = 1(1)m. \tag{4.13}$$

Step 3: Choose the preference of alternatives based on FMF model.

The following steps include the ranking of the options and assess the optimal one using the FMF procedure:

Step 3.1: Compute A_i and B_i by IVPFWG operator as

$$A_i = \left(\left[\prod_{j\in C_b}\left(\mu_{ij}^-\right)^{w_j}, \prod_{j\in C_b}\left(\mu_{ij}^+\right)^{w_j}\right], \left[\sqrt{1-\prod_{j\in C_b}\left(1-\left(\nu_{ij}^-\right)^2\right)^{w_j}}, \sqrt{1-\prod_{j\in C_b}\left(1-\left(\nu_{ij}^+\right)^2\right)^{w_j}}\right]\right), \tag{4.14}$$

$$B_i = \left(\left[\prod_{j\in C_n}\left(\mu_{ij}^-\right)^{w_j}, \prod_{j\in C_n}\left(\mu_{ij}^+\right)^{w_j}\right], \left[\sqrt{1-\prod_{j\in C_n}\left(1-\left(\nu_{ij}^-\right)^2\right)^{w_j}}, \sqrt{1-\prod_{j\in C_n}\left(1-\left(\nu_{ij}^+\right)^2\right)^{w_j}}\right]\right), \tag{4.15}$$

where A_i and B_i are IVPFNs. C_b and C_n represent the benefit and cost type challenges

Step 3.2: Estimate α_i and β_i by score function as

$$\alpha_i = \mathrm{S}^*\left(A_i\right) \text{ and } \beta_i = \mathrm{S}^*\left(B_i\right) \tag{4.16}$$

Step 3.3: Assess the overall utility for each alternative as

$$u_i = \frac{\alpha_i}{\beta_i}. \tag{4.17}$$

Step 4: Decide the final preference order of alternatives.

First, let y_i^*, d_i^*, and u_i^*, be the normalized values of RS, RP and FMF, respectively, using the vector normalization. Then the overall assessment degree (OAD) of alternative by Improved Borda Rule is given by

$$I_B\left(R_i\right)=y_i^* \cdot \frac{m-\rho\left(y_i^*\right)+1}{\left(m\left(m+1\right)/2\right)}-d_i^* \cdot \frac{\rho\left(d_i^*\right)}{\left(m\left(m+1\right)/2\right)}+u_i^* \cdot \frac{m-\rho\left(u_i^*\right)+1}{\left(m\left(m+1\right)/2\right)}, \tag{4.18}$$

where $y_i^*=\frac{y_i}{\sqrt{\sum_{i=1}^m\left(y_i\right)^2}}$, $d_i^*=\frac{d_i}{\sqrt{\sum_{i=1}^m\left(d_i\right)^2}}$, $u_i^*=\frac{u_i}{\sqrt{\sum_{i=1}^m\left(u_i\right)^2}}$ are the normalized assessment degree of RS, RP and FMF approaches and determined using Eq. (4.10), Eq. (4.13) and Eq. (4.17), respectively. $\rho\left(y_i^*\right)$, $\rho\left(d_i^*\right)$, and $\rho\left(u_i^*\right)$ are ranking order of RS, RP and FMF approaches, respectively. The optimal option has the highest value of $I_B\left(R_i\right)$, $i=1,2,\ldots,m.$

CHAPTER 5

IMPLEMENTATION OF THE PROPOSED MODEL FOR EVALUATING THE CHALLENGES OF DIGITAL TRANSFORMATION

5.1. Introduction

Because of the impreciseness of knowledge, the vagueness of human beings' minds, time limitations, and the absence of required information, the assessment, and ranking of suitable DT challenges in "sustainable financial service systems (SFSSs)" of the banking sector is an important and uncertain MCDM issue faced by the hospitals and medical centers. In addition, the fuzziness has widely occurred in the SFSSs during the process of DT. As an extension of fuzzy set, the concept of IVPFSs has a more valuable and novel tool to deal with the uncertainty of real-life problems. This motivates to consider the IVPFSs context for the assessment of DT challenges in SFSSs of the banking sector. Moreover, the literature lacks research studies on SFSS from theoretical, managerial, and societal perspectives, and no study has yet systematically identified the research challenges related to this domain. The present work attempts to determine the key challenges that may arise with the DT of sustainable financial services based on the questionnaire and data collection. Here, it provides a realistic contribution by identifying 18 key challenges related to the DT of financial services. This new conceptualization of challenges related to sustainable financial service systems (SFSSs) has contributed to the current discourse in regard to the role of ICT in financial services in general and the influence of DT upon these services in particular.

The rest part of this chapter is arranged as follows: Section 5.2 discusses the various challenges of digital transformation. Section 5.3 presents the process of questionnaire and data collection during the implementation of digital technologies. Section 5.4 implements the IVPF-Entropy-divergence-RS-MULTIMOORA methods on a case study of digital transformation challenges in sustainable financial service systems of the banking sector.

5.2. Various Challenges of Digital Transformation

Digitalization has made important contributions to the sustainable development objectives of the United Nations. Only the transformation of prevailing businesses could help to solve the economic and environmental challenges of the future in a sustainable way (Bican and Brem, 2020). Digital transformation generates novel social groups- partly human, semi-human, or non-human. Some of these groups already exist, and some could be predicted to exist soon because of the latest advances in fields such as software engineering, brain wearables, and robotics. The growth of our dependency on digital services and tools could bring about a number of challenges to both organizations and human beings (Fekete and Rhyner, 2020; Forcadell et al., 2020). Based on their findings, firms can achieve sustainability by effectively considering the customers, data processing, and innovation. However, they did not prove the considerable role of competition in the enhancement of the firms' commitment to sustainability. It was also endorsed partly in the study of Ordieres-Meré et al. (2020), where they showed the positive impacts of knowledge creation that could be facilitated by directly or indirectly applying digitalization. Technology has been found a factor capable of disrupting the financial industry, solving friction points for customers and businesses, and injecting more resilience and sustainability into the overall business. In addition, sustainable financial technology could have a great contribution to the stability of the financial system (Moro-Visconti et al., 2020). According to above discussions and current literature review, to evaluate the DT in SFSSs, 18 challenges are identified, i.e., understanding of the customers through big financial data (c_1), knowledge for open data for value co-creation (c_2), understanding changing role of traditional financial intermediaries (c_3), integrating the multi-platform services (c_4), improving platform orchestration (c_5), investigate the platforms and markets (c_6), managing the experience and quality of digital financial services (c_7), knowledge for new value creating resource configurations (c_8), understanding hybrid business models (c_9), facilitating the co-creation of value without intermediaries (c_{10}), understanding value co-creation with cryptocurrencies (c_{11}), adaptable infrastructure of financial services for cryptocurrencies (c_{12}), regulating value co-creation without intermediaries (c_{13}), rules and regulations of financial institutions (c_{14}), managing the deregulation of financial service systems (c_{15}), designing customer-centric fintech services (c_{16}), designing communities of practice (c_{17}) and developing support systems (c_{18}).

5.3. Questionnaire and Data Collection

In this study, to identify and evaluate the main digital transformation challenges in the financial service sector, this study conducted a survey approach using the current literature review and interviews with experts. To do so, in the first stage, to identify the main digital transformation challenges, a comprehensive list including 29 challenges is collected from current literature. In the next stage, this list of challenges is presented in the questionnaire format and sent to 30 experts in digitalization and finance who work in different universities using their academic emails. To identify these 30 experts, we have searched through the published papers in the areas of digital transformation, digitalization, and financial service in Google Scholar. Before sending the questionnaire to those experts, we invited them to the participants by their email; from these 30 experts, 22 of them accepted our invitations to participate in our survey study. In the next stage, we sent the questionnaire in a word file format with blank space to provide their opinions. After three weeks, we sent the reminders to those experts to provide their feedback; therefore, after a few days, we received 16 questionnaires that experts completed and provided their feedback. We repeated this reminder after another two weeks, and finally, we could collect 22 questionnaires. In the primary questionnaire, we identified 29 digital transformation challenges; after analyzing all questionnaires, we selected 18 digital transformation challenges. In the next round of our survey approach and to evaluate these selected challenges, we have conducted the second round of invitations. In this round of data collection, we have invited experts from industry and academic who are experts in financial and digital transformation with several years of experience. The main industry experts work in IT, financial, accounting and computing areas from five bank financial sectors. At this stage, we have invited four experts from industry and four experts from academics. To collect the data for evaluation and analysis, we conducted in-person interviews with these experts; although we invited eight experts, only five agreed to collaborate with us to evaluate the questionnaires. Based on the articles published by the digital transformation challenges in sustainable financial service systems domain-specific publishers, five sustainable financial service systems are identified as climate finance (R_1), environmental finance (R_2), green banking (R_3), ethical banking (R_4), ESG finance (R_5) to assess the digital transformation challenges in sustainable financial service systems. In the next stage, we have implemented the new fuzzy decision methodology called the IVPF-entropy-divergence-RS-MULTIMOORA model.

5.4. Results and Discussion

In this approach, the IVPF-entropy-divergence-RS model is applied to find the objective, subjective and integrated criteria weights, and the IVPF-entropy-divergence-RS-MULTIMOORA model are developed to obtain the preferences of SFSSs in the banking sector. Here, the presented method is applied as follows: From Eq. (4.1) and Table 5.1 (Al-Barakati et al., 2022), the DEs' weights are obtained and are given in Table 5.2. The DEs offer LDM $Z = \left(z_{ij}^{(k)}\right)$ and are given in Table 5.3. From Eq. (4.2) (or Eq. (4.3)) and Table 5.3, the AIVPF-DM is created and presented in Table 5.4.

Table 5.1: Significance rating of alternatives in the form of LVs

LVs	IVPFNs
Perfectly Good (PG)	([0.90, 0.95], [0.10, 0.15])
Very Good (VG)	([0.80, 0.90], [0.20, 0.35])
Good (G)	([0.65, 0.80], [0.40, 0.50])
Moderate Good (MG)	([0.50, 0.65], [0.50, 0.60])
Fair (F)	([0.40, 0.50], [0.60, 0.70])
Moderate Low (ML)	([0.30, 0.40], [0.70, 0.80])
Low (L)	([0.20, 0.30], [0.80, 0.85])
Very low (VL)	([0.10, 0.20], [0.85, 0.90])
Very very low (VVL)	([0.05, 0.10], [0.90, 0.95])

Table 5.2: Assessment of DE's weight

DEs	LTs	IVPFN	Weights
E_1	Good	([0.65, 0.80], [0.40, 0.50])	0.2204
E_2	Moderate Good	([0.50, 0.65], [0.50, 0.60])	0.1500
E_3	Fair	([0.40, 0.50], [0.60, 0.70])	0.0922
E_4	Very Good	([0.90, 0.95], [0.10, 0.15])	0.3170
E_5	Good	([0.65, 0.80], [0.40, 0.50])	0.2204

Table 5.3: LDM for the ratings of alternatives by DEs

Challenges	$\mathbf{R_1}$	$\mathbf{R_2}$	$\mathbf{R_3}$	$\mathbf{R_4}$	$\mathbf{R_5}$
C_1	(F,MG,MG,F,ML)	(G,G,G,MG,F)	(MG,MG,F,ML,G)	(G,MG,F,F,F)	(F,G,MG,G,F)
C_2	(ML,L,F,ML,F)	(VL,VL,VL,ML,L)	(MG,F,MG,G,G)	(VG,MG,VG,G,F)	(F,G,F,MG,ML)

C_3	(G,VG,G,ML,G)	(VG,VG,VG,F,G)	(F,MG,F,G,ML)	(MG,G,F,ML,G)	(L,F,VL,G,G)
C_4	(L,VL,VL,G,G)	(L,ML,VL,F,F)	(L,ML,VL,ML,F)	(MG,ML,F,G,MG)	(G,PG,G,MG,F)
C_5	(MG,G,F,MG,ML)	(VG,G,VG,F,ML)	(G,VG,F,G,ML)	(VL,ML,L,MG,G)	(VG,F,G,G,MG)
C_6	(VG,VG,MG,L,F)	(G,VG,VG,F,F)	(F,MG,VL,ML,F)	(MG,ML,F,MG,F)	(L,ML,ML,MG,F)
C_7	(MG,L,F,ML,G)	(PG,G,MG,G,G)	(MG,F,VG,F,ML)	(PG,G,F,MG,G)	(ML,G,F,ML,VG)
C_8	(L,L,VL,ML,MG)	(L,VL,VL,ML,ML)	(ML,ML,F,G,MG)	(L,MG,F,MG,MG)	(ML,VL,F,ML,L)
C_9	(G,G,F,G,MG)	(ML,F,VL,L,VL)	(F,G,MG,G,F)	(F,VL,ML,VL,F)	(MG,G,G,L,ML)
C_{10}	(ML,L,F,L,F)	(ML,VL,ML,VL,F)	(G,ML,ML,F,MG)	(VL,VL,L,F,G)	(L,MG,ML,ML,F)
C_{11}	(MG,MG,F,G,G)	(F,G,G,PG,MG)	(VG,MG,F,MG,F)	(L,VL,ML,F,L)	(ML,VL,L,F,G)
C_{12}	(PG,G,F,F,F)	(MG,G,VG,G,F)	(F,VG,F,G,ML)	(G,ML,MG,G,VG)	(F,ML,MG,L,ML)
C_{13}	(VG,G,VG,F,F)	(G,PG,VG,F,MG)	(MG,ML,ML,F,G)	(MG,VG,F,L,ML)	(ML,ML,F,F,F)
C_{14}	(L,ML,VL,ML,F)	(L,L,ML,MG,G)	(G,F,L,MG,MG)	(ML,L,F,MG,MG)	(VG,MG,F,G,G)
C_{15}	(G,MG,F,MG,G)	(VG,G,MG,MG,F)	(ML,G,F,MG,G)	(VL,VL,L,ML,F)	(VG,F,ML,MG,F)
C_{16}	(L,VL,L,ML,ML)	(L,VL,VL,ML,F)	(G,VG,F,ML,F)	(ML,MG,F,G,MG)	(F,G,MG,F,G)
C_{17}	(ML,F,F,MG,ML)	(ML,ML,L,F,MG)	(ML,F,MG,G,MG)	(VG,MG,L,VL,F)	(PG,G,F,ML,L)
C_{18}	(VG,G,PG,MG,F)	(VG,G,VG,MG,F)	(F,L,F,MG,G)	(VG,L,F,F,F)	(VG,G,VG,G,G)

Table 5.4: The AIVPF-DM for options

Challenges	R_1	R_2	R_3	R_4	R_5

C_1	([0.410, 0.527], [0.594, 0.694])	([0.564, 0.713], [0.469, 0.571])	([0.489, 0.632], [0.539, 0.640])	([0.490, 0.622], [0.534, 0.635])	([0.550, 0.693], [0.488, 0.590])
C_2	([0.325, 0.424], [0.681, 0.774])	([0.206, 0.301], [0.789, 0.856])	([0.581, 0.731], [0.456, 0.557])	([0.661, 0.792], [0.364, 0.495])	([0.469, 0.605], [0.551, 0.653])
C_3	([0.570, 0.705], [0.471, 0.592])	([0.691, 0.814], [0.330, 0.472])	([0.504, 0.643], [0.531, 0.633])	([0.518, 0.664], [0.521, 0.623])	([0.531, 0.678], [0.531, 0.624])
C_4	([0.514, 0.663], [0.559, 0.648])	([0.333, 0.431], [0.676, 0.763])	([0.297, 0.395], [0.709, 0.796])	([0.531, 0.679], [0.498, 0.600])	([0.651, 0.772], [0.381, 0.476])
C_5	([0.489, 0.634], [0.530, 0.631])	([0.616, 0.743], [0.414, 0.552])	([0.620, 0.760], [0.423, 0.542])	([0.455, 0.598], [0.588, 0.680])	([0.605, 0.783], [0.383, 0.506])
C_6	([0.605, 0.729], [0.430, 0.572])	([0.608, 0.737], [0.421, 0.550])	([0.374, 0.485],	([0.447, 0.581], [0.557, 0.657])	([0.386, 0.509], [0.626, 0.719])

			[0.633, 0.730])		
C_7	([0.459, 0.598], [0.578, 0.675])	([0.733, 0.847], [0.301, 0.390])	([0.478, 0.600], [0.539, 0.654])	([0.701, 0.815], [0.328, 0.419])	([0.559, 0.687], [0.481, 0.614])
C_8	([0.355, 0.416], [0.582, 0.744])	([0.315, 0.432], [0.702, 0.783])	([0.501, 0.644], [0.537, 0.639])	([0.447, 0.588], [0.564, 0.657])	([0.273, 0.370], [0.732, 0.815])
C_9	([0.606, 0.757], [0.436, 0.537])	([0.606, 0.757], [0.436, 0.537])	([0.550, 0.693], [0.488, 0.590])	([0.294, 0.387], [0.716, 0.797])	([0.453, 0.595], [0.592, 0.683])
C_{10}	([0.300, 0.398], [0.710, 0.789])	([0.300, 0.398], [0.710, 0.789])	([0.481, 0.617], [0.547, 0.649])	([0.410, 0.537], [0.641, 0.726])	([0.348, 0.460], [0.663, 0.754])
C_{11}	([0.584, 0.736], [0.451, 0.552])	([0.730, 0.830], [0.296, 0.383])	([0.582, 0.714], [0.433, 0.559])	([0.283, 0.379], [0.728, 0.802])	([0.443, 0.553], [0.614, 0.708])

C_{12}	([0.654, 0.756], [0.380, 0.474])	([0.603, 0.746], [0.431, 0.542])	([0.576, 0.710], [0.463, 0.584])	([0.655, 0.793], [0.381, 0.504])	([0.327, 0.434], [0.684, 0.771])
C_{13}	([0.625, 0.750], [0.400, 0.536])	([0.665, 0.781], [0.364, 0.468])	([0.481, 0.617], [0.547, 0.649])	([0.484, 0.610], [0.554, 0.668])	([0.367, 0.467], [0.635, 0.735])
C_{14}	([0.297, 0.395], [0.709, 0.796])	([0.459, 0.602], [0.584, 0.673])	([0.513, 0.660], [0.511, 0.609])	([0.423, 0.559], [0.588, 0.683])	([0.663, 0.801], [0.369, 0.490])
C_{15}	([0.570, 0.710], [0.461, 0.562])	([0.609, 0.744], [0.411, 0.536])	([0.531, 0.679], [0.504, 0.606])	([0.270, 0.364], [0.736, 0.816])	([0.568, 0.696], [0.451, 0.579])
C_{16}	([0.251, 0.348], [0.751, 0.830])	([0.277, 0.372], [0.730, 0.810])	([0.549, 0.680], [0.489, 0.611])	([0.522, 0.668], [0.510, 0.612])	([0.526, 0.665], [0.508, 0.609])
C_{17}	([0.401, 0.524], [0.606, 0.707])	([0.382, 0.498], [0.627, 0.724])	([0.517, 0.662],	([0.519, 0.643], [0.526, 0.647])	([0.629, 0.732], [0.426, 0.516])

			[0.553, 0.654])		
C_{18}	([0.668, 0.788], [0.355, 0.472])	([0.642, 0.773], [0.378, 0.510])	([0.489, 0.629], [0.541, 0.637])	([0.538, 0.655], [0.492, 0.619])	([0.708, 0.840], [0.322, 0.447])

From Eq. (4.4), we have calculated the objective weights using the IVPF-entropy-based procedure for each challenge to implement digital transformation in SFSSs in banking sectors. The resultant values are in Fig. 5.1.

w_j^o = (0.0101, 0.1009, 0.0267, 0.0660, 0.0309, 0.0247, 0.0698, 0.0756, 0.0503, 0.0830, 0.0845, 0.0625, 0.0416, 0.0536, 0.0518, 0.0791, 0.0256, 0.0633).

Fig. 5.1 shows the significance objective degree or weights of different each challenge to implement digital transformation in SFSSs in banking sectors with respect to the goal. Knowledge for open data for value co-creation (c_2) with a weight value of 0.1009 has come out to be the most important challenge to implement digital transformation in SFSSs in banking sectors. Understanding value co-creation with cryptocurrencies (c_{11}) with a weight value of 0.0845 is the second most important challenge to implement digital transformation in SFSSs in banking sectors. Designing customer-centric Fintech services (c_{16}) with 0.0791 has third, knowledge for new value creating resource configurations (c_8) with 0.0756 has fourth and Managing the experience and quality of digital financial services (c_7) have with significance value 0.0698 has fifth most important challenge to implement digital transformation in SFSSs in banking sectors, and others are considered crucial challenge to implement digital transformation in SFSSs in banking sectors. Next, to determine the ranking of DT challenges in SFSSs, we compute the AIVPF-DM and PF-score value of challenges to implement digital transformation in SFSSs and given in Table 5.5.

From Eq. (4.5), we have calculated the subjective weights using the IVPF-RS procedure for each challenge to implement digital transformation in SFSSs in banking sectors in Table 5.5, and presented in Fig. 5.2.

w_j^s = (0.0578, 0.0289, 0.0116, 0.0520, 0.0058, 0.0636, 0.0462, 0.0694, 0.0462, 0.0983, 0.0173, 0.0289, 0.0925, 0.0867, 0.1040, 0.0809, 0.0751, 0.0289).

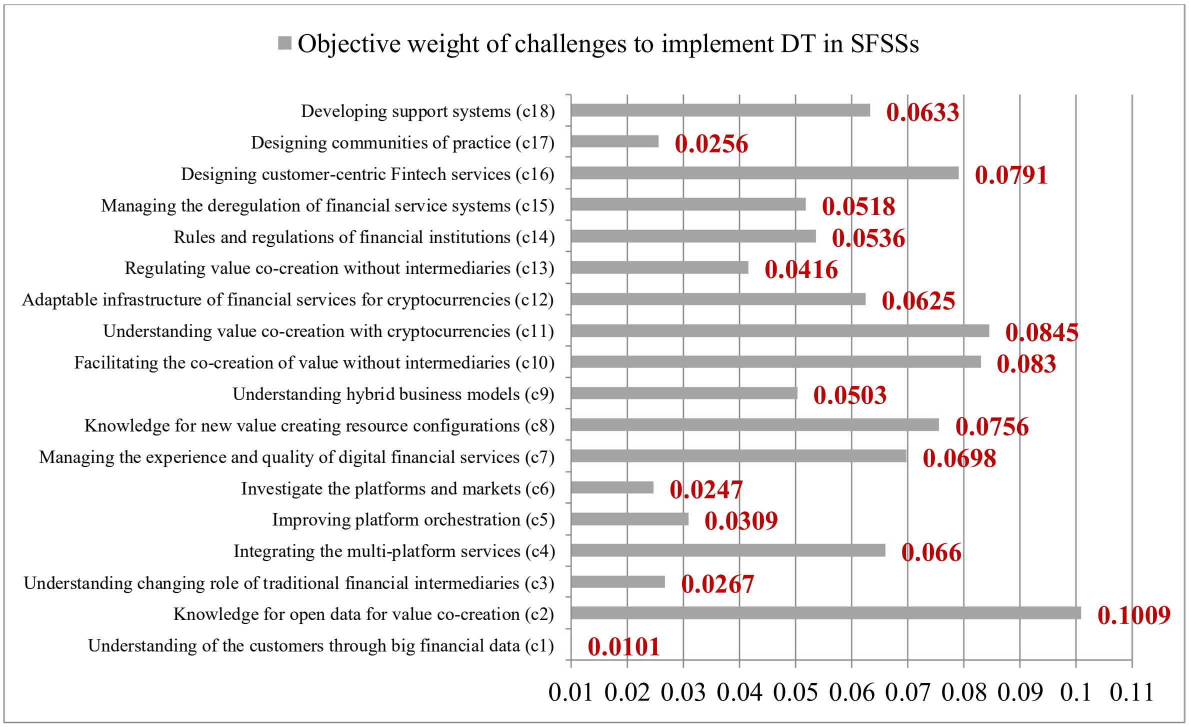

Fig. 5.1: Objective weight of challenges to implement digital transformation in SFSSs

Table 5.5: Subjective weights of challenges to implementing DT in SFSSs using IVPF-RS tool

Challenges	E_1	E_2	E_3	E_4	E_5	Aggregated IVPFNs	$S(z_{ij})$	r_j	Weight
C_1	G	MG	ML	F	MG	([0.434, 0.559], [0.569, 0.669])	0.432	9	0.0578
C_2	ML	F	MG	L	F	([0.343, 0.449], [0.669, 0.756])	0.325	14	0.0289
C_3	ML	ML	L	F	VL	([0.303, 0.399], [0.704, 0.791])	0.282	17	0.0116
C_4	F	G	ML	F	F	([0.410, 0.521], [0.592, 0.692])	0.402	10	0.0520
C_5	VL	ML	L	ML	F	([0.291, 0.387], [0.715, 0.802])	0.270	18	0.0058
C_6	F	G	MG	F	ML	([0.447, 0.572], [0.574, 0.676])	0.435	8	0.0636
C_7	ML	F	G	L	F	([0.372, 0.486], [0.655, 0.743])	0.348	11	0.0462
C_8	MG	F	L	MG	F	([0.447, 0.580], [0.559, 0.656])	0.449	7	0.0694
C_9	ML	F	MG	ML	F	([0.364, 0.471], [0.641, 0.741])	0.348	11	0.0462
C_{10}	F	G	ML	F	MG	([0.469, 0.599], [0.550, 0.651])	0.463	2	0.0983
C_{11}	L	VL	MG	ML	F	([0.316, 0.422], [0.696, 0.780])	0.296	16	0.0173

C_{12}	ML	F	MG	L	F	([0.343, 0.449], [0.669, 0.756])	0.325	14	0.0289
C_{13}	G	ML	L	MG	ML	([0.471, 0.614], [0.563, 0.662])	0.461	3	0.0925
C_{14}	G	L	ML	MG	ML	([0.468, 0.611], [0.567, 0.664])	0.457	4	0.0867
C_{15}	MG	G	L	F	MG	([0.485, 0.625], [0.535, 0.633])	0.485	1	0.1040
C_{16}	ML	G	F	F	MG	([0.460, 0.591], [0.561, 0.663])	0.452	5	0.0809
C_{17}	ML	G	F	L	G	([0.473, 0.614], [0.585, 0.677])	0.450	6	0.0751
C_{18}	L	F	MG	ML	F	([0.349, 0.456], [0.660, 0.751])	0.332	13	0.0289

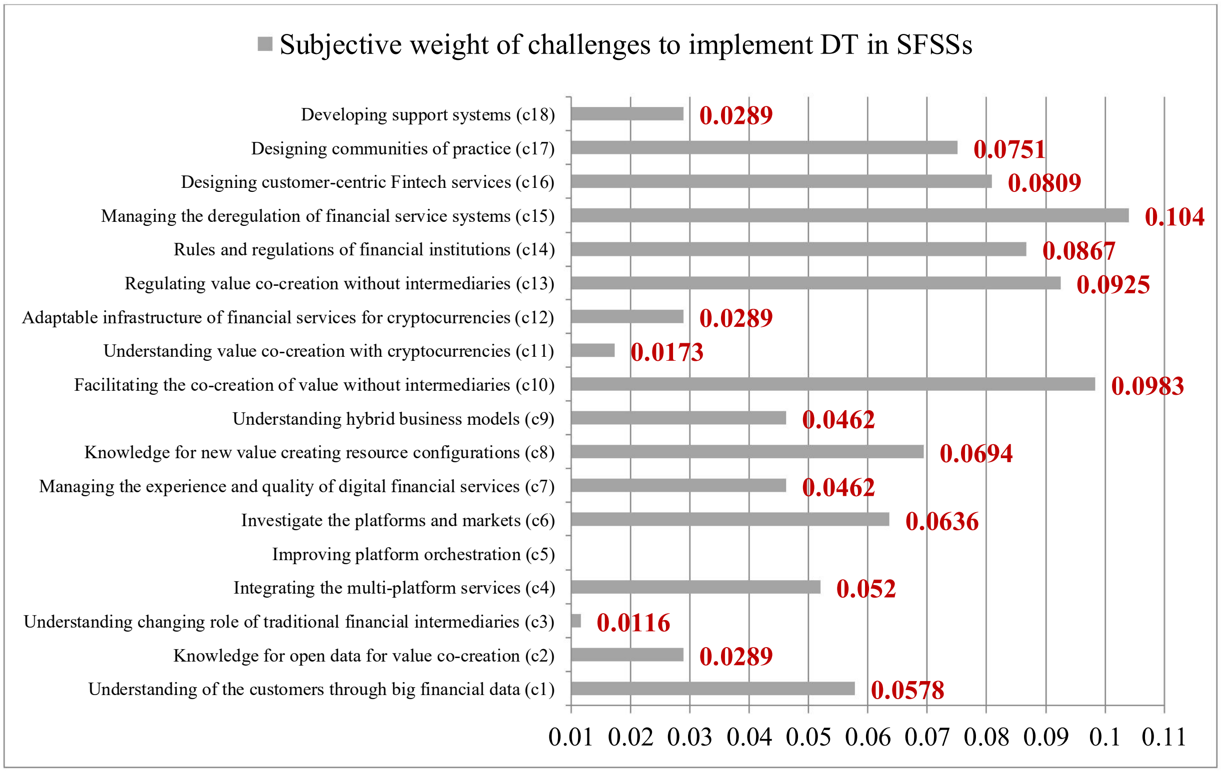

Fig. 5.2: Subjective weight of challenges to implement digital transformation in SFSSs

Here, Fig. 5.2 shows the significance subjective degree or weights of different challenge to implement digital transformation in SFSSs in banking sectors with respect to the goal. Managing the deregulation of financial service systems (c_{15}) with a weight value of 0.104 has come out to be the most important challenge to implement digital transformation in SFSSs in banking sectors. Facilitating the co-creation of value without intermediaries (c_{10}) with a weight value of 0.0983 is the second most important challenge to implement digital transformation in SFSSs in banking sectors. Regulating value co-creation without intermediaries (c_{13}) with 0.0925 has third, Rules and regulations of financial institutions (c_{14}) have fourth with a weight value of 0.0867 and Designing customer-centric Fintech services (c_{16}) with significance value 0.0809 has fifth most important challenge to implement digital transformation in SFSSs in banking sectors, and others are considered crucial challenge to implement digital transformation in SFSSs in banking sectors.

From the algorithm of proposed IVPF-entropy-RS method, we have to combining the IVPF-entropy for objective weighting and IVPF-RS for subjective weighting by using Eq. (4.6). The integrated weight for $\gamma = 0.5$ is shown in the Fig. 5.3 and given as follows:

w_j = (0.0340, 0.0649, 0.0191, 0.0590, 0.0183, 0.0441, 0.0580, 0.0725, 0.0483, 0.0906, 0.0509, 0.0457, 0.0670, 0.0701, 0.0779, 0.0800, 0.0504, 0.0461).

Fig. 5.3 shows the significance integrated degree or weights of different challenge to implement digital transformation in SFSSs in banking sectors with respect to the goal. Facilitating the co-creation of value without intermediaries (c_{10}) with a weight value of 0.0906 has come out to be the most important challenge to implement digital transformation in SFSSs in banking sectors. Designing customer-centric Fintech services (c_{16}) with a weight value of 0.08 is the second most important challenge to implement digital transformation in SFSSs in banking sectors. Managing the deregulation of financial service systems (c_{15}) with 0.0779 has third, and Knowledge for new value creating resource configurations (c_8) has fourth with 0.0725 and Rules and regulations of financial institutions (c_{14}) with significance value 0.0701 has fifth most important critical challenge to implement digital transformation in SFSSs in banking sectors, and others are considered challenge to implement digital transformation in SFSSs in banking sectors.

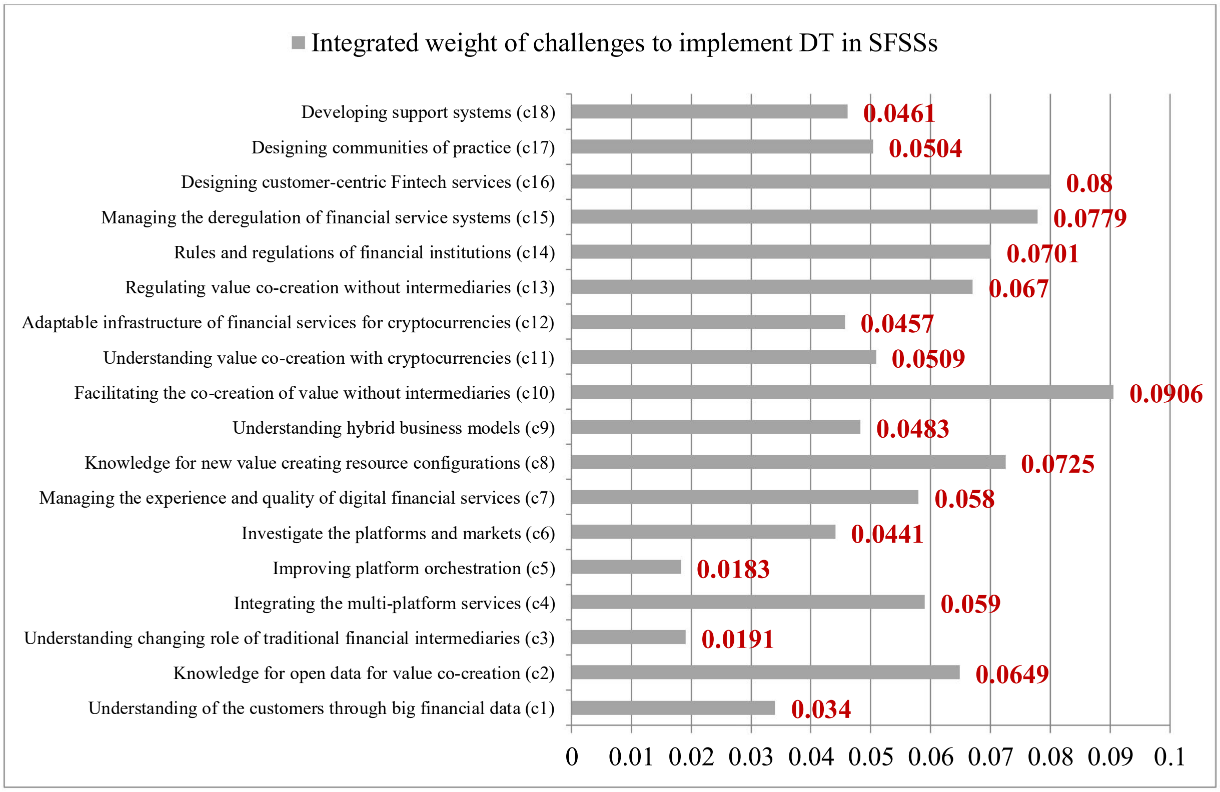

Fig. 5.3: Integrated weight of challenges to implement digital transformation in SFSSs

5.4.1. Proposed IVPF-Entropy-Divergence-RS-MULTIMOORA Tool

The implementation of IVPF-entropy-divergence-RS-MULTIMOORA framework to assess the main digital transformation challenges in SFSSs of the banking sector. In this way, we consider all the digital transformation challenges in SFSSs of the banking sector as benefit-type for evaluation.

Step 1: The RS model of the IVPF-entropy-divergence-RS-MULTIMOORA framework for the options to assess the main digital transformation challenges in SFSSs of the banking sector is evaluated by Eq. (4.7)-(4.10) and is given in Table 5.6. Since all challenges are benefit type, therefore we need to calculate only $Y_i = Y_i^+$ and IVIF-score degree $y_i = y_i^+$ to determine the prioritization of the options to assess the main digital transformation challenges in SFSSs.

Table 5.6: Prioritization of the options using RS model

Option	Y_i	y_i	Ranking
***R*₁**	([0.492,0.618], [0.544,0.652])	0.4757	5
***R*₂**	([0.535,0.663], [0.509,0.613])	0.5227	1
***R*₃**	([0.513,0.648], [0.525,0.631])	0.5024	3
***R*₄**	([0.502,0.632], [0.539,0.643])	0.4871	4
***R*₅**	([0.519,0.649], [0.521,0.628])	0.5064	2

Step 2: The IVPF-reference values of options to assess the main digital transformation challenges in SFSSs of the banking sector based on Eq. (4.11)-Eq. (13) are obtained and presented in Table 5.7.

Table 5.7: The prioritization of options using the RP model

	R_1	R_2	R_3	R_4	R_5
d_i	0.0652	0.0652	0.0478	0.0585	0.0606
Ranking	4.5	4.5	1	2	3

Step 3: The results of the IVPF-FMF model of options to assess the main digital transformation challenges in SFSSs of the banking sector are obtained with the use of Eq. (4.14)-Eq. (4.17) and presented in Table 5.8. Since all challenges are benefit type, therefore we need to calculate only $U_i = U_i^+$ and IVIF-score degree $u_i = u_i^+$ to determine the prioritization of the options to assess the main digital transformation challenges in SFSSs.

Table 5.8: The prioritization of options using the FMF model

Option	U_i	u_i	Ranking
***R*₁**	([0.438,0.551], [0.585,0.688])	0.4203	5
***R*₂**	([0.453,0.571], [0.580,0.675])	0.4349	4
***R*₃**	([0.498,0.628], [0.534,0.639])	0.4873	1
***R*₄**	([0.459,0.583], [0.573,0.671])	0.4428	3
***R*₅**	([0.476,0.588], [0.557,0.659])	0.4568	2

Table 5.9: The *OAD* of options to assess the DT challenges in SFSSs of the banking sector

Options	RS model		RP model		FMF model		$U(R_i)$	Final Ranking
	y_i^*	$\rho(y_i^*)$	d_i^*	$\rho(d_i^*)$	u_i^*	$\rho(u_i^*)$		
***R*₁**	0.426	5	0.488	4.5	0.419	5	-0.090	5
***R*₂**	0.468	1	0.488	4.5	0.433	4	0.068	4
***R*₃**	0.450	3	0.357	1	0.485	1	0.228	1
***R*₄**	0.436	4	0.437	2	0.441	3	0.088	3
***R*₅**	0.454	2	0.453	3	0.455	2	0.152	2

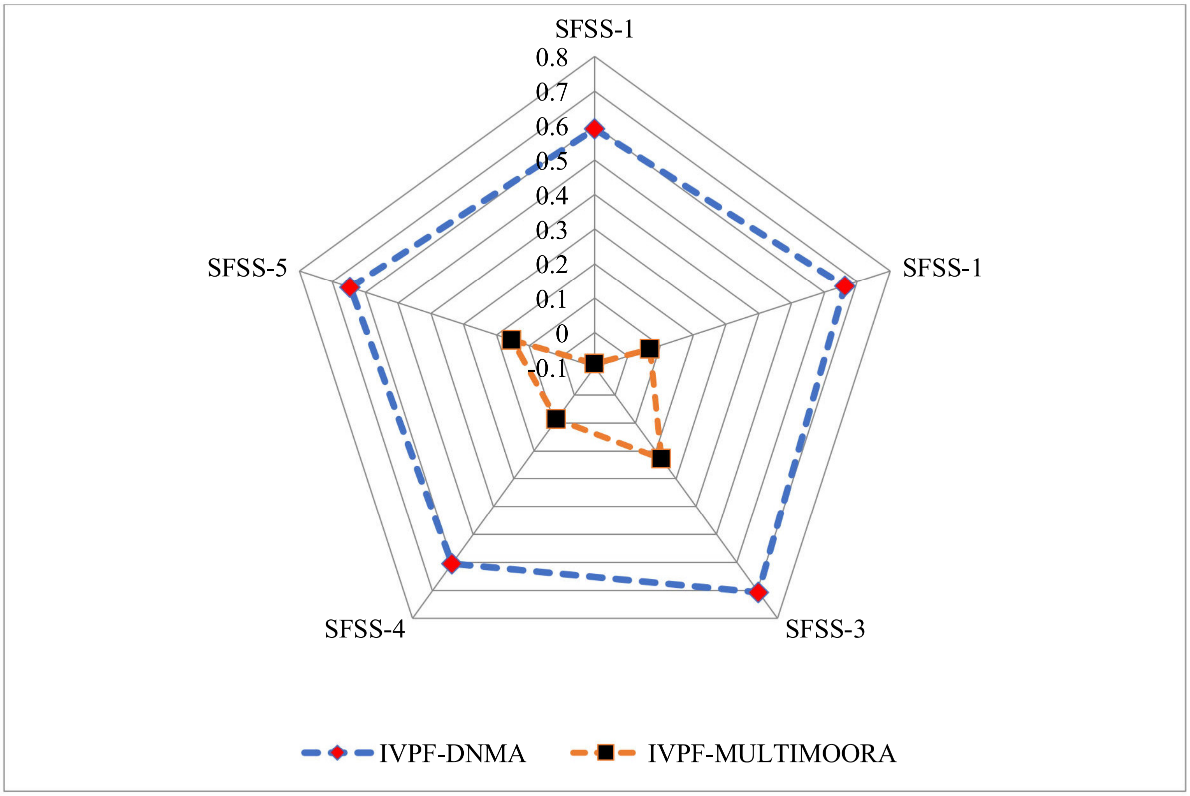

Fig. 5.5: Variation of *OUDs* of options with the different methods

Step 4: The summarization of all procedures evaluated by IVPF-entropy-divergence-RS-MULTIMOORA methodology is presented with the use of Eq. (4.18) to assess the main digital transformation challenges in SFSSs of the banking sector is depicted in Table 5.9. Finally, the preference order of the organizations to assess the main digital transformation challenges in SFSSs of the banking sector is $R_3 \succ R_5 \succ R_4 \succ R_2 \succ R_1$. Thus, the optimal organizations -III is R_3 to assess the main digital transformation challenges in SFSSs of the banking sector.

Next, the prioritization of organizations to assess the main digital transformation challenges in SFSSs of the banking sector obtained by the developed IVPF-entropy-divergence-RS-MULTIMOORA framework and IVPF-DNMA model (Rahimi et al., 2022) is discussed in Fig. 5.5. From Fig. 5.5, we find that the best organization is sector-III (R_3) to assess the main digital transformation challenges in SFSSs of the banking sector. Also, the overall assessment degree and

OUD of organizations to assess the main digital transformation challenges in SFSSs of the banking sector is presented in Fig. 5.5.

CHAPTER 6

SENSITIVITY AND COMPARATIVE DISCUSSIONS

6.1. Introduction

This chapter presents the sensitivity and comparative analyses to illustrate the validity and stability of the obtained outcomes with the developed IVPF-entropy-divergence-rank sum-MULTIMOORA model. The sensitivity analysis is conducted to validate the proposed method and their results in the case study of digital transformation challenges in SFSSs in banking sectors problem to investigate the impact of various settings of different parameters. The comparative study of the proposed IVPF-entropy-divergence- rank sum -MULTIMOORA model is twofold: the IVPF-TOPSIS model (Garg, 2017), IVPF-entropy-divergence-rank sum -DNMA (Rahimi et al., 2022) and IVPF-WASPAS (Al-Barakati et al., 2022) methods. We solve use the proposed IVPF-entropy-divergence-rank sum-MULTIMOORA model to deal with the aforementioned case study to make the comparisons more reasonable.

The rest part of this section is organized as follows: Section 6.2 discusses the sensitivity analysis with respect to values of the decision coefficient parameter. Section 6.3 implements the existing IVPF-TOPSIS method on the case study, discussed in Chapter 5. Section 6.4 implements the existing IVPF-WASPAS method on the case study, discussed in Chapter 5. Section 6.5 presents the advantages of the proposed method in comparison with the IVPF-TOPSIS and IVPF-WASPAS methods.

6.2. Sensitivity Analysis of the Proposed Method

In this section, a sensitivity analysis is conducted to validate the proposed method and their results in the case study of digital transformation challenges in SFSSs in banking sectors problem. The aim of the first sensitivity analysis is to investigate the impact of various settings of different parameters. The results and their validation are presented in two parts: 1) the effect in the variation of weights of digital transformation challenges in SFSSs over different values of γ , 2) a sensitivity investigation of the proposed

IVPF-entropy-divergence-rank sum-MULTIMOORA and IVPF-entropy-divergence-rank sum-DNMA (Rahimi et al. 2022) models considering different weighting values over parameter (γ) values of digital transformation challenges in SFSSs in banking sector. The subtleties of these three portions are discussed in the accompanying subsection as: We present analysis with the coefficient ξ values. The diverse values of ξ is useful for evaluating the sensitivity of the presented approach by altering from the SUDs to the subordinate rankings. Furthermore, variation of ξ is used to express the sensitivity of the presented approach to the distinction of weights of criteria.

Case I: In the first case, the weight/significance degree of digital transformation challenges in SFSSs in banking sectors are computed with the consideration of diverse parameter (γ) values. The changes of significance degree of digital transformation challenges in SFSSs over different parameter (γ) values are presented in Table 6.1 and Fig. 6.1. For $\gamma = 0.0$, managing the deregulation of financial service systems (c_{15}) with a weight value of 0.104 has come out to be the most important challenge to implement digital transformation in SFSSs in banking sectors. Facilitating the co-creation of value without intermediaries (c_{10}) with a weight value of 0.098 is the second most important challenge to implement digital transformation in SFSSs in banking sectors. regulating value co-creation without intermediaries (c_{13}) with 0.092 has third most important challenge to implement digital transformation in SFSSs in banking sectors, and others are considered crucial challenge to implement digital transformation in SFSSs in banking sectors.

Table 6.1: Variation of weights of digital transformation challenges in SFSSs over different values of γ

Challenges	γ = 0.0	γ = 0.1	γ = 0.2	γ = 0.3	γ = 0.4	γ = 0.5	γ = 0.6	γ = 0.7	γ = 0.8	γ = 0.9	γ = 1.0
C_1	0.058	0.053	0.048	0.044	0.039	0.034	0.029	0.024	0.020	0.015	0.010
C_2	0.029	0.036	0.043	0.051	0.058	0.065	0.072	0.079	0.087	0.094	0.101
C_3	0.012	0.013	0.015	0.016	0.018	0.019	0.021	0.022	0.024	0.025	0.027
C_4	0.052	0.053	0.055	0.056	0.058	0.059	0.060	0.062	0.063	0.065	0.066
C_5	0.006	0.008	0.011	0.013	0.016	0.018	0.021	0.023	0.026	0.028	0.031
C_6	0.064	0.060	0.056	0.052	0.048	0.044	0.040	0.036	0.032	0.029	0.025
C_7	0.046	0.049	0.051	0.053	0.056	0.058	0.060	0.063	0.065	0.067	0.070
C_8	0.069	0.070	0.071	0.071	0.072	0.073	0.073	0.074	0.074	0.075	0.076
C_9	0.046	0.047	0.047	0.047	0.048	0.048	0.049	0.049	0.049	0.050	0.050

C_{10}	0.098	0.097	0.095	0.094	0.092	0.091	0.089	0.088	0.086	0.084	0.083
C_{11}	0.017	0.024	0.031	0.037	0.044	0.051	0.058	0.064	0.071	0.078	0.084
C_{12}	0.029	0.032	0.036	0.039	0.042	0.046	0.049	0.052	0.056	0.059	0.062
C_{13}	0.092	0.087	0.082	0.077	0.072	0.067	0.062	0.057	0.052	0.047	0.042
C_{14}	0.087	0.083	0.080	0.077	0.073	0.070	0.067	0.064	0.060	0.057	0.054
C_{15}	0.104	0.099	0.094	0.088	0.083	0.078	0.073	0.067	0.062	0.057	0.052
C_{16}	0.081	0.081	0.081	0.080	0.080	0.080	0.080	0.080	0.079	0.079	0.079
C_{17}	0.075	0.070	0.065	0.060	0.055	0.050	0.045	0.040	0.036	0.031	0.026
C_{18}	0.029	0.032	0.036	0.039	0.043	0.046	0.050	0.053	0.056	0.060	0.063

For $\gamma = 0.5$, Facilitating the co-creation of value without intermediaries (c_{10}) with a weight value of 0.091 has come out to be the most important challenge to implement digital transformation in SFSSs in banking sectors. Designing customer-centric fintech services (c_{16}) with 0.08 has second most important challenge to implement digital transformation in SFSSs in banking sectors. Managing the deregulation of financial service systems (c_{15}) with a weight value of 0.078 is the third most important challenge to implement digital transformation in SFSSs in banking sectors and others are considered crucial challenge to implement digital transformation in SFSSs in banking sectors. For $\gamma = 1.0$, knowledge for open data for value co-creation (c_2) with a weight 0.1009 has the most important challenge to implement digital transformation in SFSSs in banking sectors. Understanding value co-creation with cryptocurrencies (c_{11}) with a weight value of 0.0845 is the second most important challenge to implement digital transformation in SFSSs in banking sectors. Designing customer-centric Fintech services (c_{16}) with 0.0791 has third, knowledge for new value creating resource configurations (c_8) with 0.0756 has fourth and Managing the experience and quality of digital financial services (c_7) have with significance value 0.0698 has fifth most important challenge to implement digital transformation in SFSSs in banking sectors, and others are considered crucial challenge to implement digital transformation in SFSSs in banking sectors.

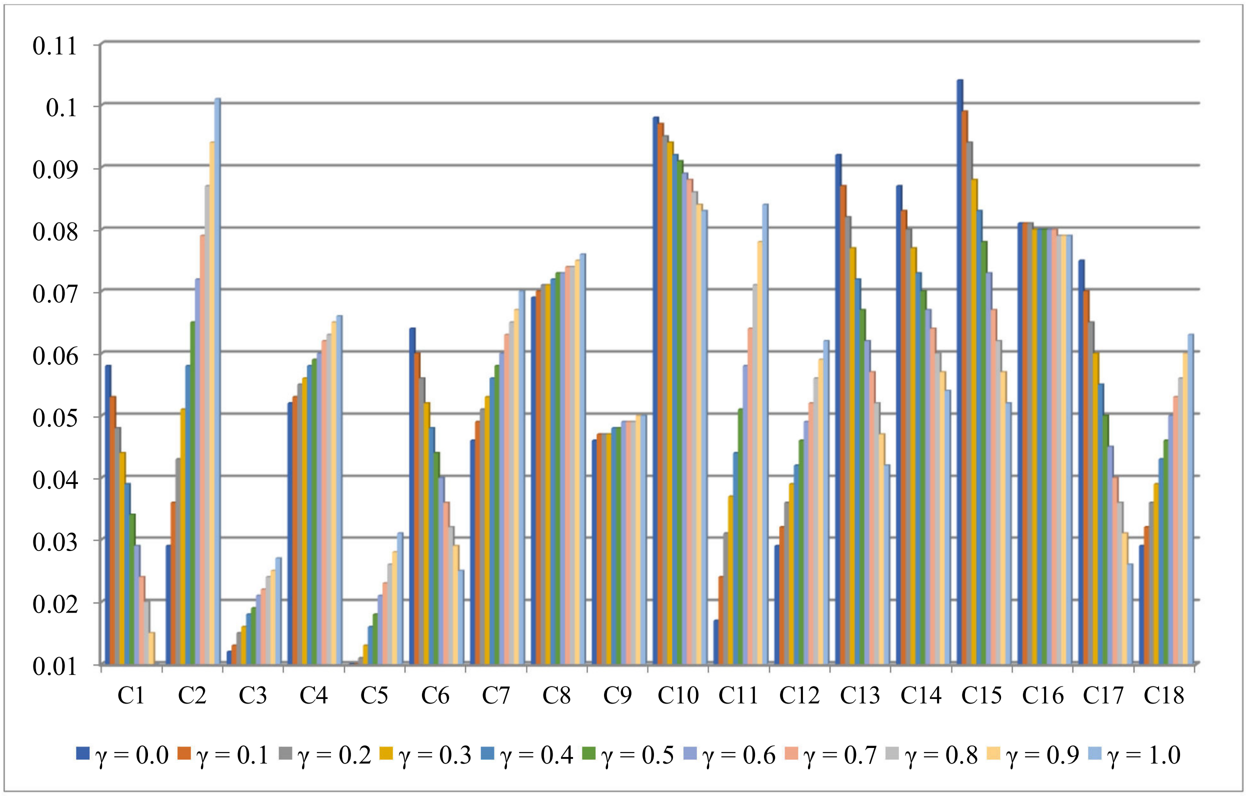

Fig. 6.1: Variation of weights of challenges of DT in SFSSs with diverse utility parameter γ values

Case II: When considering the proposed IVPF-entropy-divergence- rank sum weight-determining approach, the objective and subjective weights are obtained to give better weights/significance degrees for considered challenges of DT in SFSSs. In this first case, the weight of challenges of DT in SFSSs is considered as objective weight (for γ = 1.0) using IVPF-entropy-divergence model instead of integrated weight-determining model. The OADs of sectors/organizations for considered challenges of DT in SFSSs are estimated using the proposed IVPF-entropy-divergence-rank sum-MULTIMOORA and presented in Table 6.2 and Fig. 6.2. The ranking order of sectors/organizations for considered challenges of DT in SFSSs is given in the following form $R_3 \succ R_5 \succ R_4 \succ R_2 \succ R_1$. Next, the weight of challenges of DT in SFSSs is considered as subjective weight (for γ = 0.0) using the IVPF-RS method. The OADs of sectors/organizations for considered challenges of DT in SFSSs are estimated using the proposed IVPF-entropy-divergence-rank sum-MULTIMOORA and presented in Table 6.2 and Fig. 6.2. The ranking order of sectors/organizations for considered challenges of DT in SFSSs is given in the following form $R_3 \succ R_5 \succ R_2 \succ R_1 \succ R_4$. Furthermore, the weight of challenges of DT in SFSSs is considered as integrated weight (for γ = 0.5) using the IVPF-entropy-divergence-rank sum method. The OADs of sectors/organizations for considered challenges of DT in SFSSs are estimated using the proposed IVPF-entropy-divergence-rank sum-MULTIMOORA and presented in Table 6.2 and Fig. 6.2. The ranking order of sectors/organizations for considered challenges of DT in SFSSs is given in the following form $R_3 \succ R_5 \succ R_4 \succ R_2 \succ R_1$. As per the afore-mentioned discussion, it is found that by employing the different parameter values will improve the stability of the proposed IVPF-entropy-divergence-rank sum-MULTIMOORA method.

Table 6.2: Variation of OADs for proposed IV{F-MULTIMOORA model

γ	R_1	R_2	R_3	R_4	R_5
γ = 0.0	-0.024	0.107	0.227	-0.056	0.188
γ = 0.1	-0.053	0.164	0.227	-0.051	0.187
γ = 0.2	-0.054	0.134	0.227	-0.046	0.185
γ = 0.3	-0.055	0.103	0.228	-0.012	0.184
γ = 0.4	-0.056	0.101	0.228	-0.006	0.183
γ = 0.5	-0.090	0.068	0.228	0.088	0.152
γ = 0.6	-0.091	0.066	0.228	0.090	0.150
γ = 0.7	-0.074	0.048	0.259	0.091	0.120
γ = 0.8	-0.071	0.034	0.260	0.093	0.122
γ = 0.9	-0.071	0.028	0.261	0.064	0.155
γ = 1.0	-0.073	0.025	0.262	0.077	0.143

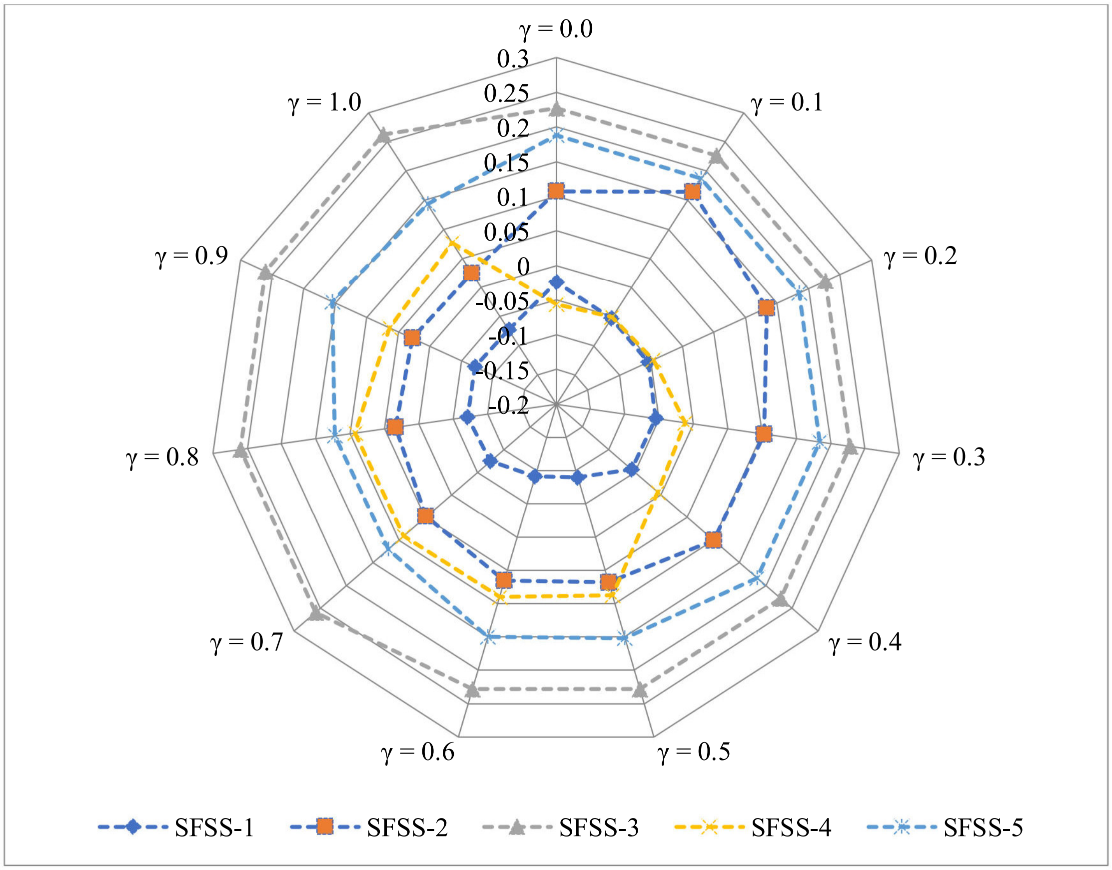

Fig. 6.2: Variation of OADs of DT in SFSSs with diverse utility parameter γ values

6.3. Comparison with IVPF-TOPSIS Model

Steps 1-4: Same as presented method

Step 5: Calculate the discriminations of each alternative from "IVPF-ideal solution (IS)" and "IVPF-anti-ideal solution (AIS)".

Here, the BD and NBD of IVPF-IS(θ^+) are discussed as 1 and 0, and defined as $\theta^+ = \langle [1,1],[0,0] \rangle_{1\times n}$.

Similarly, IVPF-A-IS is $\theta^- = \langle [0,0],\, [1,1] \rangle_{1\times n}$.

To assess the option(s) $R_i : i = 1(1)m,$ we use the distance measure as

$$d\left(R_i, \theta^+\right) = \frac{1}{4}\left[\left|\left(\overline{p}_{ij}^{(1)}\right)^2 - (1)^2\right| + \left|\left(\overline{q}_{ij}^{(1)}\right)^2 - (1)^2\right| + \left|\left(\overline{r}_{ij}^{(1)}\right)^2 - (0)^2\right| + \left|\left(\overline{s}_{ij}^{(1)}\right)^2 - (0)^2\right|\right], \tag{6.1}$$

and

$$d\left(R_i,\theta^-\right)=\frac{1}{4}\left[\left|\left(\overline{p}_{ij}^{(1)}\right)^2-(0)^2\right|+\left|\left(\overline{q}_{ij}^{(1)}\right)^2-(0)^2\right|+\left|\left(\overline{r}_{ij}^{(1)}\right)^2-(1)^2\right|+\left|\left(\overline{s}_{ij}^{(1)}\right)^2-(1)^2\right|\right]. \qquad (6.2)$$

Step 6: Obtain "closeness index (CI)".

The *CI* of an option is defined by

$$CI\left(R_i\right)=\frac{d\left(R_i,\theta^-\right)}{d\left(R_i,\theta^-\right)+d\left(R_i,\theta^+\right)}, i=1(1)m. \qquad (6.3)$$

Step 7: Determine the prioritization of alternatives

From the $CI\left(R_i\right), i=1,2,...,m,$ we find the ranking order of options, and we get the optimal option.

Following the above steps, the IVPF-TOPSIS model is applied and discussed in Table 6.3.

Table 6.3: Ranking of IVPF-TOPSIS for SFSS options

Options	$d\left(R_i,\vartheta^+\right)$	$d\left(R_i,\vartheta^-\right)$	$CI\left(R_i\right)$	Ranking
R_1	0.743	0.251	0.253	4
R_2	0.731	0.263	0.264	3
R_3	0.717	0.278	0.279	1
R_4	0.746	0.249	0.250	5
R_5	0.720	0.274	0.276	2

From Eq. (6.1)-Eq. (6.3) and Table 6.3, the most optimal SFSS option is R_3, and the priority order shows conformity with the proposed models. By analyzing Table 6.3, we get the rank as $R_3 \succ R_5 \succ R_2 \succ R_4 \succ R_1$.

6.4. Comparison with IVPF-WASPAS Model

Steps 1-4: Follow the same process as given in Chapter 4.

Step 5: Calculate the measures of WSM and WPM for i^{th} alternative using Eq. (6.4) and Eq. (6.5), respectively.

$$C_i^{(1)} = \mathop{\oplus}_{j=1}^{n} w_j \varsigma_{ij} = \left(\left[\sqrt{1 - \prod_{j=1}^{n} \left(1 - \left(\mu_{ij}^{-}\right)^2\right)^{w_j}}, \sqrt{1 - \prod_{j=1}^{n} \left(1 - \left(\mu_{ij}^{+}\right)^2\right)^{w_j}} \right], \left[\prod_{j=1}^{n} \left(\nu_{ij}^{-}\right)^{w_j}, \prod_{j=1}^{n} \left(\nu_{ij}^{+}\right)^{w_j} \right] \right). \quad (6.4)$$

$$C_i^{(2)} = \mathop{\otimes}_{j=1}^{n} \varsigma_{ij}^{w_j} = \left(\left[\prod_{j=1}^{n} \left(\mu_{ij}^{-}\right)^{w_j}, \prod_{j=1}^{n} \left(\mu_{ij}^{+}\right)^{w_j} \right], \left[\sqrt{1 - \prod_{j=1}^{n} \left(1 - \left(\nu_{ij}^{-}\right)^2\right)^{w_j}}, \sqrt{1 - \prod_{j=1}^{n} \left(1 - \left(\nu_{ij}^{+}\right)^2\right)^{w_j}} \right] \right). \quad (6.5)$$

Step 6: Compute the measure of WASPAS for each candidate.

$$C_i = \lambda C_i^{(1)} + (1-\lambda) C_i^{(2)}, i = 1, 2, ..., m, \quad (6.6)$$

where '$\lambda \in [0,1]$' denotes the decision mechanism coefficient.

Step 7: In accordance with the WASPAS measures, estimate the ranking order of the alternatives.

Step 8: End.

For the given case study of digital transformation challenges in SFSSs and using Eq. (6.4)-Eq. (6.6), the overall results of IVPF-WASPAS method are presented in Table 6.4 (for λ = 0.5). The preference ordering of the options is $R_3 \succ R_5 \succ R_2 \succ R_4 \succ R_1$ for $\lambda = 0.5$. Thus, the most appropriate alternative is SFSS-2.

Table 6.4: Computational outcomes of IVPF-WASPAS model

Options	WSM		WPM		UD	Ranking
	$C_i^{(1)}$	$S(C_i^{(1)})$	$C_i^{(2)}$	$S(C_i^{(2)})$		
R_1	([0.492,0.618], [0.544,0.652])	0.4757	([0.438,0.551], [0.585,0.688])	0.4203	0.4480	5
R_2	([0.535,0.663], [0.509,0.613])	0.5227	([0.453,0.571], [0.580,0.675])	0.4349	0.4788	3
R_3	([0.513,0.648], [0.525,0.631])	0.5024	([0.498,0.628], [0.534,0.639])	0.4873	0.4949	1
R_4	([0.502,0.632], [0.539,0.643])	0.4871	([0.459,0.583], [0.573,0.671])	0.4428	0.4649	4
R_5	([0.519,0.649], [0.521,0.628])	0.5064	([0.476,0.588], [0.557,0.659])	0.4568	0.4816	2

6.5. Advantages of Proposed IVPF-Entropy-Divergence-RS-MULTIMOORA Model

Overall, the benefits of the proposed IVPF-entropy-divergence-rank sum-MULTIMOORA model with the extant model are presented in Fig. 6.3 and discussed as follows:

- In our approach, the weights of DEs are computed with the help of the weighting formula, resulting in more correct significance degree of DEs, unlike the arbitrarily chosen criteria's weights by decision-makers in Garg (2017).
- The objective and subjective weights of digital transformation challenges in SFSSs in banking sector in the proposed IVPF-entropy-divergence-rank sum-MULTIMOORA and IVPF-entropy-divergence-rank sum-DNMA models are obtained by IVPF-entropy-divergence measure-based model and IVPF-RSWM, whereas in IVPF-TOPSIS, the criteria weights are chosen arbitrarily and IVPF-WASPAS model, author(s) only considered the objective weight of digital transformation challenges in SFSSs in banking sector for assessment.

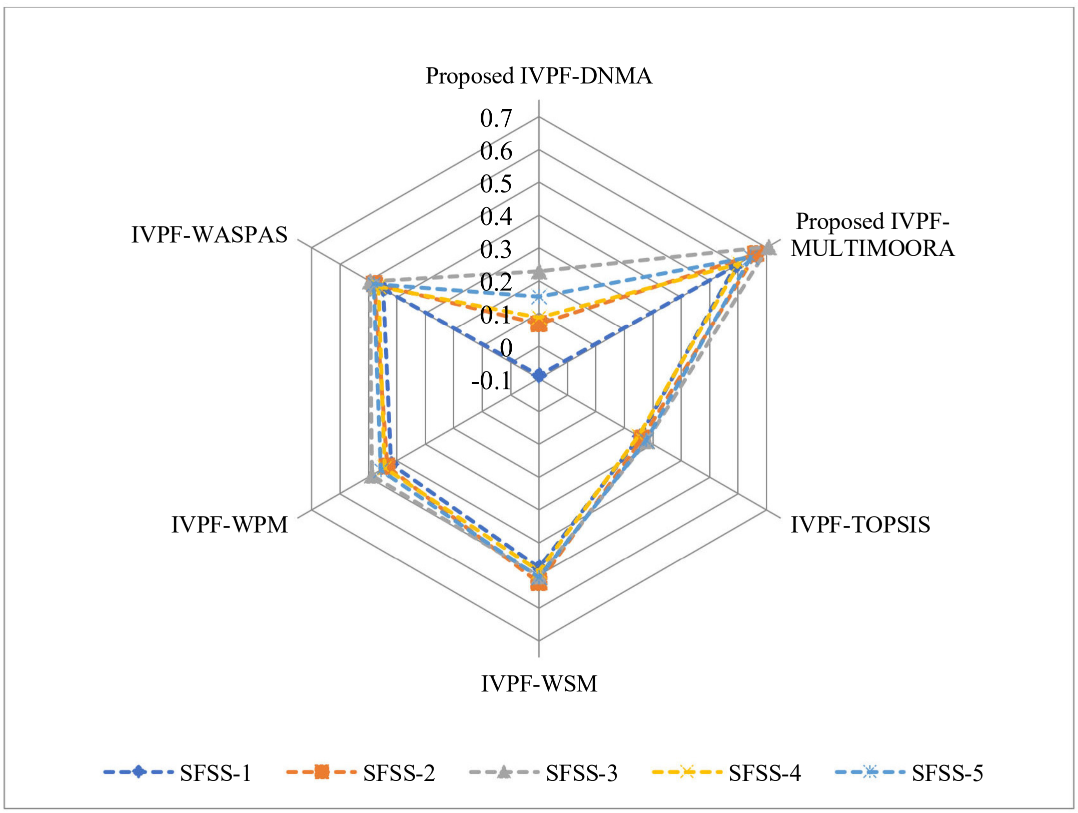

Fig. 6.3: Variation of assessment values of different options over various extant models

- In Garg (2017), the distance is calculated between the overall attribute value of an alternative R_i and the IVPF-IS $\vartheta^{+} = \langle [1,1],[0,0] \rangle_{1\times q}$ and the IVPF-AIS $\vartheta^{-} = \langle [0,0],[1,1] \rangle_{1\times q}$ to describe the *CI* of each option on the given digital transformation challenges in SFSSs in banking sectors. The IVPF-IS and IVPF-AIS may be considered two benchmarks against which the performance of the alternatives for each digital transformation challenges in SFSSs in banking sector could be assessed. Note that for the two above-mentioned benchmarks, it may be that they cannot be achieved fully in practice. IVPF-WASPAS model, the IVPFWAO and IVPFWGO are used on find the aggregated ratings of SFSSs for assessing the digital transformation challenges in SFSSs in banking sector. On the other hand, it should be noted that the proposed IVPF-entropy-divergence-rank sum-MULTIMOORA uses strength points of various methods, proposed IVPF-divergence measure and aggregation functions, and it can integrate all of them appropriately. The final integration function of the MULTIMOORA approach takes into consideration widely the subordinate OUDs using Borda rule and the ranks of options. Thus, the final ranking results from the suggested methodology could be highly reliable and more realistic as the DEs could know about the best and worst performance of alternatives on the defined attributes and compare their performance.

CHAPTER 7

CONCLUSIONS AND FUTURE SCOPE

7.1. Conclusions of the Thesis

Digitalization is a management tool and digital transformation (DT) is the process of integrating digital technologies into the value chain of activities, in order to deliver added value to both customers and broader stakeholders, which leads to improving organizational performance. The fourth industrial revolution led to a widespread use of digital technologies, which holds great potential for sustainable development. The strategic combination of sustainability and technology adoption holds immense potential for driving positive environmental and social impact while ensuring the long-term viability of the businesses in the financial sector. In addition, it is widely acknowledged that intensified competition and technological changes within a sector have the potential to give benefits to end customers through enhancing the quality of products and services and, at the same time, lowering their prices. The augmented use of digital devices and platforms is transforming the ways that the consumers do banking, shift their market expectations, and also transform the model of financial intermediation. The contemporary digitization waves in the financial service systems of banks – especially in the case of payments – and the utilization of access and network technologies have led to the creation of different opportunities for novel entrants and challenges that banks to claim some market share, but also for established banks to reassess their positions within the market and the value they propose to their clients. In such a dynamic environment, banks are able to choose either to embrace changes by using the opportunities that technology offers through making interactions with the greater ecosystem of market participants and other service providers or to take a defensive position through being concentrating on the development of competitive solutions to all customer and product segments and putting limitations on access to their systems. Numerous researchers, policy makers, and practitioners have become interested in the digital transformation in banking, which causes us to think about what the banking sector could look like after the predicted digital transformation. Motivated by the above, this study targeted to recognize and evaluate the digital transformation challenges in sustainable financial service systems of the banking sector.

This thesis develops MCDM models to analyze, rank, and evaluate the digital transformation challenges in sustainable financial service systems of the banking sector. In this regard, a hybrid MULTIMOORA approach has been proposed with the combination of decision experts' and criteria weight-determination models. New interval-valued Pythagorean fuzzy entropy and divergence measures are developed and applied to determine the criteria weights under the context of IVPFSs.

The detail contribution of the thesis is portrayed as follows:

Chapter 2 has discussed the comprehensive review of literature of the digital transformation, an interval-valued Pythagorean fuzzy set and the multi-objective optimization based on ratio analysis with the full multiplicative form (MULTIMOORA) method. Based on the literature, this Chapter has identified some research problems during the implementation of digital transformation in the sustainable financial service systems.

Chapter 3 has proposed novel entropy and divergence measures, which describe the degree of uncertainty of an interval-valued Pythagorean fuzzy set and compute the degree of discrepancy between interval-valued Pythagorean fuzzy sets, respectively. Comparative studies have discussed to illustrate the effectiveness of the proposed measures over the existing IVPF-entropy and divergence measures.

Chapter 4 has proposed IVPF-Entropy-RS-MULTIMOORA model to identify the key changes of digital transformation. The proposed models have developed by combining the objective weight of criteria through IVPF-entropy based procedure and the subjective weight of criteria through IVPF-ranking sum model. In addition, the proposed models have presented a novel weighting model for the determination of decision experts' weights.

Chapter 5 has identified some key challenges through online questionnaire and literature review, which may affect the digital transformation in the sustainable financial service systems. Moreover, the proposed models of Chapter 4 have applied to evaluate the key challenges of the digital transformation in the sustainable financial service systems.

Chapter 6 has discussed the results of the sensitivity analysis with respect to diverse values of the decision strategy parameter. Several cases have presented to illustrate the stability of the obtained outcomes. Moreover, comparative studies are presented to confirm the robustness of the proposed MCDM models.

7.2. Future Scope

This thesis proposes a MCDM models for assessing the digital transformation challenges under interval-valued Pythagorean fuzzy environment. The proposed models demonstrate more effective results in comparison with some of the existing MCDM models under IVPFSs background. However, there are several points which need to be considered in our further work.

- Because of ever-increasing competitions, the decision making has gained as one of the fastest emergent research topics related to real life problems. Most of the time, the criteria are conflicting with each other, therefore, it may not have a unique solution satisfying all the criteria concurrently. The proposed work can be further extended by developing a new MCDM model, which is suitable to evaluate the key challenges of digital transformation with respect to different conflicting criteria.
- Future work can also consider the geographical and cultural aspects of the criteria, which is one of the main limitations of this thesis.
- Moreover, the further work can consider the interdependent characteristics of the criteria and the decision experts.
- In addition, future works should be deliberated towards utilizing a wider number of global DMs who will assess the challenges and drivers affecting the digital transformation process in the sustainable financial service systems of the banking sector.

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