NONLINEAR DYNAMICS OF PLANKTON ECOSYSTEM WITH IMPULSIVE CONTROL AND ENVIRONMENTAL FLUCTUATIONS

by

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ABSTRACT

It is well known that the density of plankton populations always increases and decreases or keeps invariant for a long time, and the variation of plankton density is an important factor influencing the real aquatic environments, why do these situations occur? It is an interesting topic which has become the common interest for many researchers. As the basis of the food webs in oceans, lakes, and reservoirs, plankton plays a significant role in the material circulation and energy flow for real aquatic ecosystems that have a great effect on the economic and social values. Planktonic blooms can occur in some environments, however, and the direct or indirect adverse effects of planktonic blooms on real aquatic ecosystems, such as water quality, water landscape, aquaculture development, are sometimes catastrophic, and thus planktonic blooms have become a challenging and intractable problem worldwide in recent years. Therefore, to understand these effects so that some necessary measures can be taken, it is important and meaningful to investigate the dynamic growth mechanism of plankton and reveal the dynamics mechanisms of formation and disappearance of planktonic blooms. To this end, based on the background of the ecological environments in the subtropical lakes and reservoirs, this dissertation research takes mainly the planktonic algae as the research objective to model the mechanisms of plankton growth and evolution. In this dissertation, some theories related to population dynamics, impulsive control dynamics, stochastic dynamics, as well as the methods of dynamic modeling, dynamic analysis and experimental simulation, are applied to reveal the effects of some key biological factors on the dynamics mechanisms of the spatial-temporal distribution of plankton and the termination of planktonic blooms, and to predict the dynamics evolutionary processes of plankton growth. The main results are as follows:

Firstly, to discuss the prevention and control strategies on planktonic blooms, an impulsive reaction-diffusion hybrid system was developed. On the one hand, the dynamic analysis showed that impulsive control can significantly influence the dynamics of the system, including the ultimate boundedness, extinction, permanence, and the existence and uniqueness of positive periodic solution of the system. On the other hand, some experimental simulations were preformed to reveal that impulsive control can lead to the extinction and permanence of population directly. More precisely, the prey and intermediate predator populations can coexist at any time and location of their inhabited domain, while the top predator population undergoes extinction when the impulsive control parameter exceeds some a critical value, which can provide some key arguments to control population survival by means of some reaction-diffusion impulsive hybrid systems in the real life. Additionally, a heterogeneous environment can affect the spatial distribution of plankton and change the temporal-spatial oscillation of plankton distribution. All results are expected to be helpful in the study of dynamic complex of ecosystems.

Secondly, a stochastic phytoplankton-zooplankton system with toxic phytoplankton was proposed and the effects of environmental stochasticity and toxin-producing phytoplankton (TPP) on the dynamics mechanisms of the termination of planktonic blooms were discussed. The research illustrated that white noise can aggravate the stochastic oscillation of plankton density and a high-level intensity of white noise can accelerate the extinction of plankton and may be advantageous for the disappearance of harmful phytoplankton, which imply that the white noise can help control the biomass of plankton and provide a guide for the termination of planktonic blooms. Additionally, some experimental simulations were carried out to reveal that the increasing toxin liberation rate released by TPP can increase the survival chance of phytoplankton population and reduce the biomass of zooplankton population, but the combined effects of those two toxin liberation rates on the changes in plankton are stronger than that of controlling any one of the two TPP. All results suggest that both white noise and TPP can play an important role in controlling planktonic blooms.

Thirdly, we established a stochastic phytoplankton-toxic producing phytoplankton-zooplankton system under regime switching and investigated how the white noise, regime switching and TPP affect the dynamics mechanisms of planktonic blooms. The dynamical analysis indicated that both white noise and toxins released by TPP are disadvantageous to the development of plankton and may increase the risk of plankton extinction. Also, a series of experimental simulations were carried out to verify the correctness of the dynamical analysis and further reveal the effects of the white noise, regime switching and TPP on the dynamics mechanisms of the termination of planktonic blooms. On the one hand, the numerical study revealed that the system can switch from one state to another due to regime shift, and further indicated that the regime switching can balance the different survival states of plankton density and decrease the risk of plankton extinction when the density of white noise are particularly weak. On the other hand, an increase in the toxin liberation rate can increase the survival chance of phytoplankton but reduce the biomass of zooplankton, which implies that the presence of toxic phytoplankton may have a positive effect on the termination of planktonic blooms. These results may provide some insightful understanding on the dynamics of phytoplankton-zooplankton systems in randomly disturbed aquatic environments.

Finally, a stochastic non-autonomous phytoplankton-zooplankton system involving TPP and impulsive perturbations was studied, where the white noise, impulsive perturbations and TPP are incorporated into the system to simulate the natural aquatic ecological phenomena. The dynamical analysis revealed some key threshold conditions that ensure the existence and uniqueness of a global positive solution, plankton extinction and persistence in the mean. In particular, we determined if there is a positive periodic solution for the system when the toxin liberation rate reaches a critical value. Some experimental simulations also revealed that both white noise and impulsive control parameter can directly influence the plankton extinction and persistence in the mean. Significantly, enhancing the toxin liberation rate released by TPP increases the possibility of phytoplankton survival but reduces the zooplankton biomass. All these results can improve our understanding of the dynamics of complex of aquatic ecosystems in a fluctuating environment.

CO-AUTHORSHIP

For all the chapters in this dissertation, I was the investigator including the design of research, data collection and analysis, model building, modeling analysis and computer simulation experiments. I wrote the original manuscripts and was responsible for incorporating some comments, suggestions and feedbacks into the revised manuscripts. Dr. Min Zhao and Dr. Jianbing Li supervised the research and contributed to the research design, data analysis and the revision of the manuscripts. Dr. Chuanjun Dai and Dr. Hengguo Yu contributed to the design and implementation of experimental simulations and helped review and improve the manuscripts, so they are included in authorship on all resulting publications. Dr. Qi Wang, Dr. Aimin Hao, and Dr. Jun Kikuchi provided some useful suggestions in data collection and analysis, and they contributed to some comments on the manuscripts, so they were included in related publications.

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Chapter 1 GENERAL INTRODUCTION

1.1. Background

Plankton populations are floating organisms of many different phyla that live in the pelagic of oceans, in freshwater lakes, or in larger rivers, estuaries and reservoirs, and they are greatly influenced by water movements (Sommer, 1994, 1996). Phytoplankton are the microorganisms and commonly unicellular and microscopic in size, requiring some essential elements (such as light, carbon dioxide and a range of inorganic and organic nutrients) to photosynthetically produce biomass. Zooplankton population includes the animals in the plankton community living on phytoplankton. As the foundation of the entire aquatic food webs, plankton populations play an important role in the material circulation and energy transportation for real aquatic ecosystems, and they are great of global significance for climate regulation and biogeochemical cycling (Odum, 1971; Simo, 2001). On the one hand, phytoplankton are of fundamental importance in supporting the primary productivity of the real aquatic ecosystems (Behrenfeld and Falkowski, 1997; Field et al., 1998; Hoppe et al., 2002), which contribute to the climate regulation by absorbing carbon dioxide from surrounding environments (Duinker and Wefer, 1994) and provide the materials and energy for all life in aquatic environments (Kawecka and Eloranta, 1994). On the other hand, zooplankton can control the quantity and spatial-temporal distribution of phytoplankton by feeding on them (Martin, 1970; Lampert and Taylor, 1985; Lampert et al., 1986; Wyatt and Horwood, 1973; Levin and Segel, 1976; Griffin et al., 2001). Meanwhile, plankton populations are the main food resources for high-trophic organisms such as fish and other larger animals, and their distribution and biomass variation can directly affect the status of fishery and other aquaculture (Platt et al., 2003), which play a critical role in the structures and functioning of aquatic ecosystems (Akhurst et al., 2017). Thus, plankton play a significant role in economic and social values for aquatic ecosystems.

Due to the rapid urbanization, industrialization, intensifying agricultural production and the increase of the global population (United Nations Population Division, 2008), a large amount of industrial and agricultural wastewater, domestic sewage and farming practices are discharged into bays, rivers, lakes and oceans, enriching high level of nutrients and minerals in these water bodies, and finally resulting in water eutrophication (Webster et al., 2001). Actually, the process of eutrophication consists of a continuous increase in the contribution of essential biological nutrients, including nitrogen and phosphorus (organic loading), until it exceeds the capacity of the water bodies (i.e. the capacity of lakes, rivers or oceans to purify themselves), which may contribute to the changes of structures and functions in these water bodies. Actually, these changes strongly depend on the use of fertilizers, untreated industrial wastewater, aquaculture runoff and discharge, and the reduction of self-purification capacity of water bodies. Under the conditions of eutrophication, the rates of organic production for real aquatic ecosystems exceed those of consumption (Qin et al. 2013), causing the massive accumulation of organic matters in water bodies that may seriously destroy the ecological balance of aquatic ecosystems and greatly promote the occurrence of HABs (Jager et al., 2017). In addition, the eutrophication process can be greatly accelerated due to the effects of some factors, for example, a slow current velocity, the poor self-purification capacity (Heisler et al., 2008), a suitable temperature (for example, 25℃ or above) (O’Neil et al., 2013), an adequate light exposure (Sheiwastav et al., 2017), and point-source discharges and non-point loadings of limiting nutrients. Based on these, the Organization for Economic Cooperation and Development (OECD) that was led by 18 member countries to monitor and carry out the research of the trophic status and eutrophication in about 150 lakes, proposed the standard criteria of quantifiable indicators for eutrophication in 1982, that is, the average concentration of total phosphorus (TP) >0.035 mg/L, the average concentration of Chlorophyll-a >0.008 mg/L, and the average water transparency <3 m (OECD, 1982). The ecologist Jorgensen pointed out that the excessive growth of algae is a key process in the formation of eutrophication (Jorgensen, 1983), and a number of studies showed that the water bodies in the areas where algal blooms occur have been seriously polluted, and the levels of nutrients (i.e. nitrogen and phosphorus) have greatly exceeded the standard criteria from OECD. According to these thresholds, many lakes in China have experienced different degrees of eutrophication, which have become one of the most important ecological and environmental problems facing China's freshwater lakes (Qin et al., 2013).

Eutrophication is an enrichment of essential nutrients in water bodies, which can lead to changes in the structures and functions of aquatic ecosystems. For example, increasing production of aquatic plants (Rast and Thornton, 1996), consuming large amount of dissolved oxygen (Mu et al., 2017), deteriorating water quality (Western, 2001; Capuzzo et al., 2015), reducing the biomass of the harvestable fish and shellfish (Smith, 2003; AI Gheilani et al., 2011), increasing the biomass of consumer species (Smith, 2003), and ultimately affecting biodiversity (Nasri et al., 2008). Thus, water eutrophication has become an urgent environmental issue worldwide in recent years (Nyenje et al., 2010; Liu et al., 2011). Indeed, it was reported that over 75% of closed water bodies in the world are associated with eutrophication (Freedman, 2002). According to the survey of the state of the World's Lakes, a project promoted by the International Lake Environmental Committee (ILEC), eutrophication affects 54% of Asian lakes, 53% of lakes in Europe, 48% of lakes in North America, 41% of lakes in South America, and 28% of lakes in Africa in varying degrees. Due to the degradation of water quality caused by eutrophication, many limnological studies on lakes, rivers, and streams have been emphasized in recent years (Saxena, et al., 1988). Especially, some freshwater lakes in USA, such as 17 lakes of Western Washington, Washington (Welch and Crooke, 1987), Lake Okeechobee, Florida (the average water depth of 2.7 m) (Schelske, 1989), Lake Apopka, Florida (the average water depth of 4.7 m) (Coveney et al., 2002), and Lake City Park, Louisia (Ruleya and Rusch, 2002), have seriously suffered from eutrophication, and the estimated cost of damage caused by entrophication in USA alone was approximately $2.2 billion annually for tourism restoration (Dodds et al., 2009). In China, 25% of drinking sources are lake water or reservoir houses, but more than half of lakes have become eutrophic and hypereutrophic (Yang, 2009). For example, Lake Taihu (the average water depth of 2 m), the third largest freshwater in China, experienced the toxic cyanobacterial blooms in the summer of 2007, leading to the crisis of drinking water supplies for 2 million people in Wuxi City, Jiangsu Province (Qin et al., 2010). According to the work of Yang et al. (2008), approximately 30 billion ton of polluted water have been discharged directly into the lakes, which may cause all the urban lakes and the most of the medium-sized lakes at the urban-rural fringe areas in China facing the problem of water eutrophication by 2030. Other cases of lake eutrophication such as Lake Kastoria in Greece (Koussouris et al., 1991), Lake Biwa in Japan (Yamashiki et al., 2003), and Lake Bellandur in India (Chandrashekar et al. 2003), have been reported globally in recent years. The main reason for the deterioration of water quality in these cases is primarily associated with high nutrient enrichment derived from anthropogenic activities.

Dating back to the 1960s and 1970s, many scientists and researchers have linked the issue of algal blooms to nutrient enrichment resulting from human activities such as agriculture, industry, and sewage disposal (Schindler, 1974; Hallegraeff, 1993), and an obvious and problematic symptom of eutrophication is the increase of harmful phytoplankton biomass resulting in the occurrence of algal blooms (Tang et al., 2010). More precisely, when phytoplankton blooms occur at the surface or at specific depths in the water column, the number of a certain dominant phytoplankton species rapidly increases or almost equally rapidly decreases in aquatic ecosystems and then returns to its original low level under some conditions (Beltrami and Carroll, 1994). The dominant species of algal blooms are different during harmful algal blooms (HABs) (Smayda, 1997) due to their living aquatic environments. In this respect, HABs mainly include different types of taxa, such as diatom (e.g. *Alexandrium*), dinoflagellate (e.g. *Pfiesteria*), and cyanobacteria (or blue-green algae), among which cyanobacteria are of special importance because of their potential impact on drinking water sources or recreational water bodies (Carmichael and Wayne, 2001; Yang et al., 2012). Additionally, algal blooms are accompanied by toxin productions from HAB species (Paprdimitriou et al., 2012), such as cyanobacterial toxins, which can contaminate seafood and fish or kill other higher trophic level organisms (Penaloza et al., 1990 and Tencalla et al., 1994), and even directly or indirectly threaten human health through the food chain of aquatic ecosystems (Francis, 1878; Hallegraeff, 1993; Codd and Bell, 1996). Actually, the events of the death of marine animals or even human being caused by toxic algae have been reported globally with an increasing frequency (Naiman et al., 1995) since the first case of animal death caused by toxic cyanobacteria in 1878 (Francis, 1878). For example, many poisoning episodes caused by algal toxins occurred in the United States, Australia, France, Zimbabwe, and other countries (Falconer et al., 1983; Gugger et al., 2005; Ndebele and Magadze, 2006; Nasri et al., 2008). Significantly, HABs have dramatic consequences for the ecological balance of aquatic ecosystems, drinking water resources, economic losses, fisheries and tourism (Capenter et al., 1998; Wells et al. 2015; Yang et al., 2012). Based on these huge effects of algal blooms, it is urgent and important to find an effective way in the aquatic ecology for controlling and preventing HABs.

1.2. Statement of the problem

HABs have become one of the most severe problems in real aquatic ecosystems worldwide recently, and there have been reported globally with an increasing frequency and intensity over the last few decades (Hallegraeff, 1993). The outbreaks of algal blooms have and continue to pose a serious threat to the balance and stability of aquatic ecosystems, which not only cause profound and deleterious effects on water quality (Capuzzo et al., 2015), water landscape (Mitra and Flynn, 2006), and aquaculture development (Uye, 1986; Wyatt and Horwood, 1973; Levin and Segel, 1976), but also pose a serious health hazard to animals and human beings (Nasri et al., 2008; Falconer et al., 1983; Gugger et al., 2005; Qin et al., 2010). Moreover, excessive growth of algae can be caused by nutrients enrichment, and especially phosphorus which is the primary nutrient necessary for abundant algae and aquatic plants growth (Paerl et al., 2001; Mainstone and Parr, 2002). Once a water body is eutrophicated, it will lose its primary functions and structures and subsequently affect the sustainable development of economy and society. For a successful water restoration and the plan of eutrophication mitigation, the basic aim is to identify the sources of pollution and their subsequent remediation. Thus, many municipal authorities have passed some relevant legislations and laws to mitigate and limit the point sources and non-point sources loading of nutrients, such as prohibiting the usage of phosphate-containing laundry detergents (Kundu et al., 2015), optimizing agricultural land use and best policies (Alvarez et al., 2017; Dai et al., 2018), restoring animal and plant communities in local ecosystems (Zhang et al., 2009), and adapting domestic wastewater treatment and waste classification (Huang et al., 2017). But eutrophication and algal blooms are still prevalent on the water surface worldwide (Smith et al., 1999; Smith and Schindler, 2009) and some refinements are still going on due to the ever-increased complexity and specific problems that require wider experience. Therefore, it is important and urgent for water resource managers to understand how to minimize the intensity and frequency of algal blooms (Paerl and Paul, 2012), and the prevention and control approaches of algal blooms have become an important ecological topic in the protection of aquatic environments around the world. Actually, many researchers have investigated and developed technologies and methods for the prevention and controlling of algal blooms since the 1960s, and the purposes of these strategies are to kill harmful algae organisms or limit their growth (Schindler, 2006; Carpenter, 2008; Lewis and Wurtsbaugh, 2008; Sengco, 2009), but the algal blooms control or suppression activities are controversial and challenging. Generally, the most common types for controlling and suppressing HABs include physical, chemical, and biological technologies (Anderson, 2009).

1.2.1. Physical method

Among the algal control methods, physical method is one of the most applicable techniques in freshwater lakes. This method is a technical method to control algae pollution through physical engineering measures, such as ultrasonic algae removal, flocculants addition, shading algae removal, and other related technologies.

Ultrasonic algae removal is an important research direction of algae removal technology, and ultrasonic technology has been applied for monitoring algae growth for water and wastewater treatment (Mahvi, 2009; Wang et al., 2019). Actually, ultrasound is a sound wave with high frequencies above the limit of human hearing (22 kHz) that can control algae growth at special frequency, and the commonly used frequencies for industrial cleaning are those between 20 and 50 kHz (Suslick and Price, 1999). In addition, the process of ultrasound is to generate a sound wave layer in the top layer of a water body, and then the generated sound wave layer can directly destroy the buoyancy of algae cells, causing these cells to sink and eventually die. Generally, the ultrasound technology is advantageous to monitor the real-time water quality, and predict and control algal blooms. In past decades, a lot of works have been done by researchers and some excellent results have been achieved. For example, the work of Tang et al. (2003) showed that a high frequency of ultrasound has a great influence on the density of cyanobacteria cells at the early stage of its growth, and further concluded that the growth rate of cyanobacteria cells after irradiation can be reduced to 38.9% of the control group in a short time, while dispersive ultrasound irradiation is an effective method to inhibit the rapid growth of cyanobacteria cells. Joyce et al. (2010) studied the effect of ultrasonic frequency on the growth of Microcystis aeruginosa in an experiment, and the best results were achieved when the ultrasonic frequency was set at 580 kHz. Significantly, ultrasonic algae removal has been proven the most effective, environmentally and friendly technology to control algal blooms in lakes, reservoirs, and ponds, which provide a long-term solution for a healthy ecosystem and is harmless to humans, plants, and other aquatic life. However, most modern ultrasonic devices strongly rely on transducers which are composed of piezoelectric materials, and it is difficult to implement this technique successfully to remove algae from large areas of water bodies due to the high power and determining the specific ultrasonic frequency (NEIWPCC, 2015).

One promising control strategy may be the rapid sedimentation of HABs through flocculation with clay (Shirota, 1989; Anderson, 1997; Sengco, 2001; Sengco and Anderson, 2004). The mitigation of HABs using clay was first proposed in the 1970s (Shirota, 1989), and the addition of clays particles (such as nontoxic and inexpensive) or any flocculants can help to carry bloom to the bottom of sediments, which is the oldest and most widely used approach to control HABs (Anderson, 1997; Pierce et al., 2004; Sengco and Anderson, 2004). When sprinkled on the surface of water bodies during HABs, these tiny but dense clay particles will ‘flocculate’ or combine with other particles in water bodies, including HABs cells. The process of flocculation formation is to remove these cells through sedimentation, and the rapid sedimentation of algae cells may be caused by the flocculation (Jackson and Lochmann, 1993). And also, clays have been investigated in some countries as a means of removing harmful algae from water column (Shirota, 1989; Yu et al., 1994; Sengco and Anderson, 2004; Lee et al., 2008; Song et al., 2010). For example, South Korea, where a fish-farming industry worth hundreds of millions of dollars is seriously threatened by HABs, this control method has been proved to be a flexible and economically and socially feasible method for water treatment, and thus great progress has been made (Na et al., 1996; Lee et al., 2008; Liu et al., 2010). However, one of the limitations of this strategy may need more clay consumption (Sengco, 2001; Yu et al., 2004), resulting in excessive sediment siltation and heavy dredging work. Additionally, when algae are removed from underwater by the clay coagulant, they cannot be prevented from floating in shallow rivers and lakes, and the release of toxic substances from decomposition can cause secondary pollution (Lee et al., 2008; Orizar et al., 2013).

It is well known that the growth of algae is closely related to the photosynthesis by the light, and thus light is one of the important factors restricting algae growth and diversity (Marra and Heineman, 1982; Nanninga and Tyrrell, 1996). For example, the work of Shen (2002) showed that the growth rate of algae increases with the increasing of light intensity, and found that the best situation of algal growth is at the luminescence of 4000 lux. Also, the strong or weak light was unfavorable for phytoplankton growth, and the function together with suitable temperature, light intensity and ample sunlight encouraged algal blooms under the same water quality and hydrodynamic conditions (Cao et al., 2011). However, light-shielding can significantly inhibit the photosynthetic rate of algae and promote algae extinction due to the consumption of dissolved oxygen (Kirk, 1994). For example, Chen et al. (2009) concluded that the algae biomass can reduce rapidly under the light-shading condition, and also, the reduction efficiency is increased when the light-shading is accompanied by aeration, indicating the feasibility of reducing harmful algae by light-shading plus aeration. The work of Ye et al. (2007) illustrated that the concentration of Chlorophyll-a, turbidity and chemical oxygen demand decrease significantly after 9 days of light-shading, and their removal rates reach at 80.1%, 68.0% and 93.8%, respectively, which verifies the feasibility of the light-shading method, and further indicates that the shading of sunlight may still be a viable measure to control algal blooms in natural water purification systems. Generally, it seems to be ineffective, expensive and impractical to apply this strategy to the large scale and complex ecosystems for the proper algal control (Edmondon, 1970), and also the light-shading efficiency strongly depends on the temporal and spatial differences of different lakes and objectives of algal control.

Other physical technologies such as membrane filtration (Castaing et al., 2010; Liang et al., 2008), air flotation (Teixeira et al., 2010) and electrochemical technologies (Rodrigo et al., 2010; Hasan et al., 2016; Chaplin, 2019), have also been widely used around the world. For example, electrochemical technologies, the methods combining both physical chemistry and electronic science, can simultaneously treat multiple classes of contaminants with high removal efficiency through the production of chemicals at the electrode surfaces with low power and energy demands, which have proved to be a clean, flexible and environmentally friendly way for wastewater treatment (Giwa et al., 2019; Chaplin, 2019). However, the development of the destructive electrochemical technologies requires the manufacture of non-toxic, low-cost, and high surface area electrodes, which have long service life and can be operated without harmful toxic by-products (Tang et al., 2019; Chaplin, 2019).

1.2.2. Chemical method

Chemical treatment method involves treating wastewater with various chemical additives that inhibit or be lethal to HAB organisms, including chemical agent method, electrochemical method and photochemical degradation method. The former is the most commonly used method, while the latter two methods are less reported. The chemical agents, such as metals (Magdaleno et al., 2014), algaecides and herbicides (Nagai et al., 2016), can control the propagation of algal blooms due to its adverse effects on the other organisms and the expedited release of *Microcystins* (EPA of China, 2000). In fact, chemical control was attempted in 1957 against the Florida red tide organisms using copper sulfate delivered with crop dusting airplanes in small lakes and ponds (Rounsefell and Evans, 1958). After that many related works have been done. For example, the work of Kaya et al. (2005) found that the combination of lysine and malonic acid could selectively control toxic *Microcystis* blooms and induce the growth of *Macrophytes*. Li et al. (2007) carried out an experiment to study on the emergency treatment of toxic cyanobacteria blooms in basalt lake, and the results shown that algaecides may remove cyanobacteria temporally, but it cannot resolve the core problem of lake eutrophication. However, the most types of chemicals are not species-specific which may have a potential negative impact on ecological balance. For example, the use of algaecides, such as copper sulfate, copper chelates, chemical Endothall, has been proved to be an effective way to control and prevent HABs in small-sized lakes and ponds temporally (Boyd and Tucker, 1998). If the amount of required copper sulfate may not be prepared carefully by considering the alkalinity or acidity of the reservoir, the copper sulfate will become toxic to fish and other large aquatic animals (Anderson, 2009). Although the use of chemicals is the most common and versatile management strategy for controlling nuisance aquatic plant populations, this method is neither practical nor advised in large ecosystems, or any waters to be used for fishing, drinking and other animal and human use. Also, most chemicals are too expensive and need frequent dosing, and they may not resolve the primary cause of the problems, or even may pose risks to human being, livestock and wildlife since these chemical treatments are often associated with the changes of PH value or salinity (Murphy et al., 1999; Hullebusch et al., 2002), in addition to causing damage to non-target aquatic organisms that may reappear after treatment, and thus it is difficult to find environmentally friendly and acceptable chemical way to help to control particular harmful algae (Anderson, 2009).

1.2.3. Biological method

The biological method is a way to increase the grazing pressure on toxic phytoplankton or to reduce recycling of nutrients by using the principle of the food chains of aquatic ecosystems and the relationship between organisms, including ecological floating island technology, aquatic plants competition technology, filter-feeding fish control technology and harmful phytoplankton control technology.

Ecological floating island, a biotical-ecological restoration technology, refers to hanging biological fillers on the fixed support to make microorganisms, protozoa and small zooplankton floating on the surface of the fillers, so as to improve the wastewater purification by increasing the biomass per unit volume. Ecological floating island technology has drawn increasing attentions in biotical-ecological restoration due to its convenient operation, low environmental risks and effective treatment (for example, pollutant purification, ecosystem restoration and landscape improvement). The existing evidence showed that the effects of water spinach on nitrogen and phosphorus removal, and chemical oxygen consumption are great of significance for wastewater treatment and can improve water quality of eutrophication (Wang, 1997). In addition, Wu et al. (2010) proposed a biopond-wetland system for controlling cyanobacterial blooms in a pond at Kunming City, Western China, and shown that when the hydraulic load of the biopond-wetland system is $500\ \mathrm{m^3/d}$ on non-rainy days, the efficiencies of the overall average nutrients removal rate are 83% (Chl-a), 57% (total nitrogen), 70% (total phosphorus) and 66% (Ammonia), respectively, indicating that the biopond-wetland system can significantly control the outbreaks of toxic cyanobacterial blooms. However, it is difficult to control the hydraulic retention time and the hydraulic loading rate, as well as the selection of floating island plants for ecological floating island due to the local climate change and the surrounding environments.

The aquatic plant competition method is to use the competition between aquatic plants and harmful algae for light, nutrients, oxygen, and release allele-chemicals into water environments to inhibit the growths of harmful algae. The work of Fang et al. (2007) showed that the growth of algae can be inhibited by the cultivation of beneficial algae or higher aquatic plants in water bodies. The integrated ecological floating bed simultaneously used plants (for example, *Ipomoea aquatica*), freshwater mussels (for example, *Clam fluminea*) and biofilm carriers (in artificial and semi-soft assembly) to perform better than the other two floating beds, one of which is composed of freshwater mussels and organisms, while the other is made of plants and biofilm carriers (Li et al. 2010). However, the decomposition of aquatic plants after apoptosis reduces the concentration of dissolved oxygen in the water bodies, resulting in emitting a foul smell and accelerating the deposition and inundation of the water bodies.

Macrophytes or algae are important organisms in the aquatic ecosystems, and they are the primary food sources for herbivore fishes and other large aquatic organisms. Thus, the excessive growth of algae can be controlled by adding filter-feeding aquatic organisms, such as silver carp, nile tilapia and bighead carp. Actually, the basic premise is that secondary consumers are removed through the addition of tertiary consumers, which allows for the dominance of large-bodied, generalist grazers to control phytoplankton population. For example, the study of

Infante and Riehl (1984) indicated that nile tilapia can swallow large amounts of algae and zooplankton by the alteration of a food-chain to restore the health of the ecosystem, resulting in the rapid decrease in the biomass of plankton populations. Starling (1993) carried out a mesocosm experiment on a medium controlled ecosystem in a tropical reservoir in Brazil, and the results showed that silver carp can successfully control *Microcystis* blooms. Godlewsk and Swierzowski (2003) also found that aquatic animal community can effectively inhibit the growth of algae in the lake, which can improve water quality for the long-term, but the effect of algae control is mainly related to the selection of aquatic species, distribution and pollution degree of the water bodies. However, when toxic phytoplankton or harmful algal blooms have filtered as food resource by fishes, their toxins may accumulate in shellfish and the high level of toxins can be lethal to humans or other consumers. Significantly, the effect of filter-feeding fish on water quality is typically short-lived and most obvious in small and easily managed aquatic ecosystems.

It is widely recognized that toxin-producing phytoplankton are a group of phytoplankton that have the ability to produce toxic chemicals into the aquatic environments during harmful algal blooms, which contribute to the negative effects on the economic and ecological values, and even pose a health hazard to animals and human beings (Hallam et al., 1983). For example, *Noctiluca scintillans* (Macartney), dinofagellates *Alexandrium acatenella* (Whedon et Kofoid) Balech, diatoms *Nitzschia pungens* f. *multiseries* Hasle, etc., are harmful to planktonic organisms (Halllegraeff, 1993). Thus, many researchers have paid much attention to study the effect of TPP on the termination of planktonic blooms from their field works and modeling analysis over the past decades (Chattopadhayay, et al., 2002; Chattopadhayay, et al., 2004; Sarkar et al., 2005; Sarkar and Chattopadhayay, 2003). For example, taking the species *Noctiluca scintillans* (harmful phytoplankton), *Chaetoceros sp.* (harmful phytoplankton) and *Paracalanus sp*. (zooplankton) as the research objective, and the monitoring of plankton populations were carried out from March, 1999 to January, 2001 in the Northwest coast of Bay of Bengal by Sarkar et al. (2005), the results from the biomass distribution observed in their filed works displayed that the presence of harmful phytoplankton leads to the persistence of all species through the termination of planktonic blooms and can be used as a controlling agent for the stability of marine ecosystem (Sarkar et al., 2005). Although some excellent and interesting results have been obtained, the reason for the occurrence of planktonic blooms and their possible control mechanisms is still unclear currently. Thus, the progress of such important areas urgently requires special attention both from experimental and mathematical ecologists, which may greatly stimulate researchers to further explore the possible control mechanisms of planktonic blooms.

As the basis of food chains in oceans, lakes, and reservoirs, plankton populations are the natural and essential components of healthy aquatic ecosystems, which can provide energy and materials to aquatic life. In most water bodies, there is a fine balance between adequate irradiance and nutrient supply that determines the rate of production of phytoplankton biomass, or primary productivity (Harris and Piccinin, 1977; Reynolds, 1984; Cloern, 1999). However, unnecessary and excessive growth of phytoplankton may have a harmful impact on the aquatic ecosystems under some suitable conditions, and their density levels can significantly affect ecological balance by disrupting natural balance and reducing water quality. Thus, to control the occurrence of algal blooms and regulate the balance of aquatic ecosystems, some common methods technically, such as physical, chemical, and biological strategies, have been widely applied to control harmful algae growth and inhibit bloom formation. However, these algal blooms treatment methods all have their own pros and cons currently. To be more precise, chemical control and physical control may have the advantage of efficiency, but the practical observations show that these two control methods consume much resources (i.e., labor-intensive and high costs), and may cause secondary pollution. The effectiveness of biological control strategy may not be as fast as that of the former two control strategies, but some of biological control method are successful and environmentally friendly. Significantly, their control durability may be better, as these control methods are implemented based on the principle of ecological balance. Although all these three methods have made great progress in controlling algal blooms, the process by which harmful algal blooms occur is still not completely understood, and understanding of dynamics mechanisms of changes in plankton populations becomes much more significant. Therefore, the development of the effective methods and technologies for controlling and preventing aquatic algal blooms is needed.

1.3. Objectives of the study

In recent years, biologists, as well as ecologists, have paid increasing attention to explain the influence of some factors on the formation and disappearance of algal blooms using a number of experiments (De Baar et al., 1995; Beman et al., 2005), and some hypotheses have been established. For example, Critical Depth Hypotheses (Sverdrup, 1953) and Dilution-Recoupling Hypotheses (Behrenfeld, 2010). The results from experiments indicated that many factors affecting the dynamics of plankton growth are bound to exist, for example, iron supply (Boyd et al., 2000; Coale et al., 1996) and light (Stomp et al., 2004), but they can only qualitatively explain the dynamics of plankton growth. Due to the complexity and openness of real aquatic ecosystems and the limitations of the current technology, which contribute to the difficulty to understand the mechanisms of plankton growth only depend on experimental and field observations, and the key to solve the issue of algal blooms is to understand the dynamics mechanisms of algae growth from dynamical analysis. In this respect, mathematical modeling of plankton can provide the quantitative insight into the dynamics of plankton growth. Moreover, the study of algal blooms is to establish dynamics models for the evolutionary processes of blooms formation and dissipation according to the dynamics mechanisms of population and water physical environments, and thus mathematical modeling is now a classical way to study the planktonic blooms (Truscott and Brindley, 1994a). Generally, dynamics modeling can be used to analyze and solve the solutions of the models, understand the parameter characteristics of the dynamics systems, and finally simulate the models by the means of computer technology.

Based on the background of the ecological environments in the subtropical lakes and reservoirs, the dissertation research mainly takes the planktonic algae as the research objective, aiming to model the dynamics mechanisms of plankton growth and evolution. In this dissertation, some theories related to population dynamics, impulsive control dynamics and stochastic dynamics, as well as the methods of dynamics modeling, dynamics analysis and experimental simulation, are applied to reveal the dynamics mechanisms of the spatial-temporal distribution of plankton and predict the dynamic evolutionary processes of plankton growth. Significantly, the objectives of the dissertation are to study some nonlinear problems of population dynamics, such as the dynamics mechanisms of the termination of planktonic blooms, the dynamics mechanisms of spatial-temporal evolutionary processes of plankton growth and the dynamics mechanisms of the dynamic prevention and control of planktonic blooms. The specific objectives include:

- Establishing an impulsive reaction-diffusion hybrid system and analyzing the dynamics of the proposed system. Several simulation experimental factors, for example, impulsive control and the heterogeneity and homogeneity of environments, are taken into account to investigate the effects of these factors on the survival of species, the dynamics spatial-temporal distribution of species, and the dynamics evolutionary mechanisms of species by means of the experimental simulation technology.
- Considering a stochastic phytoplankton-zooplankton system with toxic phytoplankton. Environmental stochasticity (for example, white noise) was firstly chosen as a control parameter, aiming to explore the dynamical behaviors of plankton under the environmental fluctuations. And then the toxin liberation rate released by TPP was examined to reveal the dynamics mechanisms of the formation and disappearance of planktonic blooms, and finally the combined effects of two liberation rates on the process performance of planktonic blooms were further discussed using the

experimental simulations.

- Building a stochastic phytoplankton-toxic phytoplankton-zooplankton system under regime switching, and devoting our attention to address the issues though studying the dynamics of the system. To be more precise, studying the long-time behaviors of phytoplankton-toxic phytoplankton-zooplankton system in a fluctuating environment and revealing the effects of white noise, regime switching and TPP on the dynamics mechanisms of the termination of planktonic blooms.
- Developing a stochastic non-autonomous phytoplankton-zooplankton system involving toxin-producing phytoplankton and impulsive perturbations, and investigating the effects of these factors on the dynamics of plankton model. More precisely, we aim to study how do environmental noise and impulsive control influence the survival of plankton populations and what influence in the peak of the cyclic outbreaks of planktonic blooms in an impulsive perturbations and fluctuating aquatic environments.

1.4. Organization of the dissertation

The dissertation is organized as follows: Chapter 2 presented some literature review. In chapters 3-6, four different mathematical and biological models of plankton dynamics were proposed, respectively, aiming to study some key biological factors affecting the growth of plankton populations in aquatic ecosystems and reveal the dynamics mechanisms of the termination of planktonic blooms (see Fig. 1.1). They mainly include an impulsive reaction-diffusion hybrid system (in chapter 3), a stochastic phytoplankton-zooplankton system with toxin-producing phytoplankton (in chapter 4), a stochastic phytoplankton-toxic phytoplankton-zooplankton system under regime switching (in chapter 5), and a stochastic non-autonomous phytoplankton-zooplankton system involving toxin-producing phytoplankton and impulsive perturbations (in chapter 6). The conclusion of this research and the recommendations for future research were provided in chapter 7.

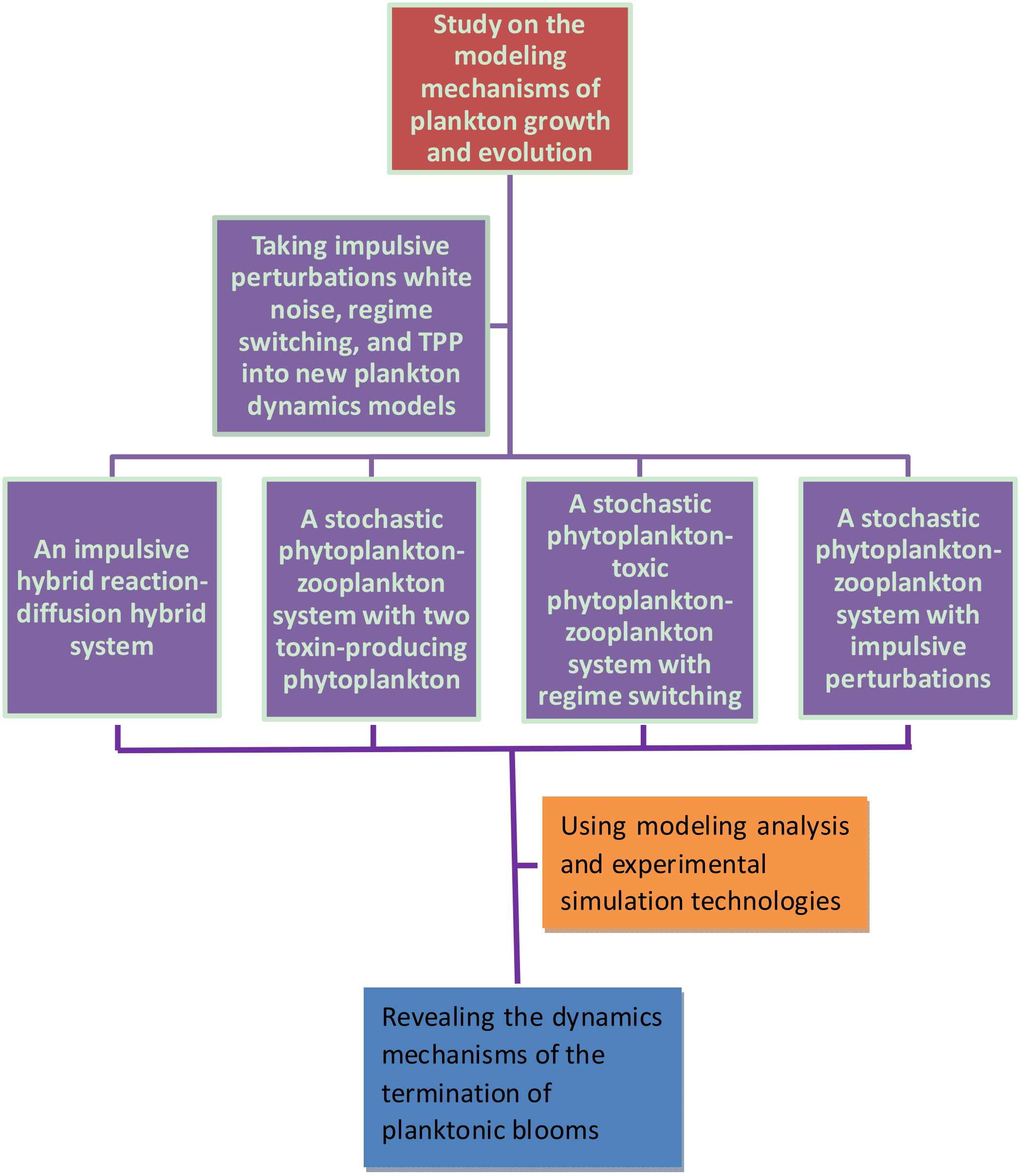

Fig. 1.1. Schematic diagram for the structure of the research.

Chapter 2 LITERATURE REVIEW

2.1. Plankton dynamics model

The dynamics of plankton studies mainly the dynamics mechanisms of plankton and the effects of some internal and external factors on the mechanisms in real-world aquatic environments. Dating back 100 years ago, many scientists and researchers have investigated the dynamics of plankton models, and the plankton research have regularly combined with field observations, laboratory experiments, as well as mathematical modeling from the beginning. Due to the strong positive correlations between zooplankton and fish abundance, fisheries and climate change have greatly stimulated the common interest of researchers in studying the dynamics of plankton populations. Actually, since the pioneering work of Fleming (1939), many mathematical models have been developed and explored extensively for the phytoplankton productivity (Ivlev 1945; Riley, 1946; Odum, 1956). Subsequently, a collection of the most frequency used models was presented by Behrenfeld and Falkowski (1997), which could be of great help in estimating the productivity of marine ecosystems. All these works have greatly lead to the rapid development of plankton dynamics models.

More precisely, the first attempt to study plankton dynamics models by using a simple ordinary differential equation can be traced back to the work of Fleming (1939), who demonstrated that zooplankton grazing plays a crucial role in the controlling of phytoplankton blooms and opened the chapter of the study of the dynamics of plankton models. Subsequently, Ivlev (1945) introduced the functional response to more realistically describe the interactions between phytoplankton and zooplankton populations. And then some other functional responses, such as Holling-type response terms (Holling, 1959, 1973), Mayzaud and Poulet (1978), have been widely considered to study the dynamics of plankton models (Steele and Henderohn, 1981, 1992; Malchow, 1993; Truscott and Brindley, 1994a; Scheffer, 2004; Garvie and Ttenchea, 2007). Other approaches the researchers are trying to construct the data-fitted functions (Riley, 1963) and apply Lotka-Volterra model to describe the interactions of prey-predator for phytoplankton-zooplankton (Dubois, 1975; Mimura and Murray, 1978; Segel and Jackson, 1972; Gierer and Meinhardt; 1972; Levin and Segel, 1976). In this respect, the understanding of plankton dynamics models has been greatly improved in recently years, which provide a deeper insight into solving some ecological problems (Huo and Li, 2004; Mukhopadhyay and Bhattacharyya, 2008; Zhang et al., 2014; Priyadarshi et al., 2017; Yu et al., 2018). In addition, due to the rapid development of population model and the idea of its construction, which have greatly lead to the establishment and development of plankton dynamics models. Thus, the following mainly introduces the related research progress of population model.

The study of population dynamics looks back over two centuries of history in the mathematical and ecological sciences, and the earliest and simplest population model is well-known Malthus population model. Actually, Thomas Robert Malthus published his "Principle of Population" thesis in 1798, and proposed the Malthus's theory that population is always growing in a geometric progression and advocated the implementation of population control (Malthus, 1798). It is assumed that the per capita growth rate of change in the number of individuals is a constant r and derived the following model of the variation of population:

$$\frac{dx}{dt} = rx \tag{2-1}$$

with an initial value $x(0) = x_0$, where x is the number of population at the time t. Noted that the trajectory of $x(t) = x_0 \exp(rt)$ is exponential. Such an exponential trajectory is of considerable theoretical significance, as it represents the long-term behaviors of age-structured population affected by constant mortality and fertility rates (Keyfitz, 1977). But the essence of this model is that each individual generates a constant number of offspring, regardless of the crowding (i.e. environmental carrying capacity) and the availability of food supply, which is not in line with the reality. In fact, the influence of external factors on the population growth is not considered in the Malthus model, which makes population growth always a constant. When the typical size of the population is small, the model can make a more accurate prediction, while it cannot reflect the real situation for a large human population base. In other words, the exponential population growth is incompatible with linear growth of food resources.

Considering that unchecked infinite exponential growth is patently unrealistic and the growth of population must be regulated by some biological factors, for example, the availability of food supply and the living environments of population. Thus, Verhulst (1838) introduced the density constraint to the basis of Malthus population model and initially derived the logistic growth model, where the per capita rate of change decreases linearly with the population size. Afterwards, Pearl applied the logistic growth equation, which has been derived independently by Lotka (1925), to model the population growth in US (Pearl, 1920). The equation can be presented as follows:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) \tag{2-2}$$

where $K(K > 0)$ is the environmental carrying capacity of population size, which means the maximum number of population that the natural environment can accommodate. Also, Verhulst assumed that the growth rate of population follows the law of logistic growth $r\left(1 - \frac{x}{K}\right)$. This implies that the population growth slows down as the number of population increases, indicting the environmental resources can significantly affect the growth of population. Other models, such as the Gompertz growth $\frac{dx}{dt} = rx\ln\left(\frac{K}{x}\right)$, can also show many of the same properties, but the logistic growth equation is arguably the best-known and most widely applied rate equation for the population growth and population invasion (Mendez et al., 2014).

It should be noted that both Eq. (2-1) and Eq. (2-2) only discussed the single population models, but species in nature are not independent of each other, for example, predation, competition, and parasitism. The first attempt to establish the multi-population prey-predator model by the work of Lotka and Volterra (Lotka, 1925; Volterra, 1926), who presented the Lotka-Volterra competition model that describes the population densities of prey x and predator y.

$$\begin{cases}\frac{dx}{dt} = x(a - a_1 y), \\ \frac{dy}{dt} = y(b_1 x - b).\end{cases} \tag{2-3}$$

The construction of the model is based on the following assumptions: the growth rate of the prey population density x, given as $x^{-1}\, dx/dt$, should be a decreasing function of the predator density y and greater than zero when the predator density is zero. Conversely, the predator growth rate $y^{-1}\, dy/dt$ should increase with the prey count, but be negative when $x = 0$. In addition, the parameters a, a_1, b, and b_1 describe the prey reproduction rate, predation rate, predator death rate, and conversion efficiency of prey into predator, respectively, and all the parameters are positive. This set of coupled ordinary differential equations gives rise to characteristic, undamped, non-linear oscillations. Generally, the classical Lotka-Volterra model can be given as the following form:

$$\begin{cases}\frac{dx}{dt} = x(a + a_1 x + a_2 y), \\ \frac{dy}{dt} = y(b + b_1 x + b_2 y).\end{cases} \tag{2-4}$$

However, the relationship between predator and prey described by classical Lotka-Volterra model is rather realistic, with the most obvious flaw being the unchecked growth of the prey in the absence of predator. In the 1930s, consequently, Gause proposed a generalized mathematical model in which the rate parameters effectively become response function of the respective species, allowing more realistic control of populations compared to the original Lotka-Volterra model (Gause, 1934; Royama, 1971). The Gause type predator-prey model is introduced as follows:

$$\begin{cases}\frac{dx}{dt} = f(x)x - g(x)y, \\ \frac{dy}{dt} = -dy + eg(x)y.\end{cases} \tag{2-5}$$

where d is the natural death rate of predator and e is the conversion efficiency of prey into predator. Function $f(x)$ denotes the growth rate function of the prey and $g(x)$ represents the predation response function. The selections of functional response are of great significance to describe the predator-prey phenomenon in different environments and many scholars have done a lot of works to construct a more accurate functional response function by field observations and laboratory experiments, for example, Michaelis-Menten uptake dynamics (Dugdale, 1967) provide a more realistic reflection of nutrient uptake dynamics (Huppert et al., 2002). In this dissertation, we mainly consider Holling-II functional response into the model. Actually, Michaelis and Menten (1913) proposed the following functional response when studying the saturation models of enzyme kinetics:

$$g(x) = \frac{\beta x}{x+h} \tag{2-6}$$

where β is the maximum growth rate of population and $h\,(h > 0)$ denotes the half-saturation constant. Actually, Holling obtained this function through the laboratory experiment (Holling, 1959) and named the function as Holling-II functional response. Fortunately, we apply the functional response to analyze the dynamics of phytoplankton-zooplankton system in this dissertation. Other functional responses such as Crowley-Martin type (1989), Beddington-DeAngelis type (Beddington, 1975; DeAngelis and Goldstein, 1975) are also considered in the following discussion.

Plankton models can be regarded as a specific application of predator-prey models, and its mathematical structure relies strongly on the generalization or improvement of the mentioned population systems. Motivated by the above works, the construction of plankton models is based on the following assumptions: the natural growth rate of phytoplankton follows the law of logistic growth; the capture of phytoplankton by zooplankton needs to be described by an appropriate functional response function and consider the natural death of zooplankton populations. Moreover, the study of plankton ecology is an important significance of the survival of our earth, but the occurrences of planktonic blooms have been reported globally with an increasing frequency over the past decades (Hallegraeff, 1993), and TPP are among the contributors in these blooms (Hallegraeff, 1993; Philips et al., 2004; Hallam and Luna, 1984). Therefore, considerable scientific attention towards harmful phytoplankton when studying phytoplankton-zooplankton systems has been paid (Estep et al., 1990; Huntley et al., 1986; Buskey and Hyatt, 1995; Wyatt and Horwood, 1973). For example, the field observation (Estep et al., 1990) and laboratory experiment (Huntley et al., 1986) shown that the toxicity may be as a strong mediator in the zooplankton feeding rate. Thus, it is necessary to consider the effect of the toxins produced by harmful phytoplankton on zooplankton when studying phytoplankton-zooplankton systems. Based on the field observation and model analysis, Chattopadhyay et al. (2002) proposed the following nonlinear coupled ordinary differential equations:

$$\begin{cases} \frac{dx}{dt} = rx\left(1-\frac{x}{K}\right) - \alpha f(x)y, \\ \frac{dy}{dt} = \beta f(x)y - dy - \gamma g(x)y. \end{cases} \tag{2-7}$$

Compared with Eq. (2-5), system (2-7) added an additional term $-\gamma g(x)y$ that describes the effect of phytoplankton toxins, where γ denotes the toxin liberation rate produced by TPP and $g(x)$ represents the distribution of toxic substances. Interestingly, the authors investigated the existence and local stability of positive equilibria and the existence of Hopf-bifurcation of the system by considering different combinations of functional response $f(x)$ and $g(x)$, and concluded that TPP may be used as a biological way to control the planktonic blooms (Chattopadhyay et al., 2002).

Based on (2-7), the dynamical behaviors of plankton systems with harmful phytoplankton, such as stability, bifurcation and chaos, have been explored extensively in recent years (Saha and Bandyopadhyay, 2009; Sarkar and Chattopadhayay, 2003; Luo, 2013; Jang, 2014; Roy et al., 2006; Upadhyay and Chattopadhyay, 2005; Chaudhuri et al., 2013). For example, the work of Saha and Bandyopadhyay (2009) shown some dynamical properties of the toxic phytoplankton-zooplankton system without time delay, and further investigated the existence of local Hopf-bifurcation induced by time delay, and studied the existence of stability switching phenomena and the stability of Hopf bifurcation and the direction of the bifurcating periodic solution. Sarkar and Chattopadhayay (2003) introduced the environmental fluctuations into the phytoplankton-zooplankton system with TPP, and the results shown that TPP and the controlling of the rapidity of environmental fluctuations are key factors influencing the termination of planktonic blooms. In this dissertation, we apply schematic few-species models to demonstrate that phytoplankton-zooplankton interactions can give rise to the dynamics mechanisms of plankton systems and predict the dynamic evolution process of plankton growth.

2.2. The research of nonlinear dynamics

The commonality of the above mentioned studies is to investigate the dynamics of the nonlinear problems in the plankton dynamics models. Generally, the research of nonlinear dynamics system includes: the study of nonlinear dynamics of nonlinear phenomena in dynamic systems and the establishment a nonlinear dynamic model respect to the practical problems. Due to the development of mathematics and computer technology, an increasing attention has been paid to the study of nonlinear dynamic systems. Especially, at the end of the 19th century, Poincaré proposed the concept of dynamics and established a qualitative theory (the theory that directly studies the behaviors of solutions without using the solutions of the equations) when studying celestial mechanics, which laid the foundation for the study of nonlinear dynamics (Poincaré, 1890). Almost at the same time, Lyapunov published his doctoral dissertation "General Theory of Motion Stability" between the year of 1882 and 1892, which greatly promoted the development of nonlinear dynamics (Lyapunov, 1966). Moreover, Lorentz was the first to discover the strange attractor (or called Lorentz attractor) and described it as the famous "Butterfly Effect" (Lorentz, 1963) in 1963, which opened the chapter of the study of chaotic dynamics. The discovery of chaos has become one of the most important achievements in the field of nonlinear dynamics (Lorenz, 1993). In addition, Prigogine (1969) put forward the theory of dissipative structure in 1969, which had a great influence on physics, chemistry, biology and other disciplines, and pushed the study of nonlinear dynamics to a new high level. Here, we do not full cover the discussions of all the branches of nonlinear science. Instead, we concentrate on the brief description of the dynamic methods of nonlinear dynamics in related to this dissertation.

2.2.1. The dynamics of stochastic system

An important problem in nonlinear and statistical physics is to under the underlying laws for various phenomena in dynamical systems forced by random environmental fluctuations. Many phenomena in nature are positively or negatively affected by environmental fluctuations, which can be called the stochastic phenomena. For example, among the modern industrial process, the social economy and other fields of a variety of dynamic systems, almost all of them are subjected to various random factors. When the intensity of random disturbance is small or the accuracy of the system is not high, the system can be generally regarded as a deterministic one. On the contrary, the interaction between nonlinearity and stochasticity of dynamical systems can give rise to unexpected phenomena which have no analogue in the deterministic case (Moss and MeClintock, 2007; Anishchenko et al., 2007). The problem is fundamental because it involves the interplay between the modern qualitative theory of deterministic dynamical systems and the stochastic analysis. For example, some properties of deterministic systems may be lost by using deterministic methods to describe some stochastic systems, and it is difficult to achieve the expected results by means of the analysis and control methods of the deterministic system theory for studying stochastic systems. Thus, it is necessary and important to consider the effects of stochastic fluctuations on the systems and take the study of the dynamic systems as stochastic systems in these cases (Mao, 1997a). Actually, stochastic fluctuations are usually referred to as random noise, and the solutions of the equations with random parameters (or stochastic dynamical systems) are called stochastic processes. Generally, deterministic systems can be described by ordinary differential equations, while the stochastic systems can be presented by stochastic differential equations (Mao, 1995, 1997a).

Since Kiyoshi Itô initiated stochastic analysis and introduced stochastic integral and stochastic differential equations in 1942, the stability theory of stochastic systems has been developed rapidly. In the middle of the last century, the study of stochastic dynamics was widely applied to the natural sciences, engineering and social sciences, as well as in the study of nonlinear stochastic problems in physics, chemistry, biology and medicine. Some related theories, such as stochastic stability, stochastic optimal control, stochastic limit cycle, stochastic bifurcation, and stochastic chaos, have been established by mathematicians, physicists and mechanics. Subsequently, many monographs, anthologies and textbooks on stochastic dynamics have been published in the world (Gihman and Skorohod, 1979; Khasminskii, 1980; Meyn and Tweedie, 1983; Mao, 1994, 1997b; Arnold, 1998; Chueshov, 2002). For example, Mao (1997b) introduced stochastic differential equation and its application theory in detail. Chueshov (2002) introduced the theory of stochastic differential dynamics system and its application. All these results have further lead to the development of the theory of dynamics of stochastic system.

In the real world, there exist many types of random noise and all species have evolved in the presence of environmental disturbance. The work of Robert et al. (1998) showed that the environmental fluctuations can be divided into the following groups: biologically based disturbance of populations (for example, overharvesting, invasion, disease, and their interactions) and physically based disturbance such as storm, volcanic eruption and forest fire. Generally, the Gaussian noises (Brownian motion or Wiener process) can describe the continuous and random amplitude at any time that often occurs in reality, which satisfies the Gaussian distribution function based on biological disturbance as a whole (Kingman and Feller, 1972). If the governing variables are or can be approximated as discrete, one can employ the well-established theory of Markov chains (Cox and Miller, 1977; Norris, 1998). There is now compelling evidence that the nature is usually assumed to respond to gradual change in a smooth way, however, many studies on lakes, coral reefs, oceans, forests and arid lands have shown that the smooth change can be interrupted by sudden drastic switches to a contrasting state (Scheffer et al., 2001). To describe such phenomenon, continuous discrete state Markov chains can be used to simulate random populations switching from one state to another or more (or regime switching) (Mao and Yuan, 2006). For example, Sun et al. (2018) revealed that the death rate of newborn Little Yellow Croaker (Larimichthys polyactis) are 0.53 and 0.84 at 22℃ and 16℃, respectively. At the same time, some biological experiments and observational evidences illustrated that many populations have scale-free characteristics in spatial distribution, which may be strongly related to the discontinuous Levy noise in the environments (Levy, 1925; Sato, 1999) (for example, micro-plankton (Reynolds, 2008) and bees (Stout and Goulson, 2001)). Therefore, many researchers have applied levy noise to describe the sudden and catastrophic interference based on physical principles (Cognata et al., 2010; Applebaum, 2009). In recent years, ecological models with random disturbances have attracted the interest of many researchers, and some excellent results have been achieved. However, most of these models are based on the classical Lotka-Volterrs models or the epidemic chamber models, and there are few literatures to study the aquatic ecological populations, especially plankton populations. Moreover, accumulating evidence showed that the environmental fluctuations can play an important role in real ecosystems (Carpenter et al., 2011; Meng et al., 2014), and the survival of plankton populations can be significantly affected by environmental stochasticity (Melbourne and Hastings, 2008). All the results could provide us a deeper understanding for the realistic aquatic ecosystems when incorporating environmental fluctuations into the systems. Here the question may arises: can environmental fluctuations such as white noise and regime switching are a cause of the termination of planktonic blooms? Therefore, in this dissertation, we mainly apply the theories of the dynamics of stochastic systems to study the temporal and spatial evolution of plankton distribution for plankton ecosystems and further reveal the distribution mechanism of plankton populations in fluctuating aquatic environments.

2.2.2. The dynamics of impulsive control system

In the real world, the states of systems will change suddenly because of the influence of some factors, such as drought, flooding, hunting, planting. Due to these changes of relatively short time interval at some fixed times, it is usually regarded as instantaneous discontinuous phenomena, which can be represented by impulsive effects that may be suitable for such phenomena (Lakshmikantham et al., 1989; Bainov and Simeonov, 1993). For example, during the outbreaks of aquatic algae, the implementation of mechanical measures, such as algae removal, will result in a short-term decrease in the density of harmful algae. In the production of fishery resources, the launch of young fish and the harvest of adult fish will cause instantaneous changes in fish density. Also, in agricultural production, people usually control pests by spraying insecticides or releasing natural enemies regularly, which lead to the rapid changes in the number of pests and natural enemies in a short period of time. These transients or almost instantaneous behaviors can indeed cause sudden and abrupt changes in the state of species, which cannot be described by continuous systems. Therefore, the greatest interest for people is what the response of systems to experiencing such sudden changes in the real world, and a relatively effective method to study such problems is to use the impulsive dynamics systems. The most prominent feature of impulsive dynamics systems is that it can fully consider the impact of instantaneous mutation on the system states, and profoundly and accurately reflect the law of action of things, which seems to be more realistic (Jannash and Mateles, 1984; Jin et al., 2004).

Impulsive dynamics system is a description of the evolution process of sudden changes of the states of systems under instantaneous disturbance using impulsive differential equations. Actually, since the pioneering work of Minman and Myshiks (Mi’man and Myshkis, 1960), impulsive dynamics systems have been the common interest among researchers, who are committed to its theoretical researches (Simeonov and Bainov, 1988, 1989; Bainov and Covachev, 1994; Samoilenko and Perestyuk, 1995; Chen, 2011, 2013). Especially, the applications of Lyapunov function and Razumikhin technique have given a good explanation of the properties of the solutions of impulsive systems, which greatly promoted the development of theory and practical applications. For example, the treatment of some diseases (Lakmeche and Arino, 2000; Panetta, 1996), pulse immune control and therapy (Donofrio, 2002a, 2002b; Shulgin et al., 1998, 2000), Microbial culture (Agur et al., 1993), population optimal control (Zhang et al., 2003; Mailleret and Grongard, 2008), pest control and management (Lu et al., 2003; Jatav and Dhar, 2014) and ecological environment management (Martins et al., 1997; Huppert et al., 2002; Liu et al., 2003; Dai et al., 2013; Wang et al., 2014). Significantly, the research on the impulsive dynamics systems is attractive and challenging because of the combined with the characteristics of continuous and discrete systems, but it is not simply a superposition of the two systems, but a new system. Additionally, the biggest difference among them is the impulsive dynamics system can describe some complex problems that cannot be described separately by the continuous or discrete dynamics systems.

Generally, impulsive dynamics system can be divided into the following systems: impulsive dynamics system in which impulse occurs at a fixed time (periodic impulsive dynamics system), impulsive dynamics system in which impulse occurs at variable time (time-varying impulsive dynamics system) and impulsive dynamics system depending on the state of system (state-dependent impulsive dynamics system) (Lakshmikantham et al., 1989; Bainov and Simeonov, 1993). In the past decades, the periodic impulsive dynamics systems have been widely explored and its theoretical systems have been improved (Liu and Chen, 2003; Chen et al., 2009; Liu et al., 2006; Wang et al., 2009; Wang et al., 2008; Yu et al., 2011; Xiang et al., 2009). In these existing literatures, they mainly studied the impulsive dynamics system in which the impulse occurs at a fixed time in the biological system, proved the existence and boundedness of the semi-trivial periodic solution of the system, and obtained the local and global asymptotic stability of the semi-trivial periodic solution, which are the key conditions ensuring the permanence of the biological populations. Significantly, the results from experimental simulations showed that chaos can occur in the process of long-time dynamic behaviors, and further discussed how to control the development of biological populations so that we can make use of biological resources and thus obtain greater economic benefits. As another important branch of impulsive control dynamics system, state-dependent impulsive control dynamics system can simulate the advantages of the controlling depending on the state of population. Recently, the studies on state-dependent impulsive differential equations have become the common interest among many researches (Tang and Cheke, 2005; Jiang et al., 2007; Dai and Zhao, 2012; He et al., 2015), and some properties of periodic solution, such as the existence of periodic solution, stability and periodicity, have been investigated (Nie et al., 2009; Dai et al., 2012). In addition, some existing literatures revealed that the existence, stability and bifurcation of periodic solutions of state-dependent impulsive differential equations using successive functions (Guo et al., 2014; Wei and Chen, 2014; Zhang et al., 2014; Ji et al., 2015; Xiao et al., 2015), indicating the easier way to obtain the economic benefits and practical value by taking the states of populations as the impulsive control parameter. For example, in the management of disaster-related problems such as pest outbreak, algal blooms and red tide, their control implementations strongly depend on the states of species. In recent years, with the frequent occurrence of algae blooms, people usually rely on their own experience to control them, but the corresponding treatments are not satisfactory and may lead to secondary contamination. Here the question may arises: does the impulsive control affect the dynamics of plankton systems and further what does role the impulsive control play in the formation and disappearance of planktonic blooms? Therefore, in this dissertation, we will establish the related plankton dynamics systems under the impulsive effects, and try to study the effect of the impulsive control on the dynamics mechanisms of the termination of planktonic blooms and predicting the dynamic evolution process of plankton growth.

2.3. Summary of literature review

Based on the dynamic models of differential equations, especially for the relationship between phytoplankton and zooplankton populations, the influence of some different key factors on the dynamics mechanisms of the termination of planktonic blooms can be revealed, which can provide a theoretical basis for exploring the dynamics mechanisms of the formation and disappearance of planktonic blooms. Although many studies on the dynamics of plankton systems have made great progress, the dynamics mechanisms of plankton growth are still not completely clear recently, especially for the detailed dynamics mechanisms of planktonic blooms. In recent years, some biological factors, for example nutrition (Huppert et al., 2002), water temperature (Zhao et al., 2020), time delay (Rehim and Imran, 2012), flow rate (Dai et al., 2015), predation pressure (Wyatt and Horwood, 1973; Uye, 1986), and harmful phytoplankton (Banerjee and Venturino, 2011), have significantly affected the dynamics mechanisms of planktonic blooms. However, other biological factors, for example population diffusion, impulsive control, environmental fluctuations and TPP, can also affect the dynamics of plankton growth, but they are rarely considered into the existing literatures. Here some questions may arise: what role does the reaction-diffusion effect play in the dynamics of spatial and temporal evolutionary process of plankton distribution, and how the impulsive control affect the survival of plankton? Also, how the impulsive control and white noise influence on the dynamics of plankton, especially the extinction and persistence in the mean of plankton in a fluctuating environment and what influence the peaks of the cyclic outbreaks of planktonic blooms under man-made factors and fluctuating environment? Moreover, what are the potential mechanisms that contribute to the extinction of plankton in an evolutionary setting when plankton is subjected to a white noise? Furthermore, predicting how the prolonged coexistence of plankton under the effects of white noise and regime switching and studying what are the mechanisms that affect the peaks of the outbreaks of planktonic blooms in a fluctuating environment? In a word, these interesting problems in the dynamics mechanisms of the formation and disappearance of the planktonic blooms can be explored in the dissertation, and it is still necessary and important to study the nonlinear problems of planktonic blooms.

Chapter 3 DYNAMIC ANALYSIS OF A REACTION-DIFFUSION IMPLUSIVE SYSTEM[1]

Abstract

In this paper, a predator-prey system with Crowley-Martin functional response, which is described by a couple of reaction-diffusion equations with impulsive, is studied analytically and numerically. The aim of this research is to analyze how the impulsive effect influences on dynamic of the system. Dynamics of the system, including the ultimate boundedness, permanence and extinction, are investigated firstly under impulsive effects. Significantly, it is found that there exists a unique positive periodic solution that is globally asymptotically stable when impulsive effects reach some critical state. Additionally, a series of numerical simulations are carried out to further study dynamics of the system, which are consistent with the analytical results.

Keywords: Crowley-Martin functional response, Reaction-diffusion equation, Permanence, Periodic solution, Impulsive effect

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3.1. Introduction

In real world, the spatial distribution of species seems to be heterogeneity in the bounded domain, and each species has a nature tendency to spread to areas where the population concentration is much smaller. Moreover, stabilization of predator-prey dynamics can be achieved through spatial heterogeneity, whose stabilizing effect of partial isolation of habitat patches is very important (Scheffer, 1998; Akhmet et al., 2006). Then, the reaction-diffusion equations may be a very useful tool to describe the spatial dispersal of species**.** Actually**,** since the pioneering work of Turing (1952), diffusion phenomena have been observed as causes of the spontaneous emergence of ordered structures, called patterns, in many non-equilibrium situations (Pearson, 1993; Bois et al., 2011; Liu et al., 2012; Hillerislambers et al., 2001). In recent decades, the studies on reaction-diffusion equations have made great progress, by which a great variety of systems with diffusion process are widely explored, and many excellent results have been obtained (Melkemi et al., 2005; Pang and Wang, 2003; Peng and Wang, 2005; Hu et al., 2014; Dai et al., 2015; Dai and Zhao, 2014). Those works mainly focus on continuous dynamic systems with diffusions. However, there exist impulsive phenomena in some diffusion processes, which cause studies on the dynamics of the system with diffusion and impulsive to be one of the significant research interests currently.

In reality, practical populations undergo inherent discontinuity of many natural and man-made factors, which lead to rapid population decrease or increase over a fixed time (for example, fire, drought, hunting, harvesting breeding, etc.). Systems with such kinds of discontinuous changes refer to impulsive differential equations, which have attracted the interest of many researches in the past decades since they provided a natural description of observed evolutionary behavior of certain real-world problems (Samoilenko and Perestyuk, 1995; Zavalishchin and Sesekin, 1997). For example, many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics, and frequency modulation models can exhibit impulsive effects, and then, impulsive effects have been extensively investigated (Nieto and 'Regan, 2009; Dai et al., 2012c; Jatav and Dhar, 2014; Chakrabory et al., 2015; Zhao et al., 2012; Wang et al., 2014). Especially, the dynamic of reaction-diffusion system with impulsive have been the common area of interest among many researchers (Akhmet et al., 2006; Wang et al., 2010; Liu et al., 2011, 2014; Zhong and Liu, 2010; Struk and Tkachenko, 2002). Akhmet et al. (2006) have investigated the dynamics of an impulsive ratio-dependent predator-prey system with diffusion, and they obtained some conditions for the permanence of the predator-prey system and for the existence of a unique globally stable periodic solution. Liu et al. generalized the impulsive reaction-diffusion system to three populations and obtained a series of interesting results (Liu et al., 2011, 2014; Zhong and Liu, 2010). All these works have lead to the development of impulsive reaction-diffusion systems.

In addition, functional response plays an important role in the population dynamic due to its characterization for the interaction between predator and prey. Holling (1959) introduced the concept of the functional response, which describes the asymptotic relationship between prey removal rate per predator and the density of prey. There exist many functional responses, such as Leslie-Gower, Holling type II, ratio-dependent, and so on. One well-known example is the Crowley-Martin functional response (Crowley and Martin, 1989), where it is assumed that predator-feeding rate decreases with higher predator density even when prey density is high. At present, the system with Crowley-Martin functional response have been explored extensively (Upadhyay and Naji, 2009; Jazar 2013; Dong et al., 2013; Shi et al., 2011; Meng et al., 2011; Liu et al., 2013; Dong and Zhang, 2015), such as a delayed Chemostat model (Dong et al., 2013), a stage-structured predator-prey model (Shi et al., 2011), a stochastic predator-prey model (Liu et al., 2013), a diffusive predator-prey system (Dong and Zhang, 2015), and so on, and many interesting results have been shown. Obviously, it is interesting and important to study the dynamics of the prey-predator systems with Crowley-Martin functional response. Additionally, impulsive effect exists in some diffusion processes in ecosystems.

However, the impulsive diffusion prey-predator system with Crowley-Martin functional response has rarely been studied, and dynamics of the impulsive prey-predator system with Crowley-Martin functional response under diffusion is still not very clear currently. Hence, the study of the dynamics on the impulsive reaction diffusion prey-predator system with Crowley-Martin functional response is the aim of this paper, especially, the influence of the impulsive effect on the dynamics of the prey-predator systems with Crowley-Martin functional response under diffusion.

Motivated by above works, a impulsive diffusion prey-predator system with Crowley-Martin functional response, which describes the interaction among a top-predator, an intermediate predator and a prey, is presented, as follows:

$$\frac{\partial u_1}{\partial \mathrm{t}} = \mu_1 \Delta u_1 + u_1[a_1(t,x) - b_1(t,x)u_1] - \frac{c_1(t,x)u_1u_2}{1+r_1(t,x)u_1+r_2(t,x)u_2+r_1(t,x)r_2(t,x)u_1u_2} \tag{3-1}$$

$$\frac{\partial u_2}{\partial \mathrm{t}} = \mu_2 \Delta u_2 + u_2[a_2(t,x) - b_2(t,x)u_2] + \frac{c'_1(t,x)u_1u_2}{1+r_1(t,x)u_1+r_2(t,x)u_2+r_1(t,x)r_2(t,x)u_1u_2} - \frac{c_2(t,x)u_2u_3}{1+r_3(t,x)u_2+r_4(t,x)u_3+r_3(t,x)r_4(t,x)u_2u_3} \tag{3-2}$$

$$\frac{\partial u_3}{\partial \mathrm{t}} = \mu_3 \Delta u_3 - a_3(t,x)u_3 + \frac{c'_2(t,x)u_2u_3}{1+r_3(t,x)u_2+r_4(t,x)u_3+r_3(t,x)r_4(t,x)u_2u_3} \tag{3-3}$$

$$u_i(t_k^+, x) = u_i(t_k, x) f_k^i\big(x, u_1(t_k,x), u_2(t_k,x), u_3(t_k,x)\big), k = 1,2,\cdots \tag{3-4}$$

and

$$\frac{\partial u_i}{\partial n} |\partial \Omega = 0, \qquad i = 1,2,3 \tag{3-5}$$

where a_1 and a_2 are the respective intrinsic growth rates of prey and an intermediate predator; b_1 and b_2 are the respective coefficients of intra-species competition of prey and intermediate predator; c_1 and c_2 are the respective maximal predator per capital consumption rates, i.e., the maximum number of prey that can be eaten by an intermediate predator and maximum number of intermediate predators can be eaten by top predators in each time and space unit; c'_1 and c'_2 are the respective conversion of prey to inter-mediate predator and intermediate predator to top predator; a_3 is the death rate of the top predator; r_1, r_3 and r_2, r_4 are the handling time and the magnitude of interference among predators, respectively; $a_i/b_i\ (i = 1,2)$ is the respective environmental carrying capacity of the prey and intermediate predator. $a_i, b_i, c_i, c'_i (i = 1,2), a_3$ and $r_j\ (j = 1,2,3,4)$ are positive functions. $\Delta u = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$ is the Laplace operator; μ_1, μ_2 and μ_3 are positive diffusion coefficients and reflect non-homogeneous population dispersion; Neumann boundary conditions (3-5) characterize the absence of migration; $\partial/\partial n$ is the outward normal derivative, $\bar{\Omega} = \Omega \cup \partial\Omega$.

In addition, it is assumed that prey and predator populations are confined to a fixed bounded space domain $\Omega \subset R^n$ with smooth boundary and are non-uniformly distributed in the domain. Furthermore, they are subjected to short-term external influences at fixed time t_k, where is a sequence of real numbers $0 = t_0 < t_1 < \cdots < t_k < \cdots$ with $\lim_{n\to\infty} t_k = +\infty$.

The rest of this paper is organized as follows: Section 3.2 presents the basic assumptions and useful theorems firstly, and then, we apply the upper and lower solution method and comparison principle of impulsive differential equations to obtain the conditions for ultimate boundedness of the solutions, the permanence of the system, and the extinction of the top predator, and finally the existence of a unique periodic solution by establishing of an appropriate auxiliary function. A series of numerical simulations are carried out to further study the dynamics of the system in Section 3.3. In Section 3.4, we summarize the results and present our conclusions.

3.2. Dynamic analysis

The population survival is an interesting topic in ecology, and the uniform persistence, from an ecological viewpoint, means all the species can coexist at any time and any location of their inhabited domain. Therefore, in this section, we firstly investigate the ultimate boundedness, permanence and extinction of the system, and then the existence, uniqueness and globally asymptotic stability of positive periodic solutions are studied from dynamic analysis.

3.2.1. Preliminaries

Let Z, N and R be the set of all integers, positive integers and real numbers, respectively, and denote $R_+ = [0,+\infty)$. Throughout this paper, we always assume that:

(C1) the functions $a_i(t,x), b_i(t,x), c_i(t,x), c'_i(t,x) (i = 1,2), r_j(t,x) (j = 1,2,3,4)$ and $a_3(t,x)$ are bounded positive-valued functions on $R \times \bar{\Omega}$, continuously differentiable in t and x;

(C2) the functions $f_k^i(x, u_1, u_2, u_3), i = 1,2,3, k \in N$ are continuously differentiable in all arguments and positive-valued;

(C3) the functions $a_i(t,x), b_i(t,x), c_i(t,x), c'_i(t,x) (i = 1,2), r_j(t,x) (j = 1,2,3,4)$ and $a_3(t,x)$ are periodic in t with a period $\tau > 0$;

(C4) there exists a number $p \in N$, such that $t_{k+p} = t_k + \tau$ for all $k \geq 1$;

(C5) the sequences f_k^i satisfy $f_{k+p}^i(x, u_1, u_2, u_3) = f_k^i(x, u_1, u_2, u_3)$ for all $i = 1,2,3$, $k \geq 1$ and x, u_1, u_2, u_3.

In order to analyze the dynamics of an ecological system, it is necessary and important to consider system with the periodicity conditions (C3)-(C5), which might be quite naturally exposed (for example, seasonal effects of weather, food supply, mating habits, etc.) (Cushing, 1977)

In the following, we introduce the following notations:

$$G = R_+ \times \Omega, \bar{G} = R_+ \times \bar{\Omega} \tag{3-6}$$

$$\Sigma_k = \{(t,x): t \in (t_{k-1}, t_k), x \in \Omega\}, k \in N, \Sigma = \cup_{k \in N} \Sigma_k \tag{3-7}$$

$$\overline{\Sigma_k} = \{(t,x): t \in (t_{k-1}, t_k), x \in \bar{\Omega}\}, k \in N, \bar{\Sigma} = \cup_{k \in N} \overline{\Sigma_k} \tag{3-8}$$

and denote

$$\Xi = \left\{ \varphi\colon \bar{G} \to R \;\middle|\; \begin{array}{l} (i)\quad \varphi(t,x) \in C^{1,2}_{t,x}(\Sigma_k),\ \varphi(t,x) \in C^{1,2}_{t,x}\left(\bar{\sum_k}\right), k \in N \\ (ii)\ \lim_{t \to t_k^-} \varphi(t,x) = \varphi(t_k, x) \quad \textit{exists} \\ (iii) \lim_{t \to t_k^+} \varphi(t,x) = \varphi(t_k^+, x) \ \textit{exists.} \end{array} \right\}. \tag{3-9}$$

A vector function $(u_1(t,x), u_2(t,x), u_3(t,x)) \in \Xi \times \Xi \times \Xi$ is called a solution of system (3-1)-(3-5) if it satisfies (3-1)-(3-3) on Σ, (3-5) for $x \in \partial\Omega$, and (3-4) for every $k \in N$.

For a continuous function $\phi(t,x)$, we define $\phi^L = \inf_{(t,x)} \phi(t,x)$ and $\phi^M = \sup_{(t,x)} \phi(t,x)$.

Definition 3.2.1.1. The solutions of system (3-1)-(3-5) are said to be ultimately bounded if there exist positive constants $M'_i (i = 1,2,3)$ such that for every solution $(u_1(t,x,u_{10},u_{20},u_{30}), u_2(t,x,u_{10},u_{20},u_{30}), u_3(t,x,u_{10},u_{20},u_{30}))$, there exists a moment of time $\bar{t} = \bar{t}(u_{10},u_{20},u_{30})$ such that $u_i(t,x,u_{10},u_{20},u_{30}) \le M'_i$ for all $i = 1,2,3, x \in \bar{\Omega}$ and $t \ge \bar{t}$.

Definition 3.2.1.2. The system (3-1)-(3-5) is called permanent if there exist positive constants m_i and $M'_i (i = 1,2,3)$ such that for every solution with non-negative initial functions $u_{10}(x), u_{20}(x)$ and $u_{30}(x)$ that are not identically zero, there exists a moment of time $\bar{t} = \bar{t}(u_{10},u_{20},u_{30})$ such that $m_i \le u_i(t,x,u_{10},u_{20},u_{30}) \le M'_i$ for all $i = 1,2,3, x \in \bar{\Omega}$ and $t \ge \bar{t}$.

Now, we discuss the existence of the solutions in system (3-1)-(3-5). According to the method of upper and lower solution method for quasi-monotone systems (Pao, 1992), it can be verified that, for continuously differentiable initial functions $u_{i0}(x)\colon \bar{\Omega} \to R_+$ and $u_{i0}(x) \not\equiv 0, i = 1,2,3$, there exists a classical solution of system (3-1)-(3-3) and (3-5), if it is of class C^2 in $x, x \in \Omega$, of class C^1 in $x, x \in \bar{\Omega}$, of C^1 in $t, t > 0$, and satisfies the system, then it is the classical one of system (3-1)-(3-3) and (3-5).

By the existence of solutions of system (3-1)-(3-3) and (3-5), we can verify the existence of solutions for the system (3-1)-(3-5). Actually, if $t \in (0, t_1]$, the solutions of the system (3-1)-(3-5) are well-defined as classical solutions of system (3-1)-(3-3) and (3-5). Impulsive condition (3-4) implies that the functions $\left(u_1(t_1^+), u_2(t_2^+), u_3(t_3^+)\right)$ are continuously differentiable in x, and satisfy boundary conditions (3-5). Hence, assuming $\left(u_1(t_1^+), u_2(t_2^+), u_3(t_3^+)\right)$ as a new initial functions, we can continue the solution on $(t_1, t_2]$. Proceeding in this way, the solution of the system (3-1)-(3-5), for all $t > 0$, can be constructed.

Based on the biological interpretation, we just take the non-negative solutions into account for the system (3-1)-(3-5), and we firstly discuss and guarantee the non-negative and positive quadrants of R^3 are positively invariant for system (3-1)-(3-5) in the following.

Suppose that $(u_1(t, x, u_{10}, u_{20}, u_{30}), u_2(t, x, u_{10}, u_{20}, u_{30}), u_3(t, x, u_{10}, u_{20}, u_{30}))$ is a solution of system (3-1)-(3-5) with the initial condition $u_{10}(x) \geq 0\ (\not\equiv 0),\ u_{20}(x) \geq 0\ (\not\equiv 0), u_{30}(x) \geq 0\ (\not\equiv 0)$, obviously, we can deduce

$$\begin{aligned}\frac{\partial u_1}{\partial \mathrm{t}} &= \mu_1 \Delta u_1 + u_1[a_1(t,x) - b_1(t,x)u_1] \\ &\quad - \frac{c_1(t,x)u_1u_2}{1 + r_1(t,x)u_1 + r_2(t,x)u_2 + r_1(t,x)r_2(t,x)u_1u_2} \\ &\leq \frac{\partial \widehat{u_1}}{\partial \mathrm{t}} - \mu_1 \Delta \widehat{u_1} - \widehat{u_1}\left[a_1^L - b_1^M \widehat{u_1} - \frac{c_1^M}{r_2^L}\right],\end{aligned}$$

and

$$\begin{aligned}\frac{\partial u_1}{\partial \mathrm{t}} &= \mu_1 \Delta u_1 + u_1[a_1(t,x) - b_1(t,x)u_1] \\ &\quad - \frac{c_1(t,x)u_1u_2}{1 + r_1(t,x)u_1 + r_2(t,x)u_2 + r_1(t,x)r_2(t,x)u_1u_2} \\ &\geq \frac{\partial \widetilde{u_1}}{\partial \mathrm{t}} - \mu_1 \Delta \widetilde{u_1} - \widetilde{u_1}\left(a_1^M - b_1^L \widetilde{u_1}\right).\end{aligned}$$

We can simply verify that $\widehat{u_1}(t,x)$ and $\widetilde{u_1}(t,x)$ are lower and upper solutions of system (3-1), respectively, where $\widehat{u_1}(t,x)$ and $\widetilde{u_1}(t,x)$ are solutions of the following equations:

$$\frac{\partial \widehat{u_1}}{\partial t} - \mu_1 \Delta \widehat{u_1} - \widehat{u_1}\left[a_1^L - b_1^M \widehat{u_1} - \frac{c_1^M}{r_2^L}\right] = 0, \quad \widehat{u_1}(0,x) = u_{10}(x), \tag{3-10}$$

and

$$\frac{\partial \widetilde{u_1}}{\partial t} - \mu_1 \Delta \widetilde{u_1} - \widetilde{u_1}(a_1^M - b_1^L \widetilde{u_1}) = 0, \qquad \widetilde{u_1}(0,x) = u_{10}(x). \tag{3-11}$$

Due to $u_{10}(x) \geq 0\ (\not\equiv 0)$, $u_{20}(x) \geq 0\ (\not\equiv 0), u_{30}(x) \geq 0\ (\not\equiv 0)$, we can get $\widehat{u_1}(t,x) > 0$ and $\widetilde{u_1}(t,x) > 0$ for $t \in (0, t_1]$ using Lemma A2 (see Appendix A). Then, we have $u_1(t,x) > 0$ for $t \in (0, t_1]$ because $u_1(t,x)$ is bounded below by positive function $\widehat{u_1}(t,x)$. Obviously, we can repeat the same argument to prove the positiveness of $u_1(t,x)$ for $t \in (t_1, t_2]$ because of the positiveness of the function f_1^1. Similarly, we can obtain that $u_1(t,x) > 0$ for $t \in (0, +\infty)$. The explanation of species $u_2(t,x) > 0$ and $u_3(t,x) > 0$ are very similar, which is omitted here. Thus, we can obtain that the non-negative and positive quadrants of R^3 are positively invariant for system (3-1)-(3-5) if the conditions (C1)-(C5) hold.

3.2.2. Permanence of population

Based on the analysis in the previous subsection, we will discuss and guarantee the ultimate boundedness of solutions for the system (3-1)-(3-5) firstly, and then investigate the permanence of the system. The following analysis shows that impulsive control can influence the permanence of the system directly.

Firstly, we discuss the ultimate boundedness of population $u_1(t,x)$. It is readily seen from Eq. (3-1) that

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= \mu_1 \Delta u_1 + u_1[a_1(t,x) - b_1(t,x)u_1] \\ &\quad - \frac{c_1(t,x)u_1u_2}{1 + r_1(t,x)u_1 + r_2(t,x)u_2 + r_1(t,x)r_2(t,x)u_1u_2} \\ &\geq \frac{\partial u_1}{\partial t} - \mu_1 \Delta u_1 - u_1\left(a_1^M - b_1^L u_1\right). \end{aligned}$$

Let $\overline{u_1}(t,x,u_{10})$ be a solution of

$$\frac{\partial \overline{u_1}}{\partial t} - \mu_1 \Delta \overline{u_1} - \overline{u_1}(a_1^M - b_1^L \overline{u_1}) = 0, \tag{3-12}$$

Then, we have

$$0 = \frac{\partial \overline{u_1}}{\partial \mathrm{t}} - \mu_1 \Delta \overline{u_1} - \overline{u_1}\left(a_1^M - b_1^L \overline{u_1}\right) \geq \frac{\partial u_1}{\partial \mathrm{t}} - \mu_1 \Delta u_1 - u_1\left(a_1^M - b_1^L u_1\right).$$

According to Lemma A1 (see Appendix A), we have $u_1(t,x,u_{10},u_{20},u_{30}) \leq \overline{u_1}(t, M_{u_1})$, if $\|u_{10}(x)\|_C = \max_{x\in\bar{\Omega}} |u_{10}(x)| \leq M_{u_1}$. From the uniqueness theorem, the solution $u_1(t, M_{u_1})$ of (3-12) with initial conditions independent of x does not depend on x for $t > 0$. Therefore, $\overline{u_1}(t, M_{u_1})$ is the solution of the following ordinary differential equation

$$\frac{\partial \overline{u_1}}{\partial \mathrm{t}} - \overline{u_1}\left(a_1^M - b_1^L \overline{u_1}\right) = 0, \quad \overline{u_1}(0, M_{u_1}) = M_{u_1}.$$

Hence

$$\begin{aligned} &\|u_1(t_k^+, x, u_{10}, u_{20}, u_{30})\|_{\mathrm{C}} \\ &\qquad = \left\|u_1(t_k, x, u_{10}, u_{20}, u_{30}) f_k^1\left(x, u_1(t_k, x), u_2(t_k, x), u_3(t_k, x)\right)\right\|_{\mathrm{C}} \\ &\qquad \leq \overline{u_1}(t, M_{u_1}) \eta_1\left(\overline{u_1}(t, M_{u_1})\right). \end{aligned}$$

Since the corresponding impulsive differential equations

$$\begin{cases} \frac{\partial \overline{u_1}}{\partial \mathrm{t}} = \overline{u_1}(a_1^M - b_1^L \overline{u_1}) \\ \overline{u_1}(t_k^+) = \overline{u_1}(t_k)\eta_1(\overline{u_1}(t, M_{u_1})) \end{cases} \tag{3-13}$$

are ultimately bounded (from Lemma B (see Appendix B)), thus, we get ultimate boundedness of solutions of Eq. (3-1) and Eq. (3-4), i.e., there exists a positive constant M'_1 such that $u_1(t,x) \leq M'_1$, starting with some time $\bar{\mathrm{t}}_1$.

Secondly, we analyze the ultimate boundedness of population $u_2(t,x)$. For population u_2, from Eq. (3-2), we have

$$\begin{aligned} \frac{\partial u_2}{\partial \mathrm{t}} = \mu_2 \Delta u_2 + u_2[a_2(t,x) - b_2(t,x)u_2] \\ + \frac{c'_1(t,x)u_1u_2}{1 + r_1(t,x)u_1 + r_2(t,x)u_2 + r_1(t,x)r_2(t,x)u_1u_2} \\ - \frac{c_2(t,x)u_2u_3}{1 + r_3(t,x)u_2 + r_4(t,x)u_3 + r_3(t,x)r_4(t,x)u_2u_3} \\ \geq \frac{\partial u_2}{\partial \mathrm{t}} - \mu_2 \Delta u_2 - u_2\left(a_2^M - b_2^L u_2 + \frac{{c'}_1^M}{r_1^L}\right). \end{aligned}$$

Since there exists a positive-valued function $\eta_2(\bar{M})$ such that $f_k^2(x, u_1, u_2, u_3) \leq \eta_2(\bar{M})$ if $k \in N, u_2 \leq M, u_1 > 0, u_3 > 0$ and $x \in \bar{\Omega}$, the same analysis with the population u_1, there exists a positive constant M'_2 such that $u_2(t,x) \leq M'_2$, starting with some time $\overline{t_2}$.

Finally, for the top predator population u_3, when $t \geq \overline{t_2}$, we can obtain that

$$\frac{\partial u_3}{\partial t} = \mu_3 \Delta u_3 - a_3(t,x)u_3 + \frac{c'_2(t,x)u_2u_3}{1 + r_3(t,x)u_2 + r_4(t,x)u_3 + r_3(t,x)r_4(t,x)u_2u_3}$$

$$\geq \frac{\partial u_3}{\partial t} - \mu_3 \Delta u_3 + a_3^L u_3 - \frac{c'^M_2 M'_2}{r_4^L},$$

and it follows that $u_3(t_k, x, u_{10}, u_{20}, u_{30}) \leq \overline{u_3}(t, M_{u_3})$, where $\overline{u_3}(t, M_{u_3})$ is a solution of the initial value problem

$$\frac{\partial u_3}{\partial t} = -a_3^L u_3 + \frac{c'^M_2 M'_2}{r_4^L}$$

with

$$\overline{u_3}(0, M_{u_3}) = M_{u_3}.$$

The linear periodic impulsive equation

$$\begin{cases} \frac{\partial \overline{u_3}}{\partial t} = -a_3^L \overline{u_3} + \frac{c'^M_2 M'_2}{r_4^L} \\ \overline{u_3}(t_k^+) = f_k^3(\overline{u_3}(t_k)) \end{cases} \tag{3-14}$$

has the general solution $\overline{u_3}(t) = X_0(t) + CX(t)$, where $X_0(t)$ is a τ-periodic piece-wise continuous function, and C is a constant and

$$X(t) = \exp\left(-a_3^L + \sum\nolimits_{0<t_k<t} \ln f_k^3\right).$$

(Samoilenko and Perestyuk, 1995). Obviously, if the condition $-\tau a_3^L + \sum_{k=1}^{p} \ln \sup_{(x,u_1,u_2,u_3)} f_k^3\,(x, u_1, u_2, u_3) < 0$ holds, then we have $X(t) \to 0$ as $t \to \infty$. Thus, all solutions of Eq. (3-14) are ultimately bounded. That is, all solutions of Eq. (3-3) and Eq. (3-4) are also ultimately bounded.

To sum up, we can conclude: if the conditions (C1)-(C5) hold, and

(i) there exists a positive-valued function $\eta_1(M)$ such that $f_k^1(x,u_1,u_2,u_3) \le \eta_1(M)$ if $k \in N, u_1 \le M, u_2 > 0, u_3 > 0$ and $x \in \bar{\Omega}$;

(ii) there exists a positive-valued function $\eta_2(\bar{M})$ such that $f_k^2(x,u_1,u_2,u_3) \le \eta_2(\bar{M})$ if $k \in N, u_2 \le M, u_1 > 0, u_3 > 0$ and $x \in \bar{\Omega}$;

(iii) the inequality

$$-\tau a_3^L + \sum_{k=1}^{p} \ln f_k^3 < 0$$

holds, where $f_k^3 = \sup_{(x,u_1,u_2,u_3)} f_k^3\,(x,u_1,u_2,u_3)$. Then all solutions of the system (3-1)-(3-5) with non-negative initial conditions are ultimately bounded.

Next, according to the ultimate boundedness of solutions for system (3-1)-(3-5), we will analyze the permanence of the system in the following, and further illustrate how the population survival under the impulsive effects.

Since the assumption that there exists a positive-valued function $\eta_1(\bar{M})$ such that $f_k^1(x,u_1,u_2,u_3) \le \eta_1(\bar{M})$ if $k \in N, u_1 \le M, u_2 > 0, u_3 > 0$ and $x \in \bar{\Omega}$, we can deduce that there exist $M'_i (i = 1,2,3)$ such that $u_i \le M'_i$, starting with a particular time. Lemma A2 (see Appendix A) implies that if $u_{10}(x) \ge 0\ (\not\equiv 0), u_{20}(x) \ge 0\ (\not\equiv 0), u_{30}(x) \ge 0\ (\not\equiv 0)$, then $u_i(t,x,u_{10},u_{20},u_{30}) > 0$ for all $i = 1,2,3, x \in \bar{\Omega}$ and $t > 0$, for some small $\varepsilon > 0$, considering the solution on the interval $t \ge \varepsilon$, we get the initial conditions have $\big(u_1(\varepsilon,x,u_{10},u_{20},u_{30}), u_2(\varepsilon,x,u_{10},u_{20},u_{30}), u_3(\varepsilon,x,u_{10},u_{20},u_{30})\big)$ separated from zero. Without loss of generality, we assume that $\min_{x\in\bar{\Omega}} u_{i0}\,(x) = m_{u_i} > 0$ for all $i = 1,2,3$. The inequality:

$$\frac{\partial u_1}{\partial \mathrm{t}} = \mu_1 \Delta u_1 + u_1[a_1(t,x) - b_1(t,x)u_1]$$

$$- \frac{c_1(t,x)u_1u_2}{1 + r_1(t,x)u_1 + r_2(t,x)u_2 + r_1(t,x)r_2(t,x)u_1u_2}$$

$$\le \frac{\partial u_1}{\partial \mathrm{t}} - \mu_1 \Delta u_1 - u_1\left[a_1^L - b_1^M u_1 - \frac{c_1^M}{r_2^L}\right]$$

is valid, so

$$0=\frac{\partial\widehat{u_1}}{\partial \mathrm{t}}-\widehat{\mu_1}\Delta\widehat{u_1}-\widehat{u_1}\left[a_1^L-b_1^M\widehat{u_1}-\frac{c_1^M}{r_2^L}\right]\le\frac{\partial u_1}{\partial \mathrm{t}}-\mu_1\Delta u_1-u_1\left[a_1^L-b_1^M u_1-\frac{c_1^M}{r_2^L}\right].$$

Now, according to Lemma A1 (see Appendix A) for $m=1, u_1(t,x,u_{10},u_{20},u_{30})\ge\widehat{u_1}(m_{u_i})$ for $t\in[0,t_1]$. Applying the last inequality for $t=t_1$, together with Eq. (3-3), we have

$$u_1(t_1^+,x,u_{10},u_{20},u_{30})\ge\widehat{u_1}(m_{u_i})\inf\nolimits_{x\in\bar{\Omega},(u_1,u_2,u_3)\in S} f_1^1\,(x,u_1,u_2,u_3).$$

Thus, the solution $u_1(t,x,u_{10},u_{20},u_{30})$ is bounded from below by a solution of periodic logistic equation with impulse:

$$\begin{cases}\frac{\partial\widehat{u_1}}{\partial \mathrm{t}}=\widehat{\mu_1}\Delta\widehat{u_1}-\widehat{u_1}\left[a_1^L-b_1^M\widehat{u_1}-\frac{c_1^M}{r_2^L}\right]\\ \widehat{u_1}(t_k^+)=\widehat{u_1}(t_k)\inf\nolimits_{x\in\bar{\Omega},(u_1,u_2,u_3)\in S} f_k^1\,(x,u_1,u_2,u_3)\end{cases}\qquad(3\text{-}15)$$

Obviously, if the condition $H_1=\sum_{k=1}^{p}\ln\inf\nolimits_{x\in\bar{\Omega},(u_1,u_2,u_3)\in E} f_k^1\,(x,u_1,u_2,u_3)+\tau\left(a_1^L-\frac{c_1^M}{r_2^L}\right)>0$ holds, where $E=\{(u_1,u_2,u_3):0<u_i<M'_i, i=1,2,3\}$, and by Theorem 2.1 (Smith, 1999), then Eq. (3-15) has a unique piece-wise continuous and strictly positive periodic solution $\widehat{u_1^*}(t)$. Thus, $u_1(t,x,u_{10},u_{20},u_{30})\ge\widehat{u_1}(m_{u_1})\to\widehat{u_1^*}(t)$ as $t\to\infty$. Therefore, there exists a positive constant σ_1^* such that $u_1(t,x,u_{10},u_{20},u_{30})\ge\sigma_1^*$ for $t\ge\overline{t_1}$.

For the population u_2. Using the inequality

$$\begin{aligned}0=\frac{\partial u_2}{\partial \mathrm{t}}&-\mu_2\Delta u_2-u_2[a_2(t,x)-b_2(t,x)u_2]\\&-\frac{c'_1(t,x)u_1u_2}{1+r_1(t,x)u_1+r_2(t,x)u_2+r_1(t,x)r_2(t,x)u_1u_2}\\&+\frac{c_2(t,x)u_2u_3}{1+r_3(t,x)u_2+r_4(t,x)u_3+r_3(t,x)r_4(t,x)u_2u_3}\\&\le\frac{\partial u_2}{\partial \mathrm{t}}-\mu_2\Delta u_2-u_2\left(a_2^L-\frac{c_2^M}{r_4^L}-b_2^M u_2\right).\end{aligned}$$

If the condition $H_2=\sum_{k=1}^{p}\ln\inf\nolimits_{x\in\bar{\Omega},(u_1,u_2,u_3)\in E} f_k^2\,(x,u_1,u_2,u_3)+\tau\left(a_2^L-\frac{c_2^M}{r_4^L}\right)>0$ holds, where $E=\{(u_1,u_2,u_3):0<u_i<M'_i, i=1,2,3\}$, and following the

same analysis as for u_1, there exists a positive constant $\sigma_2^* > 0$ such that $u_2(t, x, u_{10}, u_{20}, u_{30}) \geq \sigma_2^*$ for $t \geq \overline{t_2}$.

For the top predator population u_3. When $t \geq \overline{t_2}$, since $u_2(t, x, u_{10}, u_{20}, u_{30}) \geq \sigma_2^*$, thus,

$$0 = \frac{\partial u_3}{\partial \mathrm{t}} - \mu_3 \Delta u_3 + a_3(t, x) u_3 - \frac{c'_2(t, x) u_2 u_3}{1 + r_3(t, x) u_2 + r_4(t, x) u_3 + r_3(t, x) r_4(t, x) u_2 u_3}$$

$$\leq \frac{\partial u_3}{\partial \mathrm{t}} - \mu_3 \Delta u_3 + \left(a_3^M - \frac{c'^L_2}{r_3^M}\right) u_3$$

$$+ \frac{c'^M_2 u_3 + r_4^M u_3^2 + c'^M_2 r_3^M r_4^M \sigma_2^* u_3^2}{r_3^L + (r_3^L)^2 \sigma_2^* + r_3^L r_4^L u_3 + (r_3^L)^2 r_4^L u_3}.$$

Hence,

$$u_3(t, x, u_{10}, u_{20}, u_{30}) \geq \widehat{u_3}(t, m_{u_3}),$$

where

$$\widehat{u_3}(0, m_{u_3}) = m_{u_3}$$

is the solution of

$$\begin{cases} \frac{d\widehat{u_3}}{dt} = \left(\frac{c'^L_2}{r_3^M} - a_3^M\right) \widehat{u_3} - \frac{c'^M_2 \widehat{u_3} + r_4^M \widehat{u_3^2} + c'^M_2 r_3^M r_4^M \sigma_2^* \widehat{u_3^2}}{r_3^L + \left(r_3^L\right)^2 \sigma_2^* + r_3^L r_4^L \widehat{u_3} + \left(r_3^L\right)^2 r_4^L \widehat{u_3}} \\ \widehat{u_3}(t_k^+) = \widehat{f_3^3} \widehat{u_3}(t_k) \end{cases} \tag{3-16}$$

where $\widehat{f_3^3} = \inf_{x \in \overline{\Omega}, (u_1, u_2, u_3) \in S} f_k^3(x, u_1, u_2, u_3)$. If $\widehat{u_3}(t) \leq \sigma_3$ for some $\sigma_3 > 0$ and $t \in [0, t_1]$, then

$$\widehat{u_3}(t, m_{u_3}) \geq m_{u_3} \exp\left\{t_1 \left(\frac{c'^L_2}{r_3^M} - a_3^M - \frac{c'^M_2 + r_4^M \sigma_3 + c'^M_2 r_3^M r_4^M \sigma_2^* \sigma_3}{r_3^L + (r_3^L)^2 \sigma_2^* + r_3^L r_4^L \sigma_3 + (r_3^L)^2 r_4^L \sigma_3}\right)\right\}$$

and

$$\widehat{u_3}(t_1^+, m_{u_3}) \geq \widehat{f_3^3} m_{u_3} \exp\left\{t_1 \left(\frac{c'^L_2}{r_3^M} - a_3^M - \frac{c'^M_2 + r_4^M \sigma_3 + c'^M_2 r_3^M r_4^M \sigma_2^* \sigma_3}{r_3^L + (r_3^L)^2 \sigma_2^* + r_3^L r_4^L \sigma_3 + (r_3^L)^2 r_4^L \sigma_3}\right)\right\}.$$

Therefore, if $\widehat{u_3}(t) \leq \sigma_3$, for some $\sigma_3 > 0$ and $t \in [0, \tau]$, and

$$\widehat{u_3}(\tau, m_{u_3}) \geq m_{u_3} \exp\left\{\sum_{k=1}^{p} \ln \widehat{f_3^3} + \tau\left(\frac{c'^{L}_{2}}{r_3^M} - a_3^M - \frac{c'^{M}_{2} + r_4^M\sigma_3 + c'^{M}_{2} r_3^M r_4^M \sigma_2^* \sigma_3}{r_3^L + (r_3^L)^2\sigma_2^* + r_3^L r_4^L \sigma_3 + (r_3^L)^2 r_4^L \sigma_3}\right)\right\}.$$

If $H_3 = \sum_{k=1}^{p} \ln \inf_{x\in\overline{\Omega},(u_1,u_2,u_3)\in E} f_k^3\,(x, u_1, u_2, u_3) + \tau\left(\frac{c'^{L}_{2}}{r_3^M} - a_3^M\right) > 0$, where $E = \{(u_1, u_2, u_3){:}\, 0 < u_i < M'_i, i = 1,2,3\}$, we can take sufficiently small $\sigma_3 > 0$ such that

$$\sum_{k=1}^{p} \ln \widehat{f_3^3} + \tau\left(\frac{c'^{L}_{2}}{r_3^M} - a_3^M - \frac{c'^{M}_{2} + r_4^M\sigma_3 + c'^{M}_{2} r_3^M r_4^M \sigma_2^* \sigma_3}{r_3^L + (r_3^L)^2\sigma_2^* + r_3^L r_4^L \sigma_3 + (r_3^L)^2 r_4^L \sigma_3}\right) = \rho > 0.$$

For $\sigma_3^0 \in (0, \sigma_3)$, there exists a positive integer k_3 such that $\widehat{u_3}(k\tau, m_{u_3}) \geq e^{k_3\rho} m_{u_3} \geq \sigma_3^0$, with the additional condition $\widehat{u_3}(\tau, m_{u_3}) < \sigma_3$ for all $t \in [0, k_3\tau]$. Hence, for every solution $\widehat{u_3}(t, \widehat{u_{30}})$ of (3-16) with $\widehat{u_{30}} > 0$, there exists a time $\hat{t}$ such that $\widehat{u_3}(t, \widehat{u_{30}}) \geq \sigma_3^0$. Denote $\widehat{u_3}(t, t_0, \widehat{u_{30}})$ as the solution of Eq. (3-16) with $\widehat{u_3}(t, t_0, \widehat{u_{30}}) = \widehat{u_{30}}$, and consider a positive number

$$\sigma_3^* = \inf\{\widehat{u_3}(t, t_0, \widehat{u_{30}}){:}\, t_0 \in [0, \tau], \widehat{u_{30}} \in [\sigma_3^0, N_3], t \in [t_0, 2\tau]\}.$$

Then, for all $t \geq 2\tau$, one can obtain that,

$$\widehat{u_3}(t, t_0, \widehat{u_{30}}) \geq \sigma_3^*.$$

Let

$$\sigma_\tau = \inf\{\widehat{u_3}(t, t_0, \widehat{u_{30}}){:}\, t_0 \in [0, \tau], \widehat{u_{30}} \in [\sigma_3^0, N_3], t \in [t_0, 2\tau]\} \geq \sigma_3^*,$$

and consider a solution $\widehat{u_3}(t, \tau, \widehat{u_{30}})$ with $\widehat{u_{30}} \geq \sigma_\tau$. If $\widehat{u_3}(t, \tau, \widehat{u_{30}}) < \sigma_3$ for all $t \in [t_0, 2\tau]$, then $\widehat{u_3}(t, \tau, \widehat{u_{30}}) \geq e^{\rho}\widehat{u_3}(\tau, \tau, \widehat{u_{30}}) \geq \sigma_\tau$. If $\widehat{u_3}(t, \tau, \widehat{u_{30}}) > \sigma_3$ at time $t \in [\tau, 2\tau]$, then $\widehat{u_3}(2\tau, \tau, \widehat{u_{30}}) \geq \sigma_\tau$ from the definition of σ_τ. Therefore, it is enough to consider $\widehat{u_3}(t, 2\tau, \widehat{u_{30}}), t \geq 2\tau$ with $\widehat{u_{30}} \geq \sigma_\tau$. By construction, these solutions are bounded from below by positive constant σ_3^* for $t \in [2\tau, 3\tau]$. Proceeding in this way, it can be proved that the solutions of the system are bounded from below for $t \geq 3\tau$.

Therefore, we can conclude that, if the conditions (C1)-(C5) hold, and further suppose that

(i) the solutions of system (3-1)-(3-5) are ultimately bounded, i.e., there exist positive constants $M'_i (i = 1,2,3)$ such that for every solution $\big(u_1(t, x, u_{10}, u_{20}, u_{30}), u_2(t, x, u_{10}, u_{20}, u_{30}), u_3(t, x, u_{10}, u_{20}, u_{30})\big)$, there exists $\bar{t} = \bar{t}(u_{10}, u_{20}, u_{30}) > 0$ such that $u_i(t, x, u_{10}, u_{20}, u_{30}) \leq M'_i$ for all $i = 1,2,3, \mathrm{x} \in \bar{\Omega}$ and $t \geq \bar{t}$;

(ii) the conditions $H_1 > 0, H_2 > 0$ and $H_3 > 0$ hold.

Then, there exist positive constants $\sigma_i (i = 1,2,3)$ such that any solution of system (3-1)-(3-5) with non-negative initial functions not identically equal to zero satisfies

$$\big(u_1(t, x), u_2(t, x), u_3(t, x)\big) \in S = \{(u_1, u_2, u_3): \sigma_i < u_i < M'_i, i = 1,2,3\}$$

starting with a particular time, i.e., system (3-1)-(3-5) is permanent.

According to above analysis, we can easily induce the following results:

Remark 3.2.2.1 If there are no impulses in problems (3-1)-(3-5), then conditions $H_i > 0\ (i = 1,2,3)$ can be taken as the following form:

$$a_1^L - -\frac{c_1^M}{r_2^L} > 0, a_2^L - \frac{c_2^M}{r_4^L} > 0, \frac{c'^L_2}{r_3^M} - a_3^M > 0,$$

which are sufficient to establish permanence for the system.

3.2.3. Extinction of top predator population

Are there certain populations in the ecosystem that tend to become extinct after a certain time? The extinction of some species is often desirable, such as the extinction of pests, while some species are undesirable. For example, in order to protect the diversity of species, we should take protection measures for some endangered species. Based on the previous analysis, it is interesting to find that the top predator will undergo extinction under some a condition, and the result shows that if the condition $H_4 = \sum_{k=1}^{p} \ln \sup_{(u_1, u_2, u_3)} f_k^3\,(x, u_1, u_2, u_3) + \tau\left(\frac{c'^M_2}{r_3^L} - a_3^L\right) < 0$ holds, then the top predator tends to extinction, indicating the impulsive control can significantly affect the dynamics of system (3-1)-(3-5). Now we give a specific analysis.

Suppose M_{u_3} is a fixed positive constant such that $M_{u_3} \geq u_{30}(x)$ and $\overline{u_3}(t, M_{u_3})$ is the solution of initial value problem

$$\begin{cases} \frac{\mathrm{d}\overline{u_3}}{dt} = \left(\frac{c'^M_2}{r_3^L} - a_3^L\right)\overline{u_3} \\ \overline{u_3}(0, M_{u_3}) = M_{u_3} \end{cases}$$

From the inequality

$$\begin{aligned} \frac{\partial u_3}{\partial \mathrm{t}} &= \mu_3 \Delta u_3 + a_3(t,x)u_3 - \frac{c'_2(t,x)u_2u_3}{1 + r_3(t,x)u_2 + r_4(t,x)u_3 + r_3(t,x)r_4(t,x)u_2u_3} \\ &\geq \frac{\partial u_3}{\partial \mathrm{t}} - \mu_3 \Delta u_3 + \left(a_3^L - \frac{c'^M_2}{r_3^L}\right)u_3, \end{aligned}$$

and Lemma A1 (see Appendix A), we can deduce that, for $t \leq t_1$

$$u_3(t, x, u_{10}, u_{20}, u_{30}) \leq \overline{u_3}(t, M_{u_3})$$

and

$$u_3(t_1^+, x, u_{10}, u_{20}, u_{30}) \leq \overline{u_3}(t, M_{u_3}) \sup_{(u_1,u_2,u_3)} f_k^3\,(x, u_1, u_2, u_3)$$

if the impulsive condition (3-4) holds. Proceeding in this way, we conclude that every solution of Eq. (3-3) and Eq. (3-4) is bounded from above by the corresponding solution of the linear impulses equation

$$\begin{cases} \frac{d\overline{u_3}}{dt} = \left(\frac{c'^M_2}{r_3^L} - a_3^L\right)\overline{u_3} \\ \overline{u_3}(t_k^+) = \overline{u_3}(t_k) \sup_{(u_1,u_2,u_3)} f_k^3\,(x, u_1, u_2, u_3) \end{cases}$$

Applying the Lemma B (see Appendix B), all solutions of the last equation tend to zero as $t \to \infty$ if the inequality $H_4 < 0$ holds.

Obviously, we can obtain the following results if the system does not experience the impulsive perturbations:

Remark 3.2.3.1 Suppose that the conditions (C1)-(C3) hold, then the top predator $u_3(t,x)$ in system (3-1)-(3-3), (3-5) is extinctive if

$$\frac{c'^M_2}{r_3^L} - a_3^L < 0.$$

3.2.4. Periodic oscillations of population density

In the previous subsection 3.2.2, we have obtained the result that the system is permanent. Thus, in this subsection, we will further study the existence of periodic solutions in system (3-1)-(3-5) by constructing a proper auxiliary function in the following, which implies that all the species can coexist at a stable state for a long time.

In order to study the existence of periodic solutions in the system (3-1)-(3-5), we firstly suppose that $\left(u_1(t,x), u_2(t,x), u_3(t,x)\right)$ and $\left(\overline{u_1}(t,x), \overline{u_2}(t,x), \overline{u_3}(t,x)\right)$ are two solutions of system (3-1)-(3-5) bounded by constants σ and N from below and above, respectively. Consider the function

$$L(t) = \sum_{i=1}^{3} \int_{\Omega} \left[u_i(u_1, u_2, u_3) - \bar{u}_i(\overline{u_1}, \overline{u_2}, \overline{u_3})\right]^2 dx$$

with derivative

$$\frac{dL(t)}{dt} = 2\sum_{i=1}^{3} \int_{\Omega} (u_1 - \bar{u}_i)\left(\frac{\partial u_1}{\partial t} - \frac{\partial \overline{u_1}}{\partial t}\right) dx = I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 = 2\sum_{i=1}^{3} \mu_i \int_{\Omega} (u_1 - \bar{u}_i)\Delta(u_1 - \bar{u}_i)\, dx,$$

$$\begin{aligned} I_2 = 2\int_{\Omega} (u_1 - \overline{u_1}) \Bigg[& u_1(a_1 - b_1 u_1) - \frac{c_1 u_1 u_2}{1 + r_1 u_1 + r_2 u_2 + r_1 r_2 u_1 u_2} - \overline{u_1}(a_1 - b_1 \overline{u_1}) \\ & - \frac{c_1 \overline{u_1}\,\overline{u_2}}{1 + r_1 \overline{u_1} + r_2 \overline{u_2} + r_1 r_2 \overline{u_1}\,\overline{u_2}} \Bigg] dx, \end{aligned}$$

$$\begin{aligned} I_3 = 2\int_{\Omega} (u_2 - \overline{u_2}) \Bigg[& u_2(a_2 - b_2 u_2) - \frac{c'_1 u_1 u_2}{1 + r_1 u_1 + r_2 u_2 + r_1 r_2 u_1 u_2} \\ & - \frac{c_2 u_2 u_3}{1 + r_3 u_2 + r_4 u_3 + r_3 r_4 u_2 u_3} - \overline{u_2}(a_2 - b_2 \overline{u_2}) \\ & - \frac{c'_1 \overline{u_1}\,\overline{u_2}}{1 + r_1 \overline{u_1} + r_2 \overline{u_2} + r_1 r_2 \overline{u_1}\,\overline{u_2}} + \frac{c_2 \overline{u_2}\,\overline{u_3}}{1 + r_3 \overline{u_2} + r_4 \overline{u_3} + r_3 r_4 \overline{u_2}\,\overline{u_3}} \Bigg] dx, \end{aligned}$$

$$I_4 = 2\int_{\Omega} (u_3 - \overline{u_3})\left[-a_3 u_3 + \frac{c'_2 u_2 u_3}{1 + r_3 u_2 + r_4 u_3 + r_3 r_4 u_2 u_3} - \frac{c'_2 \overline{u_2 u_3}}{1 + r_3\overline{u_2} + r_4\overline{u_3} + r_3 r_4 \overline{u_2 u_3}}\right] dx.$$

Then, from the boundary condition (3-5),

$$I_1 = 2\sum_{i=1}^{3} \mu_i \int_{\Omega} (u_1 - \overline{u_i})\Delta(u_1 - \overline{u_i})\, dx \leq -I_1 = 2\sum_{i=1}^{3} \mu_i \int_{\Omega} |\nabla(u_1 - \overline{u_i})|^2\, dx \leq 0.$$

For the other terms I_2, I_3, I_4, since

$$uv - \bar{u}\bar{v} = uv - u\bar{v} + u\bar{v} - \bar{u}\bar{v} = u(v - \bar{v}) + \bar{v}(u - \bar{u})$$

Then

$$\begin{aligned}
I_2 + I_3 + I_4 = 2\int_{\Omega} (u_1 - \overline{u_1}) \Big[& a_1(u_1 - \overline{u_1}) - b_1(u_1 - \overline{u_1})(u_1 + \overline{u_1}) \\
& + \left(\frac{c_1 \overline{u_1 u_2}}{1 + r_1\overline{u_1} + r_2\overline{u_2} + r_1 r_2 \overline{u_1 u_2}} - \frac{c_1 u_1 u_2}{1 + r_1 u_1 + r_2 u_2 + r_1 r_2 u_1 u_2}\right)\Big] dx \\
& + 2\int_{\Omega} (u_2 - \overline{u_2}) \Big[a_2(u_2 - \overline{u_2}) - b_2(u_2 - \overline{u_2})(u_2 + \overline{u_2}) \\
& + \left(\frac{c'_1 \overline{u_1 u_2}}{1 + r_1\overline{u_1} + r_2\overline{u_2} + r_1 r_2 \overline{u_1 u_2}} - \frac{c'_1 u_1 u_2}{1 + r_1 u_1 + r_2 u_2 + r_1 r_2 u_1 u_2}\right) \\
& + \left(\frac{c_2 \overline{u_2 u_3}}{1 + r_3\overline{u_2} + r_4\overline{u_3} + r_3 r_4 \overline{u_2 u_3}} - \frac{c_2 u_2 u_3}{1 + r_3 u_2 + r_4 u_3 + r_3 r_4 u_2 u_3}\right)\Big] dx \\
& + 2\int_{\Omega} (u_3 - \overline{u_3}) \Big[-a_3(u_3 - \overline{u_3}) + \frac{c'_2 u_2 u_3}{1 + r_3 u_2 + r_4 u_3 + r_3 r_4 u_2 u_3} \\
& - \frac{c'_2 \overline{u_2 u_3}}{1 + r_3\overline{u_2} + r_4\overline{u_3} + r_3 r_4 \overline{u_2 u_3}}\Big] dx
\end{aligned}$$

$$\leq 2\int_{\Omega} (u_1-\overline{u_1})^2\left(a_1^M-b_1^L\sigma-\frac{c_1^Lr_2^L\sigma^2+c_1^L\sigma}{(1+r_1^M+r_2^M+r_1^Mr_2^MM')^2}\right)dx$$

$$+2\int_{\Omega} (u_2-\overline{u_2})^2\left(a_2^M-b_2^L\sigma+\frac{c'^M_1r_1^MM'^2+c'^M_1M'}{(1+r_1^L+r_2^L+r_1^Lr_2^L\sigma)^2}\right.$$

$$\left.-\frac{c_1^Lr_4^L\sigma^2+c_2^L\sigma}{(1+r_3^L+r_4^L+r_3^Lr_4^LM')^2}\right)dx+2\int_{\Omega} (u_3-\overline{u_3})^2\left[-a_3^L\right.$$

$$\left.+\frac{c'^M_2r_3^MM'^2+c'^M_2M'}{(1+r_3^L+r_4^L+r_3^Lr_4^L\sigma)^2}\right]dx$$

$$+2\int_{\Omega} |u_1-\overline{u_1}||u_2-\overline{u_2}|\frac{c'^M_1r_1^MM'^2+c'^M_1M^{'}-(c_1^Lr_2^L\sigma^2+c_1^L\sigma)}{(r_1^L+r_2^L+r_1^Lr_2^L\sigma)^2}dx$$

$$+2\int_{\Omega} |u_2-\overline{u_2}||u_3-\overline{u_3}|\frac{c'^M_2r_3^MM'^2+c'^M_2M'-(c_1^Lr_4^L\sigma^2+c_2^L\sigma)}{(r_3^L+r_4^L+r_3^Lr_4^L\sigma)^2}dx$$

$$\leq \lambda_M\int_{\Omega} [(u_1-\overline{u_1})^2+(u_2-\overline{u_2})^2+(u_3-\overline{u_3})^2]\,dx=\lambda_ML(t),$$

where σ and M' are minimum and maximum value of the system, respectively, λ_M is the maximal eigenvalue of the following matrix

$$\begin{pmatrix} E_{11} & E_{12} & E_{13}\\ E_{21} & E_{22} & E_{23}\\ E_{31} & E_{32} & E_{33}\end{pmatrix}$$

and

$$E_{11}=2\left(a_1^M-b_1^L\sigma-\frac{c_1^Lr_2^L\sigma^2+c_1^L\sigma}{(1+r_1^M+r_2^M+r_1^Mr_2^MM')^2}\right),$$

$$E_{22}=2\left(a_2^M-b_2^L\sigma+\frac{c'^M_1r_1^MM'^2+c'^M_1M'}{(1+r_1^L+r_2^L+r_1^Lr_2^L\sigma)^2}-\frac{c_2^Lr_4^L\sigma^2+c_2^L\sigma}{(1+r_3^L+r_4^L+r_3^Lr_4^LM')^2}\right),$$

$$E_{33}=2\left(-a_3^L+\frac{c'^M_2r_3^MM'^2+c'^M_2M'}{(1+r_3^L+r_4^L+r_3^Lr_4^LM')^2}\right),E_{13}=E_{31}=0,$$

$$E_{12}=E_{21}=\frac{c_1^Mr_1^MM'^2+c'^M_1M^{'}-(c_1^L\sigma+c_1^Lr_1^L\sigma^2)}{(r_1^L+r_2^L+r_1^Lr_2^L\sigma)^2},$$

and

$$E_{23} = E_{32} = \frac{c'^{M}_{2} r_3^M M'^2 + c'^{M}_{2} M' - (c_2^L \sigma + c_2^L r_2^L \sigma^2)}{(r_3^L + r_4^L + r_3^L r_4^L \sigma)^2}.$$

Then, one can see that

$$L(t_{j+1}) \leq L(t_j^+) \exp\left(\lambda_M (t_{j+1} - t_j)\right)$$

and

$$L(t_j^+) = \sum_{i=1}^{3} \int_{\Omega} \left[u_i f_{j+1}^i (u_1, u_2, u_3) - \overline{u_i} f_{j+1}^i (\overline{u_1}, \overline{u_2}, \overline{u_3})\right]^2 dt$$

$$\leq K_{j+1} L(t_{j+1}) \exp\left(\lambda_M (t_{j+1} - t_j)\right) L(t_j^+),$$

where

$$K_j = \max_{x \in \bar{\Omega}, u_1, u_2, u_3 \in S} 2 \left\{ \sum_{i=1}^{3} \left(f_j^i\right)^2 + \sum_{i=1}^{3} \sum_{i=1}^{3} \left(M' \frac{\partial f_j^i}{\partial u_i}\right)^2 \right\}.$$

In the following, let us estimate the variation of the function over the period. We have

$$L(t+\tau) \leq K_* L(t) = \prod_{j=1}^{p} K_j \exp(\lambda_M \tau) L(t).$$

Obviously, if the condition $\Pi = \sum_{j=1}^{p} \ln K_j + \tau \lambda_M < 0$ holds, then we have $K_* < 1$. Therefore, $L(m\tau + s) \leq K_*^m L(s) \to 0$ as $m \to \infty$, which implies that $\|u_i(t,x) - \overline{u_i}(t,x)\| \to 0$ for all $i = 1,2,3$ as $t \to \infty$, where $\|\cdot\|$ is the norm of space $L_2(\Omega)$. From Lemma C (see Appendix C), solutions of system (3-1)-(3-5) are bounded in the space C^{1+v}. Therefore,

$$\sup_{x \in \bar{\Omega}} |u_i(t,x) - \overline{u_i}(t,x)| \to 0, i = 1,2,3, t \to \infty. \tag{3-17}$$

Consider the following sequence

$$\left(u_1(k\tau, x, \overline{u_1}, \overline{u_2}, \overline{u_3}), u_2(k\tau, x, \overline{u_1}, \overline{u_2}, \overline{u_3}), u_3(k\tau, x, \overline{u_1}, \overline{u_2}, \overline{u_3})\right) = w(k\tau, w_0), k \in N.$$

From Lemma C (see Appendix C), it is compact in the space $C(\bar{\Omega} \times \bar{\Omega} \times \bar{\Omega})$. Let $\overline{w}$ be a limit point of this sequence, $\overline{w} = \lim_{n \to \infty} w(k_n \tau, w_0)$, then $w(\tau, \overline{w}) = \overline{w}$. Since the fact that

$$w\left(\tau, w(k_n \tau, w_0)\right) = w\left(k_n \tau, w(\tau, w_0)\right)$$

and as $k_n \to \infty$

$$w\left(k_n \tau, w(\tau, w_0)\right) - w(k_n \tau, w_0) \to 0.$$

Then, one can obtain that, for $n \to \infty$,

$$\begin{aligned}
&\|w(\tau,\overline{w})-\overline{w}\|_c \\
&\qquad\le \left\|w(\tau,\overline{w})-w\big(\tau,w(k_n\tau,w_0)\big)\right\|_c \\
&\qquad+\left\|w\big(\tau,w(k_n\tau,w_0)\big)-w(k_n\tau,w_0)\right\|_c+\|w(k_n\tau,w_0)-\overline{w}\|_c \to 0
\end{aligned}$$

The sequence $w(k\tau,w_0), k\in N$ has a unique limit point. Otherwise, suppose the sequence has two limit points, that is, $\overline{w}=\lim_{n\to\infty} w(k_n\tau,w_0)$ and $\widetilde{w}=\lim_{n\to\infty} w(k_n\tau,w_0)$. From (3-17) and $\widetilde{w}=w(k_n\tau,\widetilde{w})$, then we have

$$\|\overline{w}-\widetilde{w}\|_c \le \|\overline{w}-w(k_n\tau,w_0)\|_c+\|w(k_n\tau,w_0)-\widetilde{w}\|_c \to 0, n\to\infty.$$

Hence, $\overline{w}=\widetilde{w}$ and the solution $\big(u_1(t,x,\overline{u_1},\overline{u_2},\overline{u_3}),u_2(t,x,\overline{u_1},\overline{u_2},\overline{u_3}),u_3(t,x,\overline{u_1},\overline{u_2},\overline{u_3})\big)$ is the unique periodic solution of system (3-1)-(3-5), and from (3-17), it is asymptotically stable. Therefore, suppose that the conditions (C1)-(C5) and the conditions in Lemma C (see Appendix C) hold, system (3-1)-(3-5) has a unique, globally asymptotically stable, and strictly positive piece-wise continuous τ-periodic solution if $\Pi<0$ holds.

Remark 3.2.4.1 Assume that conditions (C1)-(C3) holds, and the system (3-1)-(3-3), (3-5) is permanent. Then the system (3-1)-(3-3), (3-5) has a unique, globally asymptotic stable, and strictly positive τ-periodic solution if

$$\lambda_M<0.$$

3.3. Experimental simulations

Although some interesting results of system (3-1)-(3-5) have been achieved by means of modeling analysis, it is difficult to further study the dynamical properties of the system due to its complexity. Fortunately, experimental simulations can help us provide more in-depth insights on the dynamics of the system. In this section, therefore, some numerical simulations are carried out to study the effects of the key factors on the dynamics of the system and show the effects of impulsive control and environmental heterogeneity on the survival of population and the dynamic mechanisms of the spatial and temporal distribution of population.

3.3.1 Impact of impulsive control on the dynamics of system (3-1)-(3-5)

In this subsection, some numerical simulations are carried out to study the effect of impulsive control on the dynamics of system (3-1)-(3-5). In these numerical simulations, the following parameter values are used: $r_1(t,x) = r_2(t,x) = r_3(t,x) = r_3(t,x) = 1.5, \mu_1 = \mu_2 = \mu_3 = 1, \tau = 2, \Omega = [-1,1]$, $a_1(t,x) = 1.5\cos(\pi t) + 1.5\sin(\pi x) + 10, b_1(t,x) = \sin(\pi t) + 2\sin(\pi x) + 6, c_1(t,x) = 2\cos(\pi t) + 2\cos(\pi x) + 5, a_2(t,x) = 2\cos(\pi t) + sin(\pi x) + 8, b_2(t,x) = 1.5\cos(\pi t) + 1.2\sin(\pi x) + 4, c_2(t,x) = \cos(\pi t) + 0.5\sin(\pi x) + 4, c'_1(t,x) = 0.5\cos(\pi t) + 0.5\sin(\pi x) + 4, \mathrm{c}'_2(\mathrm{t},\mathrm{x}) = 0.4\cos(\pi t) + 0.1\sin(\pi x) + 4.5, a_3(t,x) = 0.5\sin(\pi t)\sin(\pi x) + 0.1$, and other parameters are chosen as control parameters. Obviously, the conditions (C1)-(C5) in the preliminaries subsection can be verified. To study how impulsive control affect the dynamical behaviors of system (3-1)-(3-5), we firstly consider that the system (3-1)-(3-5) does not have impulsive control, that is, $f_k^i(x,u_1,u_2,u_3)(i = 1,2,3, k \in N)$ are identically equal to zero. Thus, the Remarks 3.2.2.1 and 3.2.4.1 hold by direct computation, which are sufficient to establish permanence for system (3-1)-(3-3) and (3-5), that is, all the species can coexist at a stable state under some conditions (see Fig. 3.1.).

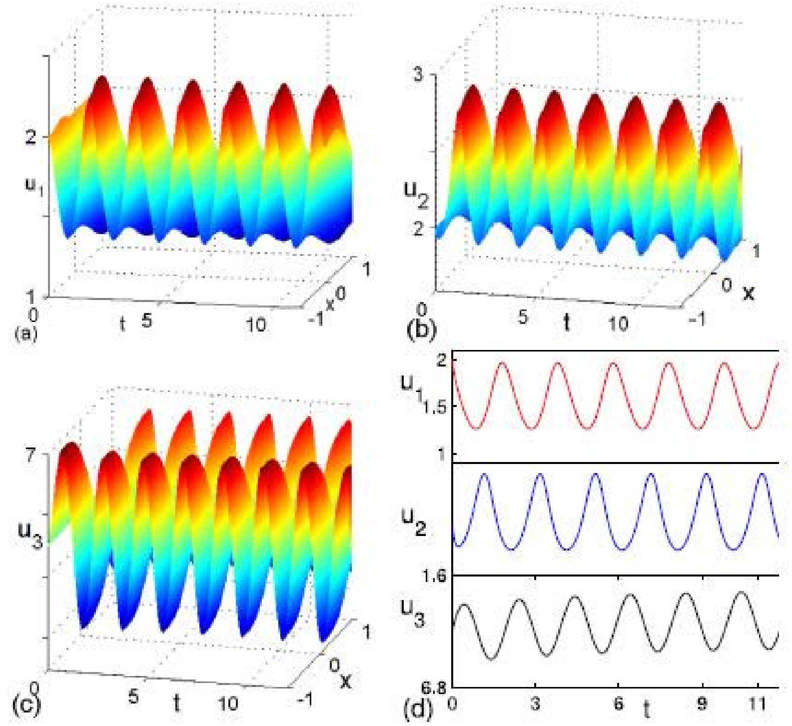

Fig. 3.1 Dynamic behaviors of species u_1, u_2 and u_3 of system (3-1)-(3-5) without impulsive effects, and the initial conditions $u_1(0,x)=2, u_2(0,x)=2, u_3(0,x)=6.85$ for all $x \in \Omega$. **(a)**: permanence of species u_1; **(b)**: permanence of species u_2; **(c)**: permanence of species u_3; **(d)**: time series of u_1, u_2 and u_3 on the population dynamics with $x=0$.

In the following, the impact of impulsive control on the dynamics of the system (3-1)-(3-5) will be shown. Choosing $f_k^1(x,u_1,u_2,u_3)=0.9, f_k^2(x,u_1,u_2,u_3)=0.95, k=1,2,\cdots, f_k^3(x,u_1,u_2,u_3)=0.95$ for all $k=1,2,\cdots,p=8$, it is not difficult to find that the species $u_1(t,x), u_2(t,x)$ and $u_3(t,x)$ can coexist at a stable state under some conditions (see Fig. 3.2). Actually, due to the conditions $H_i(i=1,2,3)>0$, all the species $u_1(t,x), u_2(t,x)$ and $u_3(t,x)$ are permanent, which are consistent with our experimental simulations. Comparing Figs. 3.1 and 3.2, it is obvious to find that the impulsive control can affect the temporal spatial dynamics of the system. That is, the impulsive control has a profound effect on population dynamic evolution mechanism. Furthermore, it is obliged to be stressed that the impulsive control not only can aggravate the emergence of pulse oscillation, but also can change the periodicity of population density. Thus, it is worth pointing out that the results from Figs. 3.1 and 3.2 can support that the reaction-diffusion impulsive hybrid system can depict the interaction mechanism between populations.

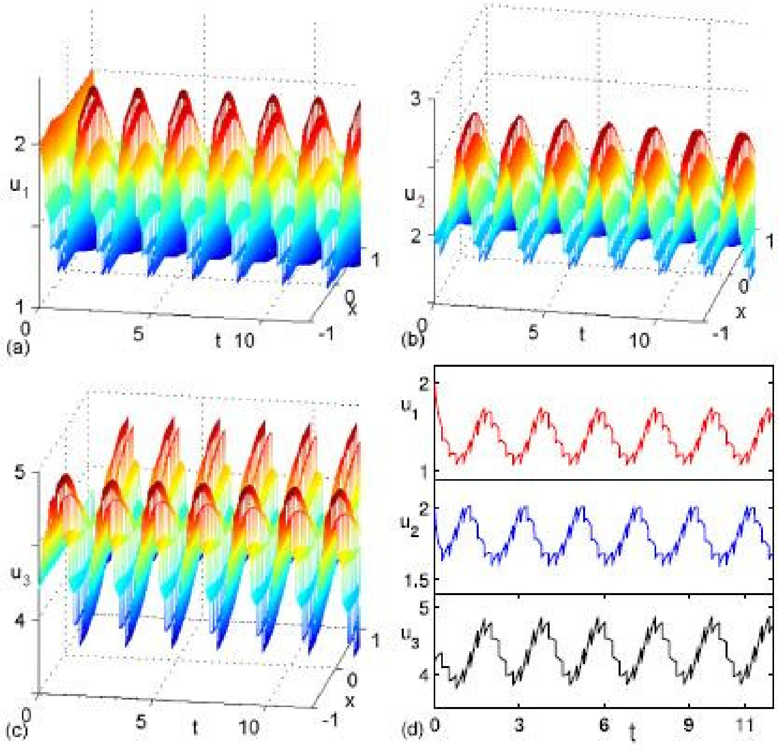

Fig. 3.2 The effects of impulsive effects on the dynamic behaviors of species u_1, u_2 and u_3 of system (3-1)-(3-5), here $f_k^1(x, u_1, u_2, u_3) = 0.9, f_k^2(x, u_1, u_2, u_3) = 0.95, k = 1,2, \cdots, f_k^3(x, u_1, u_2, u_3) = 0.95$ for all $k = 1,2, \cdots, p = 8$ and initial conditions $u_1(0, x) = 2, u_2(0, x) = 2, u_3(0, x) = 4.2$ for all $x \in \Omega$: **(a)**: permanence of species u_1 ; **(b)**: permanence of species u_2 ; **(c)**: permanence of species u_3 ; **(d)**: time series of u_1, u_2 and u_3 on population dynamics with $x = 0$ permanence of species u_1 ; **(b)**: permanence of species u_2 ; **(c)**: permanence of species u_3 ; **(d)**: time series of u_1, u_2 and u_3 on population dynamics with $x = 0$.

On the other hand, when we control some a parameter values satisfying the condition of $H_4 < 0$, then the species $u_3(t, x)$ undergoes extinction, but the other two species $u_1(t, x)$ and $u_2(t, x)$ generate periodic oscillation (permanence), as are shown in Fig. 3.3. It is easy to know from Fig. 3.3 (d) that the time series solutions for species $u_1(t, x)$ and $u_2(t, x)$ have a stable periodic oscillation over time. In contrast, species $u_3(t,x)$ will rapidly decrease to extinction over time. Moreover, comparing Figs. 3.2 and 3.3, it is obvious to find that the dynamical behaviors of the system (3-1)-(3-5) have been changed by controlling the value of p ($p = 40$). Therefore, it is necessary to point out that some critical parameters have a profound impact on the persistence and extinction of populations.

In order to further study how the impulsive parameter p in a period and impulsive perturbation constants $f_k^i(x,u_1,u_2,u_3)$ $(i = 1,2,3, k \in N)$ affect the population density dynamic evolution trend, it is necessary to present a more in-depth analysis of the relationship between the parameter value and the population density. Fig. 3.4 depicts the variation of species $u_1(t,x), u_2(t,x)$ and $u_3(t,x)$ with increasing p, where the red lines represent the initial value of the corresponding population. It is obvious to survey from Fig. 3.4 that the maximum density of species $u_1(t,x)$ and $u_2(t,x)$ can remain unchanged, but species $u_3(t,x)$ will rapidly decline and finally will undergo extinction when p increases. Thus, it is easy to see that impulsive parameter p has a great influence on the maximum of species $u_3(t,x)$, but it has almost no effect on the maximum density of species $u_1(t,x)$ and $u_2(t,x)$. Fig. 3.5 depicts how the impulsive perturbation constants $f_k^3(x,u_1,u_2,u_3)(k \in N)$ affect the density distribution of species u_3, where $r > 0$ denotes the number of species $u_3(t,x)$ released each time, and the red line represents the initial value of species $u_3(t,x)$. When $r = 0$, species $u_3(t,x)$ will undergo extinction as $t \to \infty$ without impulsive perturbation $f_k^3(x,u_1,u_2,u_3)$ $(k \in N)$. However, when $0 \le r < 0.5$, species $u_3(t,x)$ will develop a periodic oscillation. Thus, it is obvious that the impulsive perturbation $f_k^3(x,u_1,u_2,u_3)(k \in N)$ has a profound effect on extinction of species $u_3(t,x)$. These results suggested that the impulsive perturbation constants $f_k^3(x,u_1,u_2,u_3)$ $(k \in N)$ and impulsive parameter p can play a restrictive role for survival of species $u_3(t,x)$, which also verifies that the system has potential to be applied for actual biological control issues.

3.3.2 Impact of environmental heterogeneity on the dynamics of system (3-1)-(3-5)

The distribution of population relies strongly on the heterogeneity and homogeneity of their living environments, thus, the dynamical behaviors of population distribution in a homogeneous environment are studied. It is obvious to find that the system (3-1)-(3-5) exists spatially homogeneous periodic solutions and species $u_1(t,x), u_2(t,x)$ and $u_3(t,x)$ are distributed uniformly over space at the same time, as shown in Fig. 3.6. However, when the environment is heterogeneous, the system (3-1)-(3-5) can generate a solution which is heterogeneous in space direction and periodic in time direction, as it is shown in Fig. 3.7. In such circumstances, species $u_1(t,x), u_2(t,x)$ and $u_3(t,x)$ is heterogeneously distributed in space with high density on the edge and cyclical variations over time. From the comparative analysis of Figs. 3.6 and 3.7, it is not difficult to find that the living environment plays an import role in the spatial distribution of populations.

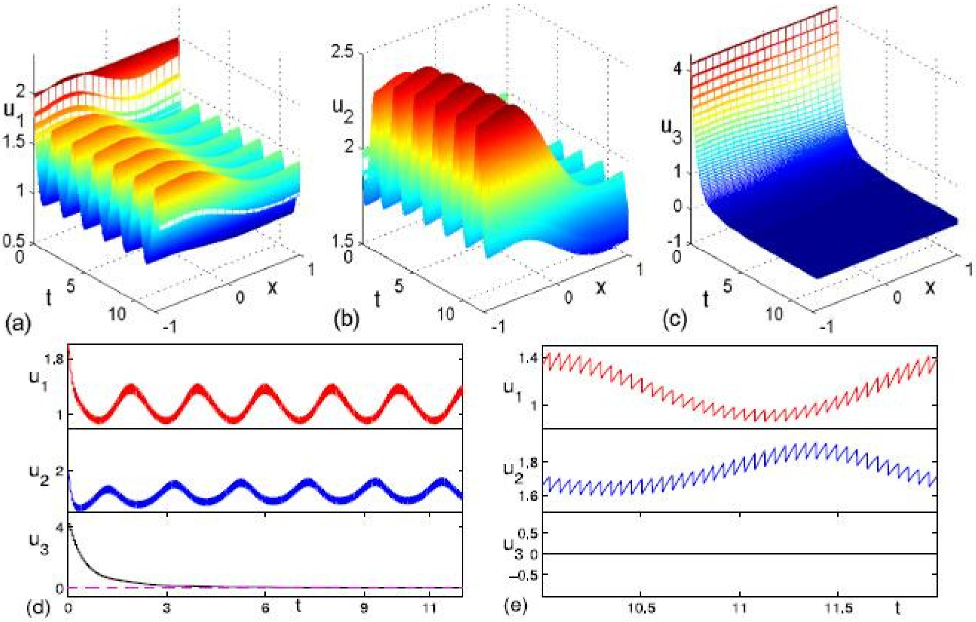

Fig. 3.3 The effects of impulsive control on the dynamic behaviors of species u_1, u_2 and u_3 of system (3-1)-(3-5), here $p = 40$ and initial conditions $u_1(0,x) = 2, u_2(0,x) = 2, u_3(0,x) = 4.2$ for all $x \in \Omega$: **(a)**: permanence of species u_1 ; **(b)**: permanence of species u_2 ; **(c)**: extinction

of species u_3 as $t \to \infty$; **(d)**: time series of u_1, u_2 and u_3 on population dynamics with $x = 0$.

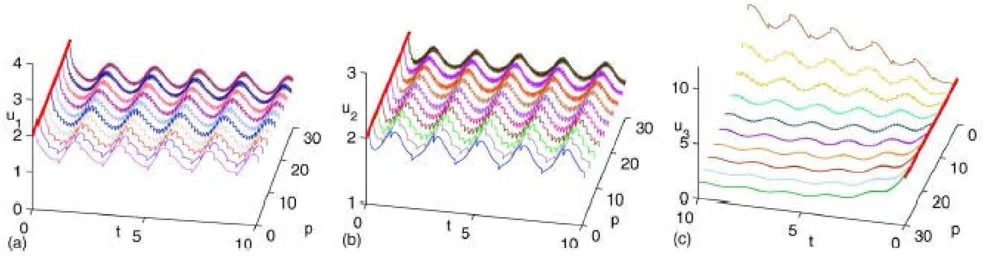

Fig. 3.4 The effect of impulsive control parameter $p\ (0 \le p \le 30)$ on the density of all species, her the initial conditions $u_1(0,x) = 2, u_2(0,x) = 2, u_3(0,x) = 4.2$ for all $x \in \Omega$.

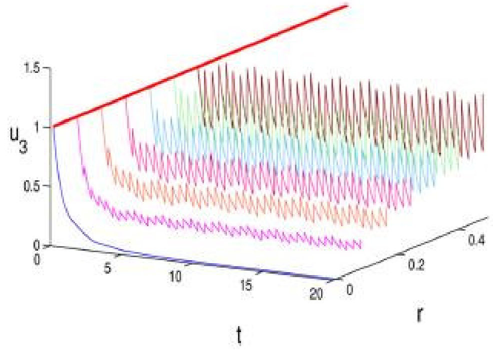

Fig. 3.5 The effect of impulse perturbation constants $f_k^i (i = 1,2,3, k \in N)$ the density of species u_3 with fixed $p = 1$, $f_k^3(x, u_1, u_2, u_3) = 1 + \frac{r}{u_3} (0 \le r \le 0.5)$ for all $k = 1,2,\cdots,$; $c'_2(t,x) = 0.4\cos(\pi t) + 0.1\sin(\pi x) + 3, a_3(t,x) = 0.5\sin(\pi t)\sin(\pi x) + 2$ and the initial conditions $u_1(0,x) = 2, u_2(0,x) = 2, u_3(0,x) = 1$ for all $x \in \Omega$.

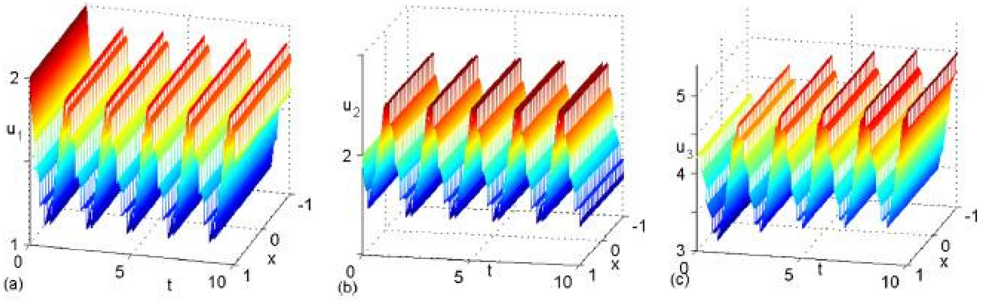

Fig. 3.6 Numerical simulation of spatially homogeneous stable periodic solutions of the system (3-1)-(3-5) without spatial representation for $p = 8, a_1(t,x) = 1.5\cos(\pi t) + 10, b_1(t,x) = \sin(\pi t) + 6, c_1(t,x) = 2\cos(\pi t) + +5, a_2(t,x) = 2\cos(\pi t) + 8, b_2(t,x) = 1.5\cos(\pi t) +$

$4, c_2(t,x) = \cos(\pi t) + 4, c'_1(t,x) = 0.5\cos(\pi t) + 4, c'_2(t,x) = 0.4\cos(\pi t) + 4,$ $a_3(t,x) = 0.5\sin(\pi t) + 0.1$ and the initial conditions $u_1(0,x) = 2$, $u_2(0,x) = 3, u_3(0,x) = 1$ for all $x \in \Omega$.

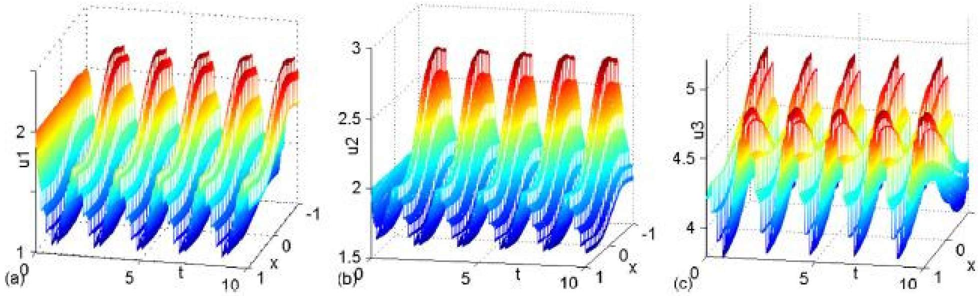

Fig. 3.7 Numerical simulation of spatially non-homogeneous periodic solutions of the system (3-1)-(3-5) with $p = 8$, $a_1(t,x) = 1.5\cos(\pi t) + 1.5\sin(\pi x) + 10, b_1(t,x) = \sin(\pi t) + 2\sin(\pi x) + 6, c_1(t,x) = 2\cos(\pi t) + \cos(\pi x) + 5, a_2(t,x) = 2\cos(\pi t) + \sin(\pi x) + 8, b_2(t,x) = 1.5\cos(\pi t) + 1.2\sin(\pi x) + 4, c_2(t,x) = \cos(\pi t) + 0.5\sin(\pi x) + 4, c'_1(t,x) = 0.5\cos(\pi t) + 0.5\sin(\pi x) + 4, c'_2(t,x) = 0.4\cos(\pi t) + 0.1\sin(\pi x) + 4.5, a_3(t,x) = 0.5\sin(\pi t)\sin(\pi x) + 0.1$ and initial conditions $u_1(0,x) = 2, u_2(0,x) = 2, u_3(0,x) = 4.2$ for all $x \in \Omega$.

4.3. Conclusions

An impulsive reaction-diffusion predator-prey system with Crowley-Martin functional response is proposed, and we study the dynamics of the system analytically and numerically in this thesis. Using the upper and lower solution method and comparison theory of differential equations, the boundedness, persistence, and extinction are analyzed theoretically, which provided some sufficiency conditions. To prove the existence, uniqueness, and globally asymptotic stability of positive periodic solutions, compactness theory and a method based on constructing a proper auxiliary function are applied. Numerical analysis verifies the theoretical results and further indicates that the change of population density is periodic oscillation in time direction whether their spatial distribution is heterogeneous or not, but the heterogeneous environment indeed has influence on the spatial distribution of populations.

Furthermore, it is investigated how the dynamics of system (3-1)-(3-5) strongly depends on pulse parameter p and the impulsive perturbation constants $f_k^i (i = 1,2,3, k \in N)$, as well as other parameters. Choosing p and $f_k^i (i = 1,2,3, k \in N)$ as control parameters, the dynamics of the system were analyzed numerically, particularly species extinction and permanence. According to the Figs. 3.1 and 3.2, it is obvious to find that the impulsive control can significantly affect the temporal and spatial dynamics of the system. From Figs. 3.2 and 3.3, by controlling the impulsive control parameter p, the dynamical behaviors of the system (3-1)-(3-5) can be greatly changed. That is, when $p > p^*$, the prey and intermediate predator can coexist, while top-predator undergone extinction rapidly, where p^* is some a critical value of p. It should be stressed that the top-predator will trend to extinct without impulsive effects and a periodic oscillation can be generated by controlling the released number r under some conditions. From an ecological viewpoint, uniform persistence implies that the prey, intermediate predator and top-predator populations can coexist at any time and any location of the inhabited domain. Therefore, we can utilize some key arguments to control population permanence and extinction by means of some reaction-diffusion impulsive hybrid systems, which are expected to be useful in the studies on the dynamic complexity of ecosystems.

Chapter 4 DYNAMICS INDUCED BY ENVIRONMENTAL STOCHASTICITY IN A PHYTOPLANKTON-ZOOPLANKTON SYSTEM WITH TOXIC PHYTOPLANKTON[2]

Abstract

Environmental stochasticity and TPP are the key factors that affect the real aquatic ecosystems. To investigate the effects of environmental stochasticity and TPP on the dynamics of plankton populations, a stochastic phytoplankton-zooplankton system with two TPP is studied theoretically and numerically in this thesis. Theoretically, we first prove that the system possesses a unique and global positive solution with any given positive initial values, and then derive some sufficient conditions guaranteeing the extinction and persistence in the mean of the system. Significantly, it is shown that the system has a stationary distribution when toxin liberation rate reaches some a critical value. Additionally, numerical analysis shows that the white noise can affect the survival of plankton populations directly. Furthermore, it has been observed that the increasing one toxin liberation rate can increase the survival chance of phytoplankton and reduce the biomass of zooplankton, but the combined effects of two liberation rates on the changes in plankton populations are stronger than that of controlling any one of the two TPP.

Keywords: Toxin-producing phytoplankton, Phytoplankton-zooplankton system, White noise, Extinction, Stationary distribution

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4.1. Introduction

Plankton, the organisms that the freely floating and weakly swimming in aquatic environments, occupy the first trophic level and the second trophic level of any aquatic food chains. Phytoplankton are the photosynthetic microorganisms and commonly unicellular and microscopic in size, and zooplankton are the heterotrophic plankton that live on phytoplankton. In addition to recognizing the importance of plankton for the wealth of the aquatic ecosystems and ultimately for the planet itself (Huppert et al., 2002), the variation of plankton biomass is an important factor influencing the real aquatic environments, and understanding of plankton dynamics can be helpful to estimate the productivity of aquatic ecosystems (Behrenfeld and Falkowski, 1997; Hoppe et al., 2002) and regulate the balance of plankton ecosystems. However, planktonic blooms can occur under some conductive environments, which may cause seriously environmental issues and threat to human health. But the processes underlying the formation of planktonic blooms are not yet well understood. In this respect, thus, the great effort has been made towards the understanding of the complex dynamics of plankton, and then mathematical models can be acted as a useful tool to investigate the dynamics of plankton ecosystems, which can provide a deeper understanding of the dynamic mechanisms of changes in plankton populations.

Actually, many mathematical models have been constructed to study the dynamical behaviors of plankton since the pioneering work of Riley et al. (1949), and many physical and biological processes underlying the mechanisms of plankton dynamics in the aquatic environments have been investigated (Huppert et al. 2002; Dai et al., 2016; Caperon, 1969; Guo et al., 2019; Lin et al., 2005; Liao et al., 2020; Zhao et al., 2020). For example, in order to study how the nutrient affects the dynamics of phytoplankton blooms, Huppert et al. (2002) presented a simple nutrient-phytoplankton model and identified an important threshold effect that a bloom will only be triggered when nutrients exceed a certain defined level using mathematical model analysis. Caperon (1969) concluded that the time-lag effect exists in the growth process of phytoplankton, and further suggested that models play an important role in understanding the growth dynamics of phytoplankton characterized by time delays. Lin et al. (2005) used a nutrient-phytoplankton-zooplankton model to examine the patterns and consequences of adaptive changes in plankton body size and suggested that evolutionary interactions between phytoplankton and zooplankton may have contributed to observed changes in phytoplankton sizes and associated biogeochemical cycle over geological time scales.

In recent years, the dynamical behaviors of phytoplankton-zooplankton systems with various biological factors, such as stability, bifurcation and spatiotemporal pattern, have been explored extensively (Caperon, 1969; Zhao et al., 2020; Zhao et al., 2016; Han and Dai, 2019). Nevertheless, some phytoplankton species are harmful phytoplankton that can produce potent toxic or allelopathic substances during phytoplankton blooms (Hallam and Luna, 1984), which can affect species interaction by suppressing the growth and establishment of other phytoplankton species (Hallam et al., 1983). Moreover, some laboratory experiments (Huntley et al., 1986; Nejstgaard and Solberg, 1996), as well as field observation (Estep et al., 1990), have suggested that the toxicity may be as a strong mediator in the zooplankton feeding rate. As a result, some researchers have taken this important factor of toxic production released by TPP into account when studying the phytoplankton-zooplankton systems (Scotti et al. 2015; Banerjee and Venturino, 2011; Khare et al., 2010). For example, Scotti et al. (2015) indicated that a toxic phytoplankton may destabilize the spatially homogeneous coexistence and trigger the formation of spatial pattern, and further concluded that local blooms more likely occur when the strength of the toxicity is of a certain level. Additionally, some results from field observations and model analysis concluded that the toxic substances can affect the interaction between phytoplankton and zooplankton and reduce the growth of zooplankton, indicating TPP may act as a biological control way for the termination of planktonic blooms (Chattopadhyay et al., 2002; Chattopadhyay et al., 2002; Sarkar and Chattopadhyay, 2003; Chattopadhyay et al., 2004). Sarkar et al. (2005) proposed the following mathematical model consisting of two harmful phytoplankton and zooplankton species:

$$\begin{cases}\frac{dP_1(t)}{dt} = r_1P_1(t)\left(1-\frac{P_1(t)}{K_1(t)}\right) - a_1(t)P_1(t)P_2(t) - \alpha P_1(t)Z(t),\\ \frac{dP_2(t)}{dt} = r_2P_2(t)\left(1-\frac{P_2(t)}{K_2(t)}\right) - a_2(t)P_1(t)P_2(t) - \beta P_2(t)Z(t),\\ \frac{dZ(t)}{dt} = mP_1(t)Z(t) + nP_2(t)Z(t) - \frac{\gamma P_1(t)Z(t)}{a+P_1(t)} - \frac{\delta P_2(t)Z(t)}{b+P_2(t)} - dZ(t),\end{cases} \tag{4-1}$$

where $P_1(t), P_2(t)$ and $Z(t)$ are the population densities of two harmful phytoplankton and zooplankton species at time t, respectively; r_1 and r_2 denote the intrinsic growth rates of two TPP, respectively; K_1 and K_2 are their corresponding environment carrying capacities; a_1 and a_2 represent be the inhibitory effects of two harmful phytoplankton; α and β are the maximum zooplankton ingestion rates for both two TPP species, respectively; m and n are the maximum zooplankton conversion rates, respectively; d is the natural death rate of zooplankton; γ and δ are the rates of toxin liberation by two TPP species, respectively; a and b denote the half-saturation constants for two TPP species. The authors studied the asymptotic stability of the system (4-1) and claimed that the presence of two harmful phytoplankton populations has a positive impact for the termination of planktonic blooms (Sarkar, 2005).

In the real world, however, the unpredictability and ubiquity of environmental fluctuations in the natural aquatic ecosystems, for example, the necessary nutrient availability, water temperature, light and turbulence, can greatly cause the growths of plankton populations to experience random fluctuations. Systems with such kinds of environmental fluctuations can be described by stochastic differential equations, which play a significant role in the population dynamics as they can provide some additional degree of realism compared to their corresponding deterministic counterparts (Renshaw, 1991). Thus, stochastic population systems, as an important application in ecological and biological systems, have attracted increasing attention (Sun et al., 2020; Deng and Liu, 2020; Jiang et al., 2020; Yu et al., 2019; Gao and Wang, 2019; Liu et al., 2017; Liu and Liu, 2019). Especially, stochastic plankton systems with white noise have been the common area of interest among researchers

(Liao et al., 2020; Sarkar and Chattopadhyay, 2003a, 2003b; Yu et al., 2019; Zhao et al., 2017; Chen et al., 2020; Chen et al., 2020; Liao et al., 2019; Xia et al., 2020; Zhao et al., 2020; Yu et al., 2020; Wang and Liu, 2020) in recent years and many interesting results have been shown. For example, Sarkar and Chattopadhayay (2003b) proposed a toxic phytoplankton-non-toxic phytoplankton-zooplankton with stochastic perturbation around the positive equilibrium, and they concluded that TPP and stochastic fluctuations can significantly affect the coexistence of species. Yu et al. (2018) investigated a nutrient-phytoplankton system with TPP under environmental fluctuations, and they obtained some conditions for extinction, persistence and the existence of ergodic stationary distribution. All these works greatly stimulate researchers to explore the way how environmental stochasticity and toxin production affect the coexistence and survival prospect of plankton populations in the presence of harmful phytoplankton. Obviously, it is meaningful to further incorporate the environmental fluctuations into the underlying model (4-1). Moreover, there are few literatures to study the dynamics of the stochastic phytoplankton-zooplankton system with two harmful phytoplankton populations, and the dynamics of the stochastic phytoplankton-zooplankton system with two harmful phytoplankton is still not very clear currently. Hence, we mainly present the influence of the effects of environmental white noise and toxic liberation rates produced by two TPP on the dynamics of phytoplankton-zooplankton system in this paper. Motivated by the works above, we assume that the intrinsic growth rates of two harmful phytoplankton and the death rate of zooplankton are influenced by the environmental fluctuations effect, and thus introduce the white noise into underlying system (4-1), resulting in the following form:

$$\begin{cases} \frac{dP_1(t)}{dt} = r_1 P_1(t)\left(1 - \frac{P_1(t)}{K_1(t)}\right) - a_1(t)P_1(t)P_2(t) - \alpha P_1(t)Z(t) \\ \qquad\qquad +\sigma_1(t)P_1(t)dB_1(t), \\ \frac{dP_2(t)}{dt} = r_2 P_2(t)\left(1 - \frac{P_2(t)}{K_2(t)}\right) - a_2(t)P_1(t)P_2(t) - \beta P_2(t)Z(t) \\ \qquad\qquad +\sigma_2(t)P_2(t)dB_2(t), \\ \frac{dZ(t)}{dt} = mP_1(t)Z(t) + nP_2(t)Z(t) - \frac{\gamma P_1(t)Z(t)}{a+P_1(t)} - \frac{\delta P_2(t)Z(t)}{b+P_2(t)} - dZ(t) \\ \qquad\qquad +\sigma_3(t)Z(t)dB_3(t), \end{cases} \tag{4-2}$$

where $B_i(t)$ are mutually independent standard Brownian motions with $B_i(0) = 0$ (Gikhman and Skorokhod, 1979), and $dB_i(t)$ are standard white noise and $\sigma_i^2(t)$ are their intensities, $i = 1,2,3$.

By now, we have successful introduced a stochastic phytoplankton-zooplankton system with two toxic phytoplankton focusing on the effects of environmental stochasticity and TPP, and our research questions include: (i) How does the environmental stochasticity affects the dynamics of plankton populations? (ii) What influences the peak of the outbreaks of planktonic blooms in a fluctuating environment? The rest of this paper is organized as follows: Section 4.2 presents the basic assumptions firstly, and then we investigate the existence and uniqueness of global positive solutions, and apply the Ito's formula to obtain the sufficient conditions for the extinction and persistence in the mean of system (4-2), and the existence of a unique ergodic stationary distribution by establishing a appropriate stochastic Lyapunov function. A series of numerical simulations are carried out to further study the dynamics of system (4-2) in Section 4.3. In Section 4.4, we summarize the results and present our conclusions.

4.2. Dynamic analysis

The survival of plankton populations is a interesting topic in the biology and ecology, and phytoplankton population plays an important role in the balance of aquatic ecosystems, but the rapid growth of harmful phytoplankton can lead to the occurrence of HABs and pose threat to our living environments. Thus, it is necessary to discuss the survival chance of plankton populations under some conditions in the model. In this respect, we firstly investigate the existence and uniqueness of global positive solutions, and then discuss the evolution process of plankton growth, especially for the extinction and persistence in the mean of system (4-2), and the positive recurrence and ergodic property of system (4-2) in this section. Before it, we introduce some preliminaries for the following discussion.

4.2.1. Preliminaries

Denote $\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{R}_+^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_i > 0, i = 1,2, \cdots, n\}$ and $|x| = \sqrt{\sum_{i=1}^n x_i^2}$. Throughout this paper, unless otherwise indicated, we always assume that $(\Omega, \mathcal{F}_t, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ is a completed probability space with a filtration $\{\mathcal{F}_t\}$ satisfying the usual normal conditions (i.e. it is right-continuous and increasing while $\{\mathcal{F}_0\}$ contains all $\mathcal{P}$-null sets). For convenience, if $\varphi(t)$ is a integrable function on $\mathbb{R}_+$, we define $\langle\varphi\rangle = \frac{1}{T}\int_0^T \varphi(s)\, ds, T > 0$.

Generally, we consider the n-dimentional stochastic differential equation:

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t), \quad t \in [t_0, T] \tag{4-3}$$

with initial value $x(t_0) = x_0 \in \mathbb{R}^n$, while $B(t)$ is n-dimentional standard Brownian motion defined on the completed probability space $(\Omega, \mathcal{F}_t, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$. Denote by $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+, \mathbb{R})$ the family of all non-negative functions $V(x, t)$ defined on $\mathbb{R}^n \times \mathbb{R}_+$ such that they are continuously twice differentiable in x and once in t. Define a differential operator L associated with Eq. (4-3) by (Mao, 1997b) as follows:

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i(x, t)\frac{\partial}{\partial X_i} + \frac{1}{2}\sum_{i,j=1}^{n} [g^T(x, t)g(x, t)]_{ij}\frac{\partial^2}{\partial x_i \partial x_j}.$$

Let $LV(x, t) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$, then

$$LV = V_t + V_x f(x, t) + \frac{1}{2} trace[g^T(x, t)V_{xx}g(x, t)],$$

where

$$V_t = \frac{\partial V}{\partial t}, V_x = \left(\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \cdots, \frac{\partial V}{\partial x_n}\right), V_{xx} = \left(\frac{\partial^2}{\partial x_i \partial x_j}\right)_{n\times n}.$$

By Itô's formula, if $x(t) \in \mathbb{R}^n$, then

$$dV = LV(x(t),t)dt + V_x(x(t),t)g(x(t),t)dB(t).$$

Next, we introduce the criterion of positive recurrent and the ergodic properties. Before it, we consider the stochastic equation:

$$dx(t) = f\big(x(t)\big)dt + \textstyle\sum_{r=1}^{k} \sigma_r(x)dB_r(t) \tag{4-4}$$

where $x(t)$ is a homogeneous Markov process in n-dimentional Euclidean space $\mathbb{R}^n$. The diffusion matrix is $D(x) = \big(d_{ij}(x)\big)$, and $d_{ij}(x) = \sum_{r=1}^{k} \sigma_r^i(x)\,\sigma_r^j(x)$. Thus, a lemma which describes the criterion of stationary distribution can be given.

Lemma 4.2.1.1 (Khasminskii, 2012) Suppose that there exists a bounded open set $E \subset \mathbb{R}^n$ with a smooth regular boundary Γ satisfying the following conditions:

(i) the diffusion matrix $D(x)$ is strictly positive definite for all $x \in E$;

(ii) there exists a non-negative C^2-function $V(x)$ and a positive constant M such that $LV(x) \le -M$ for $\forall x \in \mathbb{R}^n \,/\, E$.

Then there exists a solution $x(t)$ of the system (4-4) which is a stationary Markov process with a stationary distribution $\mu(\cdot)$ and for any given integrable function $g(\cdot)$ with respect the measure μ, we have

$$\mathcal{P}\left(\lim_{t\to\infty} \frac{1}{t}\int_0^t g\big(x(t)\big)dt = \int_{\mathbb{R}^l} g(x)\mu(dx)\right) = 1.$$

4.2.2. Existence and uniqueness of global positive solutions

Before investigating the stochastic dynamics of system (4-2), we should first guarantee whether the solution of the system is global and positive. Therefore, based on the biological interpretation, in this subsection, we just take the non-negative solutions into account for system (4-2) and discuss the existence of global positive solutions in system (4-2) for any given initial values in the following.

Actually, from the method of the Lemma 2.1 in the work of Ji et al. (2009), it is obvious to obtain that, for any given initial values $\big(P_1(0), P_2(0), Z(0)\big) \in \mathbb{R}_+^3$, all the coefficients of system (4-2) are locally Lipschitz continuous and the system admits a unique local solution $\left(P_1(t), P_2(t), Z(t)\right)$ on $t \in [0, \tau_e)$, where τ_e represents the explosion time. In the following, thus, we need to illustrate the solution is global, that is, we only need to prove $\tau_e = \infty, a.s.$ Let $n_0 \geq 1$ large enough such that $\left(P_1(0), P_2(0), Z(0)\right) \in \left[\frac{1}{n_0}, n_0\right]$ and for each integer $n \geq n_0$, we define the stopping time by the following form:

$$\tau_n = \inf\left\{t \in [0, \tau_e): \max\ \{P_1(t), P_2(t), Z(t)\} \leq \frac{1}{n} \text{ or } \min\{P_1(t), P_2(t), Z(t)\} \geq n\right\},$$

and set $\inf \emptyset = \infty$ ($\emptyset$ denotes the empty set). Obviously, one can obtain that τ_n is increasing as $n \to \infty$ and $\tau_n < \tau_e$. Thus, let $\tau_\infty = \lim_{n \to +\infty} \tau_n$, then $\tau_\infty \leq \tau_e$ a.s. If we can show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ and $\left(P_1(t), P_2(t), Z(t)\right) \in \mathbb{R}_+^3$ a.s. for all $t \geq 0$. In other words, to complete the proof, we only need to proof $\tau_\infty = \infty$ a.s. If the statement is false, then there exist two constants $T > 0$ and $\varepsilon \in (0,1)$ such that $\mathcal{P}\{\tau_\infty \leq T\} > \varepsilon$. Hence, for all $n \geq n_1$, there exists an integer $n_1 \geq n_0$ such that $\mathcal{P}\{\tau_\infty \leq T\} \geq \varepsilon$.

Define a C^2-function $\bar{\mathrm{V}}: \mathbb{R}_+^3 \to \overline{\mathbb{R}}_+$ by

$$\bar{V}(P_1, P_2, Z) = \tfrac{m}{\alpha}(P_1 - 1 - \log P_1) + \tfrac{n}{\beta}(P_2 - 1 - \log P_2) + (Z - 1 - \log Z).$$

Obviously, the function $\bar{V}(P_1, P_2, Z)$ is non-negative since the inequality $x - 1 - \log x > 0$ holds for all $x > 0$. Applying the Itô's formula to $\bar{V}(P_1, P_2, Z)$ yields

$$d\bar{V}(P_1, P_2, Z) = L\bar{V}(P_1, P_2, Z)dt + \frac{m}{\alpha}\sigma_1(P_1 - 1)dB_1(t) + \frac{n}{\beta}\sigma_2(P_2 - 1)dB_2(t) + \sigma_3(Z - 1)dB_3(t),$$

where $L\bar{V}: \mathbb{R}_+^3 \to \mathbb{R}$ is defined by

$$
\begin{aligned}
L\bar{V}(P_1,P_2,Z) = {} & \frac{m}{\alpha}(P_1-1)\left(r_1\left(1-\frac{P_1}{K_1}\right)-a_1P_2-\alpha Z\right)+\frac{m}{\alpha}\frac{\sigma_1^2}{2} \\
& +\frac{n}{\beta}(P_2-1)\left(r_2\left(1-\frac{P_2}{K_2}\right)-a_2P_1-\beta Z(t)\right)+\frac{n}{\beta}\frac{\sigma_2^2}{2} \\
& +(Z-1)\left(mP_1+nP_2-d-\frac{\gamma P_1}{a+P_1}-\frac{\delta P_2}{b+P_2}\right)+\frac{\sigma_3^2}{2} \\
& \le \left(d-\frac{r_2 n}{\beta}-\frac{r_1 m}{\alpha}+\frac{m\sigma_1^2}{2\alpha}+\frac{n\sigma_2^2}{2\beta}+\frac{\sigma_3^2}{2}+\gamma+\delta\right) \\
& +\frac{m}{\alpha}\left(r_1+\frac{r_1}{K_1}+a_2-m\right)P_1-\frac{mr_1}{\alpha K_1}P_1^2+\frac{n}{\beta}\left(r_2+\frac{r_2}{K_2}+a_1-n\right)P_2 \\
& -\frac{nr_2}{\beta K_2}P_2^2+|m+n-d|Z\le M+|m+n-d|Z,
\end{aligned}
$$

where

$$
\begin{aligned}
M = {} & \left(d-\frac{r_2 n}{\beta}-\frac{r_1 m}{\alpha}+\frac{m\sigma_1^2}{2\alpha}+\frac{n\sigma_2^2}{2\beta}+\frac{\sigma_3^2}{2}+\gamma+\delta\right) \\
& +\max_{P_1\in(0,+\infty)}\left\{\frac{m}{\alpha}\left(r_1+\frac{r_1}{K_1}+a_2-m\right)P_1-\frac{mr_1}{\alpha K_1}P_1^2\right\} \\
& +\max_{P_2\in(0,+\infty)}\left\{\frac{n}{\beta}\left(r_2+\frac{r_2}{K_2}+a_1-n\right)P_2-\frac{nr_2}{\beta K_2}P_2^2\right\}.
\end{aligned}
$$

Notice that $Z\le 2(Z-1-\log Z)+2\log 2\le 2\bar{V}(P_1,P_2,Z)+2\log 2$ for all $Z>0$, then one can obtain that

$$
L\bar{V}\le M+2|m+n-d|\log 2+2|m+n-d|\bar{V}\le \Upsilon(1+\bar{V})
$$

where

$$
\Upsilon=\max\{M+2|m+n-d|\log 2\,,2|m+n-d|\}.
$$

The remainder of the discussion follows that in the Theorem 3.3 (Chen et al., 2020), here, we omit it. Therefore, for any given initial values $\left(P_1(0),P_2(0),Z(0)\right)\in\mathbb{R}_+^3$, system (4-2) exists a unique solution $\left(P_1(t),P_2(t),Z(t)\right)$ on $\mathbb{R}_+$ and the positive solution will remain in $\mathbb{R}_+^3$ with probability one, that is, $\left(P_1(t),P_2(t),Z(t)\right)\in\mathbb{R}_+^3$ for all $t\ge 0$ almost surely.

4.2.3. Extinction and persistence induced by white noise

The extinction and persistence in the mean are two important aspects in a population system and it is vital to discover whether the white noise has important effect on the survival of population (Tapaswi and Mukhopadhyay, 1999). Therefore, based on the previous analysis, we will further discuss and analyze the extinction and persistence in the mean of plankton populations in system (4-2). The following analysis shows that white noise intensity can significantly affect the dynamical behaviors of stochastic system (4-2). Here we give a detailed analysis:

Firstly, suppose that $\left(P_1(t), P_2(t), Z(t)\right)$ be the solution of system (4-2) with the initial value $\left(P_1(0), P_2(0), Z(0)\right) \in \mathbb{R}_+^3$. Applying the Itô's formula to system (4-2) yields

$$d\ln P_1(t) = \left[r_1\left(1 - \frac{P_1(t)}{K_1}\right) - a_1P_2(t) - \alpha Z(t) - \frac{1}{2}\sigma_1^2\right]dt + \sigma_1 dB_1(t),$$

$$d\ln P_2(t) = \left[r_2\left(1 - \frac{P_2(t)}{K_2}\right) - a_2P_1(t) - \beta Z(t) - \frac{1}{2}\sigma_2^2\right]dt + \sigma_2 dB_2(t),$$

and

$$d\ln Z(t) = \left[mP_1(t) + nP_2(t) - d - \frac{\gamma P_1(t)}{a+P_1(t)} - \frac{\delta P_2(t)}{b+P_2(t)} - \frac{1}{2}\sigma_3^2\right]dt + \sigma_3 dB_3(t),$$

Integrating the above from 0 to t and dividing t on both sides, we have

$$\frac{1}{t}\frac{\ln P_1(t)}{\ln P_1(0)} = r_1 - \frac{1}{2}\sigma_1^2 - \frac{r_1}{K_1}\frac{1}{t}\int_0^t P_1(s)ds - a_1\frac{1}{t}\int_0^t P_2(s)ds - \alpha\frac{1}{t}\int_0^t Z(s)ds + \frac{M_1(t)}{t} \tag{4-5}$$

$$\frac{1}{t}\frac{\ln P_2(t)}{\ln P_2(0)} = r_2 - \frac{1}{2}\sigma_2^2 - \frac{r_2}{K_2}\frac{1}{t}\int_0^t P_2(s)ds - a_2\frac{1}{t}\int_0^t P_1(s)ds - \beta\frac{1}{t}\int_0^t Z(s)ds + \frac{M_2(t)}{t} \tag{4-6}$$

and

$$\frac{1}{t}\frac{\ln Z(t)}{\ln Z(0)} = -\left(d + \frac{1}{2}\sigma_3^2\right) + m\frac{1}{t}\int_0^t P_1(s)ds + n\frac{1}{t}\int_0^t P_2(s)ds - \frac{1}{t}\int_0^t \frac{\gamma P_1(s)}{a+P_1(s)}ds - \frac{1}{t}\int_0^t \frac{\delta P_2(s)}{b+P_2(s)}ds + \frac{M_3(t)}{t} \tag{4-7}$$

where

$$M_i(t) = \int_0^t \sigma_i P_i(s)dB_i(s), i = 1,2, M_3(t) = \int_0^t \sigma_3 Z(s)dB_3(s).$$

Moreover, the quadratic variation of $M_i(t)(i = 1,2,3)$ satisfy

$$\langle M_i(t), M_i(t)\rangle_t = \int_0^t \sigma_i^2\, ds \le (\sigma_i^*)^2 t, i = 1,2,3.$$

By the strong law of large numbers for martingales (Khasminskii, 2012) yields

$$\lim_{t\to\infty} \frac{M_i(t)}{t} = 0 \quad a.s., i = 1,2,3. \tag{4-8}$$

Thus, according to (4-5), we have

$$\limsup_{t\to\infty} \frac{\ln P_1(t)}{t} \le A = r_1 - \frac{1}{2}\sigma_1^2 \ a.s.$$

Obviously, we can obtain that $\lim_{t\to\infty} P_1(t) = 0 \ a.s.$ if $A < 0$, that is, the toxic phytoplankton population $P_1(t)$ tends toward extinction.

Next, we analyze the persistence in the mean of the toxic phytoplankton population $P_1(t)$. By making some estimation of (4-5), we have

$$\frac{1}{t}\frac{\ln P_1(t)}{\ln P_1(0)} \ge A - \frac{r_1}{K_1}\frac{1}{t}\int_0^t P_1(s)ds - \frac{a_1 K_2 B}{r_2} - \alpha D + \frac{M_1(t)}{t} \tag{4-9}$$

where B and D will be determined later. In addition, since the fact that

$$\lim_{t\to+\infty} \frac{\ln P_1(0)}{t} = \lim_{t\to+\infty} \frac{M_1(t)}{t} = 0.$$

Thus, from the properties of the limit, for arbitrary $\epsilon_2 > 0$, there exists a constant $T_2 > 0$ such that

$$\langle P_2(t)\rangle \le \frac{K_2 B}{r_2} + \frac{\epsilon_2}{a_1}, \langle Z(t)\rangle \le D + \frac{\epsilon_2}{\alpha}, \frac{\ln P_1(0)}{t} \ge -\frac{\epsilon_2}{2}, \text{and } \frac{M_1(t)}{t} \ge -\frac{\epsilon_2}{2}.$$

Substituting above inequalities into (4-9) and for all $t \ge T_2$, we have

$$\frac{1}{t}\ln P_1(t) \ge \theta_1 t - \frac{r_1}{K_1}\frac{1}{t}\int_0^t P_1(s)ds + \frac{M_1(t)}{t},$$

where $\theta_1 = A - \frac{a_1 K_2}{r_2}B - \alpha D$, $B = r_2 - \frac{1}{2}\sigma_2^2$, $C = d + \frac{1}{2}\sigma_3^2$, and $D = \frac{m K_1 A}{r_1} + \frac{n K_2 B}{r_2} - C$ From the Lemma 4 (Liu and Wang, 2011), it is obvious to find that, if $\theta_1 > 0$, we have

$$\lim_{t\to\infty} \inf \frac{1}{t}\int_0^t P_1(s)\, ds \ge \frac{K_1}{r_1}\theta_1 > 0.$$

This implies that the toxic phytoplankton $P_1(t)$ is persistence in the mean.

For the toxic phytoplankton $P_2(t)$, the same to analysis of $P_1(t)$, we have the toxic phytoplankton $P_2(t)$ is extinct if $B < 0$, while it is persistence in the mean under the condition $\theta_2 > 0$, where $\theta_2 = B - \frac{a_2 K_1}{r_1} A - \beta D$.

In view of the zooplankton, we have

$$\limsup_{t\to\infty} \langle P_1(t)\rangle \le \frac{K_1 A}{r_1},\ \limsup_{t\to\infty} \langle P_2(t)\rangle \le \frac{K_2 B}{r_2} \tag{4-10}$$

According to Eq. (4-7), we have

$$\frac{1}{t}\frac{\ln Z(t)}{\ln Z(0)} \le -C + m\langle P_1(t)\rangle + n\langle P_2(t)\rangle + \frac{M_3(t)}{t}. \tag{4-11}$$

Combining with (4-8) and (4-10), and taking upper limit on both sides of (4-11) yields

$$\limsup_{t\to\infty} \frac{\ln Z(t)}{t} \le \frac{mK_1 A}{r_1} + \frac{nK_2 B}{r_2} - C = D, a.s.$$

Obviously, if the condition $D < 0$, we have $\lim_{\mathrm{t}\to\infty} Z(t) = 0$. This means that the zooplankton is extinct.

Now, we show the persistence in the mean of zooplankton. Computing (4-5) $\times \frac{K_1}{r_1} m$ +(4-6) $\times \frac{K_2}{r_2} n$ +(4-7), we can observe that

$$\frac{K_1}{r_1} m \frac{1}{t}\ln\frac{P_1(t)}{P_1(0)} + \frac{K_2}{r_2} n \frac{1}{t}\ln\frac{P_2(t)}{P_2(0)} + \frac{1}{t}\ln\frac{Z(t)}{Z(0)} \ge$$

$$\left(\frac{K_1}{r_1} m - \frac{K_1}{r_1}\frac{K_2}{r_2} n a_2\right) A + \left(\frac{K_2}{r_2} n - \frac{K_1}{r_1}\frac{K_2}{r_2} m a_1\right) B - C + \gamma + \delta -$$

$$\left(\frac{K_1}{r_1} m\alpha + \frac{K_2}{r_2} n\beta\right) \int_0^{\mathrm{t}} Z(s)\,\mathrm{ds} + \frac{mK_1}{r_1}\frac{M_1(t)}{t} + \frac{nK_2}{r_2}\frac{M_2(t)}{t} + \frac{M_3(t)}{t} \tag{4-12}$$

By the strong law of large numbers for martingales, we can derive that

$$\lim_{\mathrm{t}\to\infty} \frac{mK_1}{r_1}\frac{M_1(t)}{t} = 0, \lim_{\mathrm{t}\to\infty} \frac{nK_2}{r_2}\frac{M_2(t)}{t} = 0, \lim_{\mathrm{t}\to\infty} \frac{M_3(t)}{t} = 0, a.s.$$

From the Lemma 2.3 (Zhao, 2016), we can obtain that

$$\lim_{\mathrm{t}\to\infty} \sup \frac{1}{t}\ln\frac{P_1(t)}{P_1(0)} \ge 0, \lim_{\mathrm{t}\to\infty} \sup \frac{1}{t}\ln\frac{P_2(t)}{P_2(0)} \ge 0, \lim_{\mathrm{t}\to\infty} \sup \frac{1}{t}\ln\frac{Z(t)}{Z(0)} \ge 0.$$

Thus, taking the limit superior in (4-12) and from the lemma 4 (Liu and Bai, 2016), and if $\varXi = \frac{K_1}{r_1}\left(m - \frac{n a_2 K_2}{r_2}\right)A + \frac{K_2}{r_2}\left(n - \frac{m a_1 K_1}{r_1}\right)B - C - \gamma - \delta > 0$, one can see that

$$\lim_{\mathrm{t}\to\infty} \inf Z(t) \geq \left(\frac{K_1}{r_1} m\alpha + \frac{K_2}{r_2} n\beta\right)^{-1} \times$$
$$\left[\frac{K_1}{r_1}\left(m - \frac{n a_2 K_2}{r_2}\right)A + \frac{K_2}{r_2}\left(n - \frac{m a_1 K_2}{r_2}\right)B - C - \gamma - \delta\right] > 0,$$

which implies

$$\lim_{\mathrm{t}\to\infty} \inf Z(t) > 0.$$

That is, the zooplankton is persistence in the mean.

To sum up, for any initial values $\left(P_1(0), P_2(0), Z(0)\right) \in \mathbb{R}_+^3$, we can obtain the following conclusion:

I. the harmful phytoplankton $P_1(t)$ is

(i) extinct if $A < 0$;

(ii) persistent in the mean if $\theta_1 > 0$.

II. the harmful phytoplankton $P_2(t)$ is

(i) extinct if $B < 0$;

(ii) persistent in the mean if $\theta_2 > 0$.

III. the zooplankton $Z(t)$ is

(i) extinct if $D < 0$;

(ii) persistent in the mean if $A > 0$, $B > 0$, and

$$\varPi = \frac{K_1}{r_1}\left(m - \frac{n a_2 K_2}{r_2}\right)A + \frac{K_2}{r_2}\left(n - \frac{m a_1 K_1}{r_1}\right)B - C - \gamma - \delta > 0.$$

4.2.4. The ergodic stationary distribution of plankton

It is well known that ergodicity is one of the most important and significant characteristics for population systems described by stochastic differential equations, which means that the system has a unique stationary distribution, providing a biological perspective of cycling phenomena of a population system for long time. In this subsection, therefore, we discuss the ergodic property for stochastic system (4-2) by constructing a suitable Lyapunov function. The following modeling analysis shows that system (4-2) admits the stationary distribution which is ergodic when the white noise intensity and toxin released rate are not particularly large.

Actually, to prove ergodic stationary distribution, we need to verify the conditions of Lemma 4.2.1.1. According to the Lemma 4.2.1.1 in preliminaries subsection, it is obvious to obtain that the diffusion matrix $d_{ij}(x,K) = \text{diag}\,(\sigma_1^2, \sigma_2^2, \sigma_3^2)$ of system (4-2) is positive definite, which implies that the condition (i) in Lemma 4.2.1.1 holds.

Next, we prove the condition (ii) in Lemma 4.2.1.1. By constructing a C^2-function $\bar{V}: \mathbb{R}_+^3 \to \mathbb{R}$ as follows:

$$\bar{V}(P_1, P_2, Z) = \frac{1}{p+1}(\frac{m}{\alpha}P_1 + \frac{n}{\beta}P_2 + Z)^{p+1} - M\left(\frac{mK_1}{r_1}\ln P_1 + \frac{nK_2}{r_2}\ln P_2 + \ln Z\right) - Z,$$

where p is positive constant satisfying $0 < p < \frac{d}{3^p 2[\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2]}$ and $M > 0$ will be determined later. Since the function $\bar{V}(P_1, P_2, Z)$ is continuous, then there exists a unique point $(\overline{P_1}, \overline{P_2}, \bar{Z})$ in $\mathbb{R}_+^3$ which is the minimum value of $\bar{V}(P_1, P_2, Z)$. Thus, construct a non-negative C^2-function defined $V: \mathbb{R}_+^3 \to \mathbb{R}$ by

$$\begin{aligned} V(P_1, P_2, Z) &= \bar{V}(P_1, P_2, Z) - \bar{V}(\overline{P_1}, \overline{P_2}, \bar{Z}) \\ &= \frac{1}{p+1}(\frac{m}{\alpha}P_1 + \frac{n}{\beta}P_2 + Z)^{p+1} - M\left[\left(\frac{mK_1}{r_1}\ln P_1 + \frac{nK_2}{r_2}\ln P_2 + \ln Z\right)\right] \\ &\quad - Z - \bar{V}(\overline{P_1}, \overline{P_2}, \bar{Z}) = V_1(P_1, P_2, Z) + V_2(P_1, P_2, Z) + V_3(P_1, P_2, Z). \end{aligned}$$

Using the generalized Itô's formula, one can see that

$$\begin{aligned} LV_1(P_1, P_2, Z) = {} & (\frac{m}{\alpha}P_1 + \frac{n}{\beta}P_2 + Z)^p \left[\frac{mr_1}{\alpha}P_1 - \frac{mr_1}{\alpha K_1}P_1^2 - \frac{ma_1}{\alpha}P_1P_2 + \frac{nr_2}{\beta}P_2 - \frac{nr_2}{\beta K_2}P_2^2 \right. \\ & \left. - \frac{na_2}{\beta}P_1P_2 - \frac{\delta P_1 Z}{a + P_1} - \frac{\gamma P_2 Z}{b + P_2} - dZ\right] \\ & + \frac{p}{2}(\frac{m}{\alpha}P_1 + \frac{n}{\beta}P_2 + Z)^{p-1}\left[\sigma_1^2\left(\frac{m}{\alpha}P_1\right)^2 + \sigma_2^2\left(\frac{n}{\beta}P_2\right)^2 + \sigma_3^2 Z^2\right] \\ \leq {} & \frac{mr_1}{\alpha}P_1(\frac{m}{\alpha}P_1 + \frac{n}{\beta}P_2 + Z)^p - \frac{m^2 r_1}{\alpha^2 K_1}P_1^{p+2} + \frac{nr_2}{\beta}P_2(\frac{m}{\alpha}P_1 + \frac{n}{\beta}P_2 + Z)^p \\ & - \frac{n^2 r_2}{\beta^2 K_2}P_2^{p+2} - dZ^{p+1} + \frac{p}{2}(\frac{m}{\alpha}P_1 + \frac{n}{\beta}P_2 + Z)^{p+1}[\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2]. \end{aligned}$$

Noting that the inequality $|\sum_{i=1}^{n} c_i|^n \leq k^{n-1}\sum_{i=1}^{n}|c_i|^n$, we can obtain that

$$LV_1(P_1,P_2,Z) \leq -\frac{m^2r_1}{2\alpha^2K_1}P_1^{p+2} - \frac{n^2r_2}{2\beta^2K_2}P_2^{p+2} - \frac{d}{2}Z^{p+1} - \frac{d}{4}Z^{p+1}$$

$$+\frac{3^p p}{2}\left(P_1^{p+1}+P_2^{p+1}+Z^{p+1}\right)[\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2] - \frac{m^2r_1}{2\alpha^2K_1}P_1^{p+2}$$

$$-\frac{n^2r_2}{2\beta^2K_2}P_2^{p+2} - \frac{d}{4}Z^{p+1} + \frac{mr_1}{\alpha}P_1(\frac{m}{\alpha}P_1 + \frac{n}{\beta}P_2 + Z)^p$$

$$+\frac{nr_2}{\beta}P_2(\frac{m}{\alpha}P_1 + \frac{n}{\beta}P_2 + Z)^p$$

$$= -\frac{m^2r_1}{2\alpha^2K_1}P_1^{p+2} - \frac{n^2r_2}{2\beta^2K_2}P_2^{p+2} - \frac{d}{2}Z^{p+1} - \left(\frac{d}{4} - \frac{3^p p}{2}[\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2]\right)Z^{p+1}$$

$$+H \leq -\frac{m^2r_1}{2\alpha^2K_1}P_1^{p+2} - \frac{n^2r_2}{2\beta^2K_2}P_2^{p+2} - \frac{d}{2}Z^{p+1} + H,$$

where

$$H = \sup_{(P_1,P_2,Z)\in\mathbb{R}_+^3}\left\{-\frac{m^2r_1}{2\alpha^2K_1}P_1^{p+2} - \frac{n^2r_2}{2\beta^2K_2}P_2^{p+2} - \frac{d}{4}Z^{p+1}\right.$$

$$+\frac{mr_1}{\alpha}P_1(\frac{m}{\alpha}P_1 + \frac{n}{\beta}P_2 + Z)^p + \frac{nr_2}{\beta}P_2(\frac{m}{\alpha}P_1 + \frac{n}{\beta}P_2 + Z)^p$$

$$\left.+\frac{3^p p}{2}\left(P_1^{p+1}+P_2^{p+1}\right)[\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2]\right\} < \infty.$$

and

$$LV_2(P_1,P_2,Z) = -\frac{mK_1}{r_1}\ln P_1 - \frac{nK_2}{r_2}\ln P_2 - \ln Z$$

$$= -\left[\frac{mK_1}{r_1}A + \frac{nK_2}{r_2}B - \gamma - \delta - C\right] + a_2\frac{nK_2}{r_2}P_1 + a_1\frac{mK_1}{r_1}P_2$$

$$+\left(\alpha\frac{mK_1}{r_1} + \beta\frac{nK_2}{r_2}\right)Z$$

$$= -\Pi + a_2\frac{nK_2}{r_2}P_1 + a_1\frac{mK_1}{r_1}P_2 + \left(\alpha\frac{mK_1}{r_1} + \beta\frac{nK_2}{r_2}\right)Z,$$

$$LV_3(P_1,P_2,Z) = -mP_1Z - nP_2Z + \frac{\gamma P_1Z}{a+P_1} + \frac{\delta P_2Z}{b+P_2} + dZ \leq \frac{\gamma}{a}P_1Z + \frac{\delta}{b}P_2Z + dZ.$$

Thus,

$$LV(P_1,P_2,Z) \leq -\Pi M + \frac{na_2K_2M}{r_2}P_1 + \frac{ma_1K_1M}{r_1}P_2 + \left[\left(\frac{m\alpha K_1}{r_1} + \frac{n\beta K_2}{r_2}\right)M + d\right]Z$$
$$+\frac{\gamma}{a}P_1Z + \frac{\delta}{b}P_2Z - \frac{m^2r_1}{2\alpha^2K_1}P_1^{p+2} - \frac{n^2r_2}{2\beta^2K_2}P_2^{p+2} - \frac{d}{2}Z^{p+1} + H.$$

Considering the following compact subset E:

$$E = \left\{(P_1,P_2,Z) \in \mathbb{R}_+^3 : \varepsilon < P_1 < \frac{1}{\varepsilon}, \varepsilon < P_2 < \frac{1}{\varepsilon}, \varepsilon < Z < \frac{1}{\varepsilon}\right\}$$

where ε is a sufficiently small constant satisfying the following conditions:

$$-\Pi M + \left(\frac{na_2K_2M}{r_2} + \frac{\gamma}{a}\right)\varepsilon + S_1 \leq -1, \tag{4-13}$$

$$-\Pi M + S_2 - \frac{m^2r_1}{4\alpha^2K_1}\varepsilon^{-p-2} \leq -1, \tag{4-14}$$

$$-\Pi M + \left(\frac{ma_1K_1M}{r_1} + \frac{\delta}{b}\right)\varepsilon + S_3 \leq -1, \tag{4-15}$$

$$-\Pi M + S_4 - \frac{n^2r_2}{4\beta^2K_2}\varepsilon^{-p-2} \leq -1, \tag{4-16}$$

$$-\Pi M + \left(\left(\frac{\alpha m K_1}{r_1} + \frac{\beta n K_2}{r_2}\right)M + d + \frac{\gamma}{a} + \frac{\delta}{b}\right)\varepsilon + S_5 \leq -1, \tag{4-17}$$

$$-\Pi M + S_6 - \frac{d}{4}\varepsilon^{-p-1} \leq -1, \tag{4-18}$$

where

$$S_1 = \sup_{(P_1,P_2,Z)\in\mathbb{R}_+^3}\left\{\frac{ma_1K_1M}{r_1}P_2 + \frac{\delta}{2b}P_2^2 + \left[\left(\frac{\alpha mK_1}{r_1} + \frac{\beta nK_2}{r_2}\right)M + d\right]Z + \left(\frac{\gamma\varepsilon}{a} + \frac{\delta}{2b}\right)Z^2 + H\right\},$$

$$S_2 = \sup_{(P_1,P_2,Z)\in\mathbb{R}_+^3}\left\{\frac{na_2K_2M}{r_2}P_1 + \frac{\gamma}{2a}P_1^2 + \frac{ma_1K_1M}{r_1}P_2 + \frac{\delta}{2b}P_2^2 + \left[\left(\frac{\alpha mK_1}{r_1} + \frac{\beta nK_2}{r_2}\right)M + d\right]Z + \left(\frac{\gamma}{2a} + \frac{\delta}{2b}\right)Z^2 - \frac{m^2r_1}{4\alpha^2K_1}P_1^{p+2} - \frac{n^2r_2}{2\beta^2K_2}P_2^{p+2} - \frac{d}{2}Z^{p+1} + H\right\},$$

$$S_3 = \sup_{(P_1,P_2,Z)\in\mathbb{R}_+^3}\left\{\frac{na_2K_2M}{r_2}P_1 + \frac{\gamma}{2a}P_1^2 + \left[\left(\frac{\alpha mK_1}{r_1} + \frac{\beta nK_2}{r_2}\right)M + d\right]Z + \left(\frac{\gamma}{2a} + \frac{\delta}{b}\varepsilon\right)Z^2 + H\right\},$$

$$S_4 = \sup_{(P_1,P_2,Z)\in\mathbb{R}_+^3}\left\{\frac{na_2K_2M}{r_2}P_1 + \frac{\gamma}{2a}P_1^2 + \frac{ma_1K_1M}{r_1}P_2 + \frac{\delta}{2b}P_2^2 + \left[\left(\frac{\alpha mK_1}{r_1} + \frac{\beta nK_2}{r_2}\right)M + d\right]Z + \left(\frac{\gamma}{2a} + \frac{\delta}{2b}\right)Z^2 - \frac{m^2r_1}{2\alpha^2K_1}P_1^{p+2} - \frac{n^2r_2}{4\beta^2K_2}P_2^{p+2} - \frac{d}{2}Z^{p+1} + H\right\},$$

$$S_5 = \sup_{(P_1,P_2,Z)\in\mathbb{R}_+^3}\left\{\frac{na_2K_2M}{r_2}P_1 + \frac{\gamma\varepsilon}{a}P_1^2 + \frac{ma_1K_1M}{r_1}P_2 + \frac{\delta\varepsilon}{b}P_2^2 + H\right\},$$

$$S_6 = \sup_{(P_1,P_2,Z)\in\mathbb{R}_+^3}\left\{\frac{na_2K_2M}{r_2}P_1 + \frac{\gamma}{2a}P_1^2 + \frac{ma_1K_1M}{r_1}P_2 + + \frac{\delta}{2b}P_2^2 + \left[\left(\frac{\alpha mK_1}{r_1} + \frac{\beta nK_2}{r_2}\right)M + d\right]Z + \left(\frac{\gamma}{2a} + \frac{\delta}{2b}\right)Z^2 - \frac{d}{4}Z^{p+1} - \frac{m^2r_1}{2\alpha^2K_1}P_1^{p+2} - \frac{n^2r_2}{2\beta^2K_2}P_2^{p+2} + H\right\}.$$

Then

$$\mathbb{R}_+^3/E = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6$$

with

$$E_1 = \{(P_1,P_2,Z) \in \mathbb{R}_+^3 : 0 < P_1 \le \varepsilon\},\ E_2 = \left\{(P_1,P_2,Z) \in \mathbb{R}_+^3 : P_1 \ge \frac{1}{\varepsilon}\right\},$$

$$E_3 = \{(P_1,P_2,Z) \in \mathbb{R}_+^3 : 0 < P_2 \le \varepsilon\},\ E_4 = \left\{(P_1,P_2,Z) \in \mathbb{R}_+^3 : P_2 \ge \frac{1}{\varepsilon}\right\},$$

$$E_5 = \{(P_1,P_2,Z) \in \mathbb{R}_+^3 : 0 < Z \le \varepsilon\},\ E_6 = \left\{(P_1,P_2,Z) \in \mathbb{R}_+^3 : Z \ge \frac{1}{\varepsilon}\right\}.$$

Now, under the condition $\Pi = \frac{mK_1}{r_1}A + \frac{nK_2}{r_2}B - \gamma - \delta - C > 0$, we need to prove the negativity of $LV(P_1,P_2,Z)$ for any $(P_1,P_2,Z) \in \mathbb{R}_+^3/E$ in the following cases:

Case I If $(P_1,P_2,Z) \in E_1$, then $P_1Z \le \varepsilon Z \le \varepsilon(1 + Z^2)$, and one can obtain that

$$
\begin{aligned}
LV(P_1,P_2,Z) \le{} & -\Pi M + \frac{na_2K_2M}{r_2}P_1 + \frac{ma_1K_1M}{r_1}P_2 + \left[\left(\frac{\alpha mK_1}{r_1} + \frac{\beta nK_2}{r_2}\right)M + d\right]Z \\
& + \frac{\gamma}{a}P_1Z + \frac{\delta}{b}P_2Z - \frac{m^2r_1}{2\alpha^2K_1}P_1^{p+2} - \frac{n^2r_2}{2\beta^2K_2}P_2^{p+2} - \frac{d}{2}Z^{p+1} + H \\
& \le -\Pi M + \frac{na_2K_2M}{r_2}\varepsilon + \frac{ma_1K_1M}{r_1}P_2 + \left[\left(\frac{\alpha mK_1}{r_1} + \frac{\beta nK_2}{r_2}\right)M + d\right]Z \\
& + \frac{\gamma}{a}\varepsilon(1+Z^2) + \frac{\delta}{b}\frac{P_2^2 + Z^2}{2} - \frac{m^2r_1}{2\alpha^2K_1}P_1^{p+2} - \frac{n^2r_2}{2\beta^2K_2}P_2^{p+2} - \frac{d}{2}Z^{p+1} \\
& + H \le -\Pi M + \left(\frac{na_2K_2M}{r_2} + \frac{\gamma}{a}\right)\varepsilon + S_1 \le -1.
\end{aligned}
$$

By (4-13), we have $LV(P_1,P_2,Z) \le -1$ for any $(P_1,P_2,Z) \in E_1$.

Case II If $(P_1,P_2,Z) \in E_2$, we have

$$
\begin{aligned}
LV(P_1,P_2,Z) \le{} & -\Pi M + \frac{na_2K_2M}{r_2}P_1 + \frac{ma_1K_1M}{r_1}P_2 + \left[\left(\frac{\alpha mK_1}{r_1} + \frac{\beta nK_2}{r_2}\right)M + d\right]Z \\
& + \frac{\gamma}{a}P_1Z + \frac{\delta}{b}P_2Z - \frac{m^2r_1}{2\alpha^2K_1}P_1^{p+2} - \frac{n^2r_2}{2\beta^2K_2}P_2^{p+2} - \frac{d}{2}Z^{p+1} + H \\
& \le -\Pi M + \frac{na_2K_2M}{r_2}P_1 + \frac{ma_1K_1M}{r_1}P_2 + \left[\left(\frac{\alpha mK_1}{r_1} + \frac{\beta nK_2}{r_2}\right)M + d\right]Z \\
& + \frac{\gamma}{a}\frac{P_1^2 + Z^2}{2} + \frac{\delta}{b}\frac{P_2^2 + Z^2}{2} - \frac{m^2r_1}{4\alpha^2K_1}P_1^{p+2} - \frac{m^2r_1}{4\alpha^2K_1}P_1^{p+2} \\
& - \frac{n^2r_2}{2\beta^2K_2}P_2^{p+2} - \frac{d}{2}Z^{p+1} + H \le -\Pi M + S_2 - \frac{m^2r_1}{4\alpha^2K_1}\varepsilon^{-p-2} \le -1,
\end{aligned}
$$

By (4-14), we have $LV(P_1,P_2,Z) \le -1$ for any $(P_1,P_2,Z) \in E_2$.

Case III If $(P_1,P_2,Z) \in E_3$, the similar analysis to case I, we can obtain that

$$
LV(P_1,P_2,Z) \le -\Pi M + \left(\frac{ma_1K_1M}{r_1} + \frac{\delta}{b}\right)\varepsilon + S_3 \le -1.
$$

By (4-15), we have $LV(P_1,P_2,Z) \le -1$ for any $(P_1,P_2,Z) \in E_3$.

Case IV If $(P_1,P_2,Z) \in E_4$, the similar analysis to case II, we have

$$
LV(P_1,P_2,Z) \le -\Pi M + S_4 - \frac{n^2r_2}{4\beta^2K_2}\varepsilon^{-p-2} \le -1,
$$

which follows from (4-16), we have $LV(P_1,P_2,Z) \le -1$ for any $(P_1,P_2,Z) \in E_4$.

Case V If $(P_1,P_2,Z) \in E_5$, then $P_1Z \le \varepsilon P_1 \le \varepsilon(1 + P_1^2), P_2Z \le \varepsilon P_2 \le \varepsilon(1 + P_2^2)$, we have

$$
\begin{aligned}
LV(P_1,P_2,Z) \le & -\varPi M + \frac{na_2K_2M}{r_2}P_1 + \frac{ma_1K_1M}{r_1}P_2 + \left[\left(\frac{\alpha mK_1}{r_1} + \frac{\beta nK_2}{r_2}\right)M + d\right]Z \\
& + \frac{\gamma}{a}P_1Z + \frac{\delta}{b}P_2Z - \frac{m^2r_1}{2\alpha^2K_1}P_1^{p+2} - \frac{n^2r_2}{2\beta^2K_2}P_2^{p+2} - \frac{d}{2}Z^{p+1} + H \\
& \le -\varPi M + \frac{na_2K_2M}{r_2}P_1 + \frac{ma_1K_1M}{r_1}P_2 + \left[\left(\frac{\alpha mK_1}{r_1} + \frac{\beta nK_2}{r_2}\right)M + d\right]\varepsilon \\
& + \frac{\gamma}{a}\varepsilon(1+P_1^2) + \frac{\delta}{b}\varepsilon(1+P_2^2) - \frac{m^2r_1}{2\alpha^2K_1}P_1^{p+2} - \frac{n^2r_2}{2\beta^2K_2}P_2^{p+2} - \frac{d}{2}Z^{p+1} \\
& + H \le -\varPi M + \left(\left(\frac{\alpha mK_1}{r_1} + \frac{\beta nK_2}{r_2}\right)M + d + \frac{\gamma}{a} + \frac{\delta}{b}\right)\varepsilon + S_5 \le -1.
\end{aligned}
$$

According to (4-17), we have $LV(P_1,P_2,Z) \le -1$ for any $(P_1,P_2,Z) \in E_5$.

Case VI If $(P_1,P_2,Z) \in E_6$, we have

$$
\begin{aligned}
LV(P_1,P_2,Z) \le & -\varPi M + \frac{na_2K_2M}{r_2}P_1 + \frac{ma_1K_1M}{r_1}P_2 + \left[\left(\frac{\alpha mK_1}{r_1} + \frac{\beta nK_2}{r_2}\right)M + d\right]Z \\
& + \frac{\gamma}{a}P_1Z + \frac{\delta}{b}P_2Z - \frac{m^2r_1}{2\alpha^2K_1}P_1^{p+2} - \frac{n^2r_2}{2\beta^2K_2}P_2^{p+2} - \frac{d}{2}Z^{p+1} + H \\
& \le -\varPi M + \frac{na_2K_2M}{r_2}P_1 + \frac{ma_1K_1M}{r_1}P_2 + \left[\left(\frac{\alpha mK_1}{r_1} + \frac{\beta nK_2}{r_2}\right)M + d\right]Z \\
& + \frac{\gamma}{a}\frac{P_1^2+Z^2}{2} + \frac{\delta}{b}\frac{P_2^2+Z^2}{2} - \frac{d}{4}Z^{p+1} - \frac{m^2r_1}{2\alpha^2K_1}P_1^{p+2} - \frac{n^2r_2}{2\beta^2K_2}P_2^{p+2} \\
& - \frac{d}{4}Z^{p+1} + H \le -\varPi M + S_6 - \frac{d}{4}\varepsilon^{-p-1} \le -1.
\end{aligned}
$$

From (4-18), we have $LV(P_1,P_2,Z) \le -1$ for any $(P_1,P_2,Z) \in E_6$.

Hence, the condition (ii) of Lemma 4.2.1.1 is verified. Thus, for any given initial value $\big(P_1(0),P_2(0),Z(0)\big) \in \mathbb{R}_+^3$, we can obtain that, if $\varPi > 0$, the system (4-2) admits a unique ergodic stationary distribution.

Note: When the environmental fluctuations are not particularly large, there exists a unique ergodic stationary distribution, implying that the biomass of phytoplankton and zooplankton can be persistent and coexistence in the long term. Additionally, it also shows that the adaptability of plankton to environmental fluctuations is limited.

4.3. Experimental simulations

In the previous subsections, we have analyzed the effects of environmental fluctuations and TPP on the dynamics of plankton ecosystem and verified the intensity of white noise and the liberation rate produced by TPP can lead to the extinction, persistence in the mean and even the stationary distribution of plankton populations by modeling analysis. In order to further study how the white noise and TPP affect the dynamics mechanisms of the formation and evolution process of planktonic blooms, we perform some experimental simulations for system (4-2) based on the Milstein's Higher Order Method (Higham, 2001) in this section. In the following experimental simulations, unless otherwise specified, we take the initial value as $\left(P_1(0), P_2(0), Z(0)\right) = (0.5,0.5,0.5)$, and the parameter values are always used in Table 4.1 and other parameters are chosen as control parameters.

4.3.1 Impact of white noise on the dynamics of system (4-2)

Due to in a phytoplankton-zooplankton system that takes toxic phytoplankton into consideration, the intrinsic growth rates of two toxic phytoplankton and the death rate of zooplankton are parameters that are most susceptible to environmental influences and are relatively important. Therefore, we only consider the intrinsic growth rates and death rate affected by white noise. In order to study how the white noise and two TPP affect the dynamics of system (4-2), we firstly consider the system does not experience the white noise, that is, system (4-2) becomes its corresponding deterministic system. According to the work of Sarkar et al. (2005), we can obtain that the system (4-1) possesses a unique positive interior equilibrium $E^*(0.8178,1.8066,2.1071)$ which is locally asymptotically stable, depicting the coexistence of all three species at a stable state and indicating the changes in the biomass of plankton populations are static.

Table 4.1 Parameter values

Parameters	Biological meaning	Unit	Value	Sources

$\boldsymbol{r_1}$	Growth rate of one TPP	day^{-1}	0.55	Sarkar et al. (2005)
$\boldsymbol{r_2}$	Growth rate of other TPP	day^{-1}	0.5	Sarkar et al. (2005)
$\boldsymbol{a_1}$	Inhibitory effect	ml/nos.	0.005	Sarkar et al. (2005)
$\boldsymbol{a_2}$	Inhibitory effect	ml/nos.	0.004	Sarkar et al. (2005)
$\boldsymbol{\alpha}$	Zooplankton ingestion rate	ml/nos.	0.15	Sarkar et al. (2005)
$\boldsymbol{\beta}$	Zooplankton ingestion rate	ml/nos.	0.15	Sarkar et al. (2005)
$\boldsymbol{m}$	the conversion of one TPP into zooplankton	ml/nos. /day	0.09	Sarkar et al. (2005)
$\boldsymbol{n}$	the conversion of other TPP into zooplankton	ml/nos. /day	0.075	Sarkar et al. (2005)
$\boldsymbol{d}$	Death rate of zooplankton	day^{-1}	0.09	Sarkar et al. (2005)
$\boldsymbol{a}$	Half-saturation constant	ml/nos.	0.1	Sarkar et al. (2005)
$\boldsymbol{b}$	Half-saturation constant	ml/nos.	0.12	Sarkar et al. (2005)
$\boldsymbol{K_1}$	Environmental capacity of one TPP	nos./ml	2	Estimated
$\boldsymbol{K_2}$	Environmental capacity of other TPP	nos./ml	5	Estimated

Note: the estimated values in the table indicate that the parameter has been self-estimated in this thesis or has been selected as a control parameter in experimental simulations.

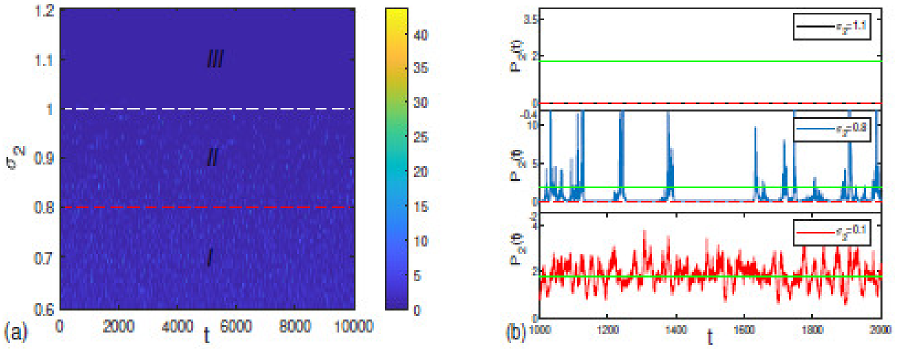

Fig. 4.1 The effect of σ_2 on the stochastic dynamical behaviors of the system (4-2) with $\sigma_1 = 0.1, \sigma_3 = 0.1, \gamma = 0.06, \delta = 0.07, 0 \leq \sigma_2 \leq 1.2$. **(a)** The dynamical behaviors of species $P_2(t)$ change from persistence in the mean to extinction in different areas of I, II and III for $0 \leq \sigma_2 \leq 1.2$. **(b)** The same path of species $P_2(t)$ for system (4-2) with respect to Fig. 4.1(a) for $\sigma_2 = 0.1, \sigma_2 = 0.8, \sigma_2 = 1$ and t on $[1000{,}2000]$ and its corresponding deterministic system (4-1).

Next, we explore the impact of the white noise on the stochastic dynamics of system (4-2). We first fix $\sigma_1 = 0.1$ and $\sigma_3 = 0.1$, and let $\sigma_2 (0 \leq \sigma_2 \leq 1.2)$ vary to see how the white noise influences the survival of plankton populations. According to the condition of $B = r_2 - \frac{\sigma_2^2}{2}$, we can obtain that all three species of system (4-2) will undergo extinction when the white noise reaches some a critical value. Obviously, we can find from Fig. 4.1 (a) that the species $P_2(t)$ of system (4-2) is always persistence in the mean in the area of I and persistence in the mean or extinct alternating in the area of II, but species $P_2(t)$ dies out rapidly in the space III when σ_2 is beyond $\sigma_2 = 1$. Fig. 4.1 (b) depicts that the stochastic dynamical behaviors of species $P_2(t)$ with respect to Fig. 4.1 (a) for $\sigma_2 = 0.1, \sigma_2 = 0.8, \sigma_2 = 1.1$ and t on $[1000{,}2000]$. Moreover, we can observe from Fig. 4.1 (b) that, with the increase in the magnitude of the environmental fluctuations, the random variation of plankton density becomes more significant, which implies that white noise can accelerate the stochastic oscillation of plankton density. For example, let $\sigma_2 = 0.1$, it is not difficult to find that the two TPP and zooplankton of system (4-2) can coexist at a relatively stable state and their densities exhibit oscillation around the deterministic steady state values

$P_1^* = 0.8178, P_2^* = 1.8066$, and $Z^* = 2.1071$, respectively (see Fig. 4.2 (a), (b) and (c)). Actually, following the $\theta_1 > 0$, $\theta_2 > 0$, and $\Xi > 0$, then system (4-2) is persistence in the mean and has a unique ergodic stationary distribution, which is consistent with our experimental simulations. From the stationary distribution of all three species, it can be seen clearly that they are distributed normally around the values 0.8, 1.8 and 2.1, respectively, which illustrates that the standard deviation $\sigma_{1,}\sigma_2$ and σ_3 can keep processes $P_1(t)$, $P_2(t)$ and $Z(t)$ moving around the solution of deterministic system (4-1). In other words, system (4-2) can preserve some stability in the random sense when the intensities of white noise are relatively weak. Now let σ_2 vary within some a level, it can be concluded that the weaker the environmental fluctuations are, the closer the solutions of system (4-2) are to steady state E^* (The Figures here are not given due to the similarity to Fig. 4.2). However, when we increase the density of environmental forcing $\sigma_2 > 1$, $\sigma_2 = 1.1$, for example, we can easily to get that $B = -0.105 < 0$, which implies that the species $P_2(t)$ tends to go rapid extinct, even if its corresponding deterministic system (4-1) still presents obvious stability, indicating a different phenomenon from its deterministic system (see Fig. 4.3). This also shows that white noise intensity can help to control the density of toxic phytoplankton. Comparing Figs. 4.1, 4.2 and 4.3, it is obvious to find that the intensity of white noise cannot only aggravate the stochastic oscillation of plankton density, but also significantly change the dynamics of the plankton system. That is, a high-level white noise intensity can accelerate the extinction of the plankton populations, which implies that the white noise can help control the biomass of plankton populations and may provide a guide for us to the termination of planktonic blooms. This is consistent with the results obtained by the work of Sarkar and Chattopadhayay (2003), who demonstrated the controlling of planktonic blooms by artificial eutrophication or the intensity of white noise from their experimental and field observations. Thus, it is worth pointing out that the results from the Figs. 4.1, 4.2 and 4.3 can support that the plankton systems incorporating white noise can better simulate planktonic blooms than its corresponding deterministic counterparts.

Similarly, if we impose the intensities of white noise on species $P_1(t)$ and $Z(t)$, respectively, we can easily obtain the similar results, thus here we omit it.

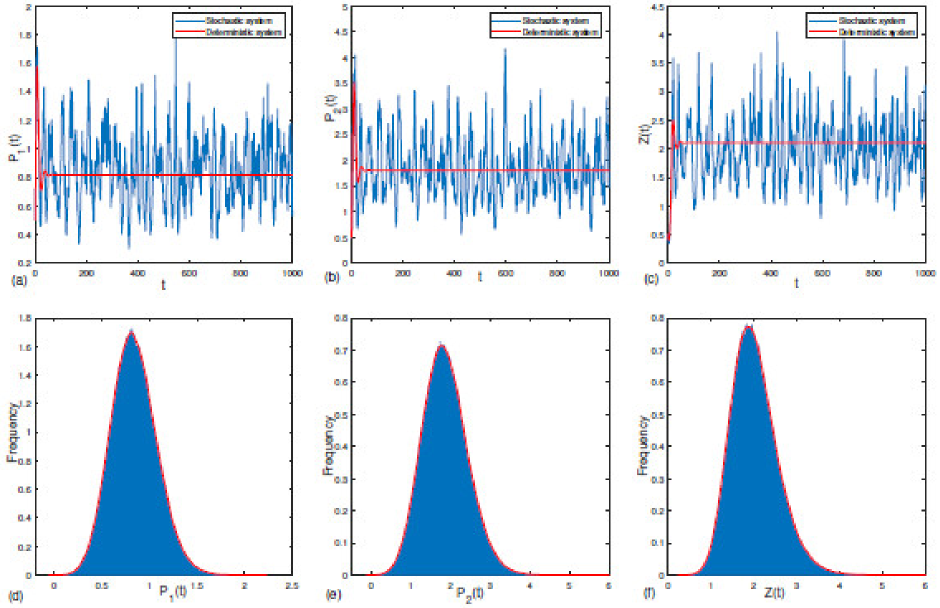

Fig. 4.2 (a), (b), (c) the solution trajectories of system (4-2) and its corresponding deterministic system (4-1). **(d), (e), (f)** the probability density function diagrams of $P_1(t), P_2(t)$ and $Z(t)$ for the system (4-2) with $\sigma_1 = \sigma_2 = \sigma_3 = 0.1,\ \gamma = 0.06, \delta = 0.07$, and the red smoothed carves are probability density functions for system (4-2).

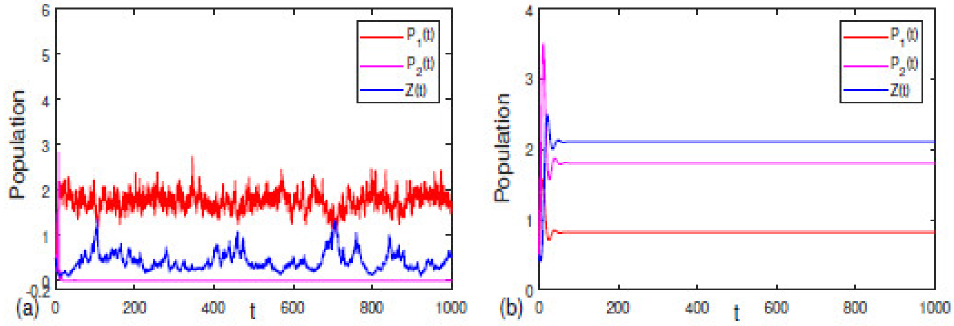

Fig. 4.3 Stochastic dynamical behaviors of system (4-2) with $\sigma_1 = \sigma_3 = 0.1,\ \gamma = 0.06, \delta = 0.07, \sigma_2 = 1.1$, and its corresponding deterministic counterparts on $t \in [0,1000]$. **(a)** The persistence in the mean of species $P_1(t)$ and $Z(t)$ and extinction of species $P_2(t)$ of stochastic system (4-2). **(b)** The persistence of deterministic system (4-1).

4.3.2 Impact of TPP on the dynamics of system (4-2)

In order to study how the effect of one toxin liberation rate on population density dynamic evolution trend under the environmental fluctuations, we choose γ as a control parameter and all other parameters are the same as Fig. 4.2. Clearly, we can observe from Fig. 4.4 that the species $P_1(t)$ and $P_2(t)$ are persistence in the mean and their biomass will increase as the increasing value of γ, while the species $Z(t)$ undergoes extinction when γ beyond a certain value, here the color-bars denote the biomass of species $P_1(t)$, $P_2(t)$ and $Z(t)$, respectively. Actually, it is easy to obtain that $\Pi > 0$ under the condition of $0 < \gamma < 0.3846$ and $\Pi = 0$ if and only if $\gamma \approx 0.3846$, which indicates that system (4-2) is persistence in the mean and has the stationary distribution under the condition of $\gamma < 0.3846$, in contrast, species $Z(t)$ of system (4-2) will die out. Therefore, it can be asserted that TPP can significantly affect the coexistence of all the three species. For precisely, we take three different values of γ ($\gamma = 0.1, 0.2, 0.35$), then system (4-2) has a unique stationary distribution. Fig. 4.5 depicts the relative frequency density of $P_1(t)$, $P_2(t)$ and $Z(t)$ with these different values, respectively, where the smoothed curves are the probability density functions of system (4-2). More importantly, we can obtain the result from the Fig. 4.5 that with the increasing value of γ, the distributions of two

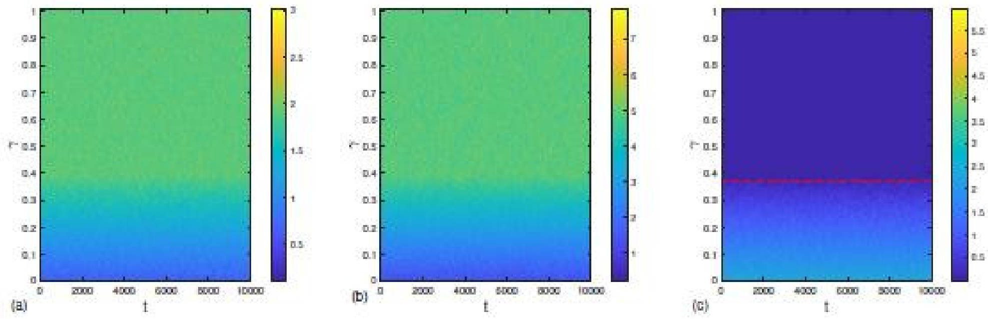

Fig. 4.4 The effect of toxin rate γ produced by population $P_1(t)$ on the stochastic dynamic behaviors of the system (4-2) with $\sigma_1 = \sigma_2 = \sigma_3 = 0.1$, $0 \le \gamma \le 1$. **(a)-(b)** The persistence in the mean of species $P_1(t)$ and $P_2(t)$; **(c)** The persistence in the mean of population $Z(t)$ for $0 \le \gamma \le 0.3846$ and extinction for $0.3846 \le \gamma \le 1$.

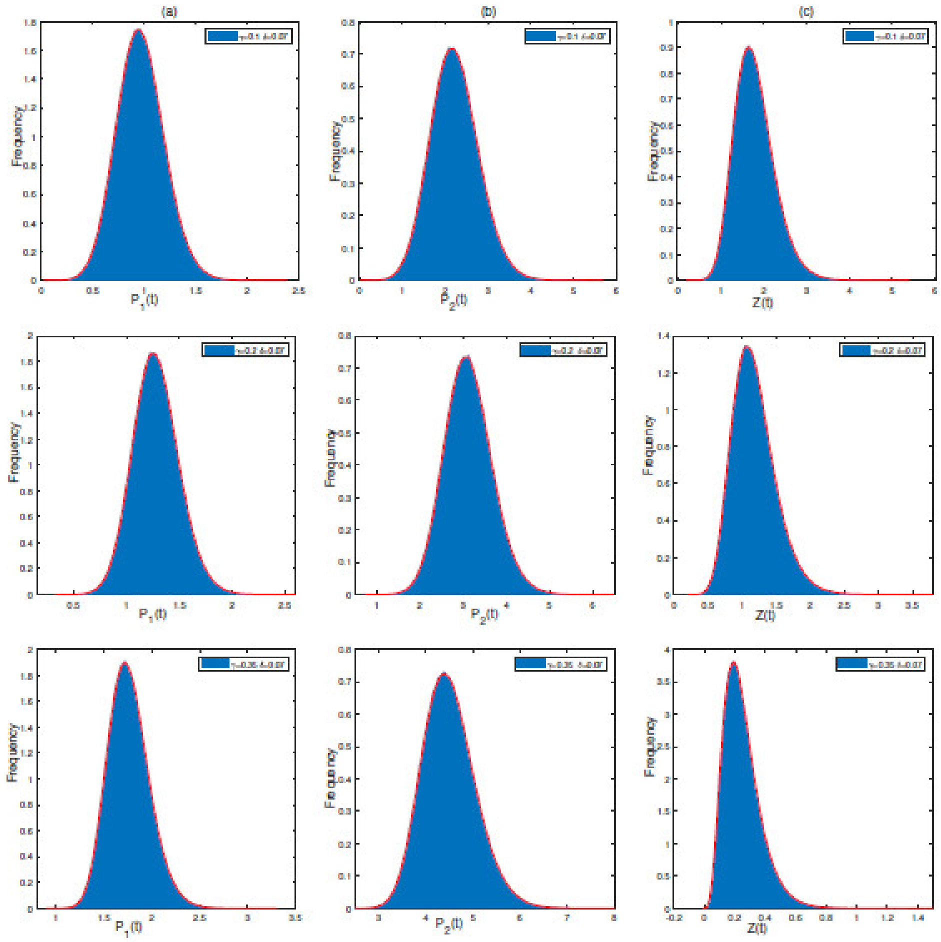

Fig. 4.5 The effect of one toxin rate γ on the dynamics of system (4-2). **(a), (b), (c)** The histograms of probability density functions for $P_1(t), P_2(t)$ and $Z(t)$ with different values of γ. Here, $\delta = 0.07$, and $\sigma_1 = \sigma_2 = \sigma_3 = 0.1$.

TPP appear closer to the normal distribution, but the distribution of zooplankton becomes more skewness, which implies that the increase of the liberation rate γ can increase the survival chance of two harmful phytoplankton but decrease the biomass of zooplankton. Additionally, it can be seen from Fig. 4.5 that the peak value of the probability density functions of system (4-2) will be higher as γ increases. All these results indicate that for the value of the toxin liberation rate γ satisfying the conditions of $\Pi > 0$, its enhancement will contribute to the persistence in the mean of system (4-2) though the termination of planktonic blooms. In addition, from the theoretical analysis, we want to know what happens if $\Pi < 0$? Selecting $\gamma = 0.39$, we can easily check $\Pi \approx -0.0054 < 0$ and $\Gamma \approx -0.0194 < 0$. Although the conditions of $\Gamma > 0$ and $\Pi > 0$ are not satisfied, the system (4-2) has a stationary distribution, indicating all the three species are persistence in the mean (see Fig. 4.6). However, when we choose $\gamma = 0.42$, similarly, we can obtain that $\Pi \approx -0.0354 < 0$ and $\Gamma \approx -0.0494 < 0$. Obviously, the two toxic phytoplankton can coexist while the zooplankton tends to go extinct (see Fig. 4.7). For the case of $\gamma = 0.06$ and changing the value of δ, we can easily get the similar results, which are omitted here.

In additionally, the combined effects of two toxin liberation rates on the dynamics of system (4-2) are studied as well. Fig. 4.8 depicts how the combined role of γ and δ affect the dynamics of system (4-2), where the red smoothed curves are probability density functions of system (4-2). By a straightforward computation, the condition of $\Pi > 0$ can be verified, which means system (4-2) has a unique stationary distribution (see Fig. 4.8). Comparing Figs. 4.5 and 4.8, one can see that the mean values of two harmful phytoplankton populations are larger than the case of $\gamma = 0.2, \delta = 0.07$, while that of zooplankton is smaller than that case. Thus, we can obtain that by controlling any one of the harmful phytoplankton, the mean values of both the harmful phytoplankton are smaller than the value observed when considering in the case of both two harmful phytoplankton populations, while that of zooplankton is larger than the case of both harmful phytoplankton are present. Therefore, the introduction of two harmful phytoplankton can be contribute to persistence of the system (4-2) and play an important role in the termination of planktonic blooms.

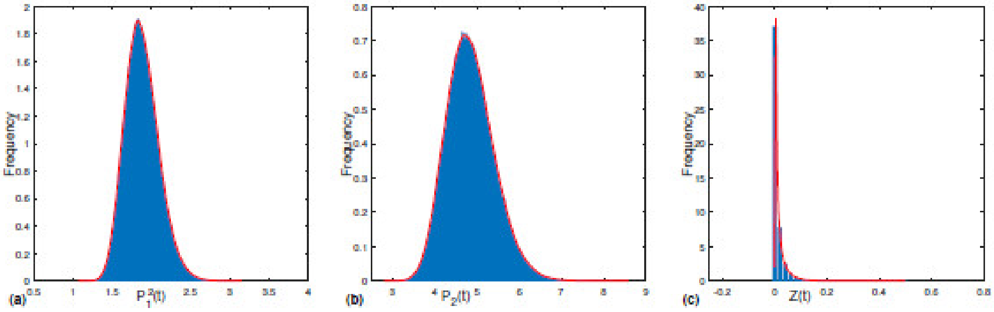

Fig. 4.6 The effect of one toxin rate γ on the dynamics of system (4-2). **(a), (b), (c)** The histograms of probability density functions for $P_1(t), P_2(t)$ and $Z(t)$ with different values of γ.

Here, $\gamma = 0.39$, $\delta = 0.07$, and $\sigma_1 = \sigma_2 = \sigma_3 = 0.1$.

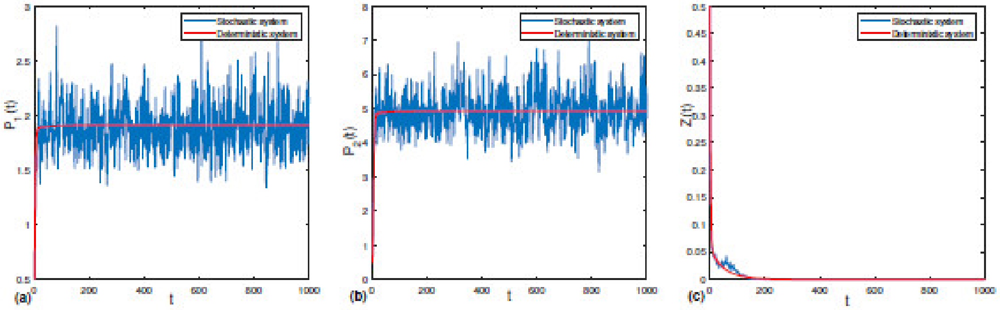

Fig. 4.7 Stochastic dynamical behaviors of system (4-2) and its corresponding deterministic counterparts on $t \in [0,1000]$. Here, $\gamma = 0.42, \sigma_1 = \sigma_2 = \sigma_3 = 0.1$, $\delta = 0.07$.

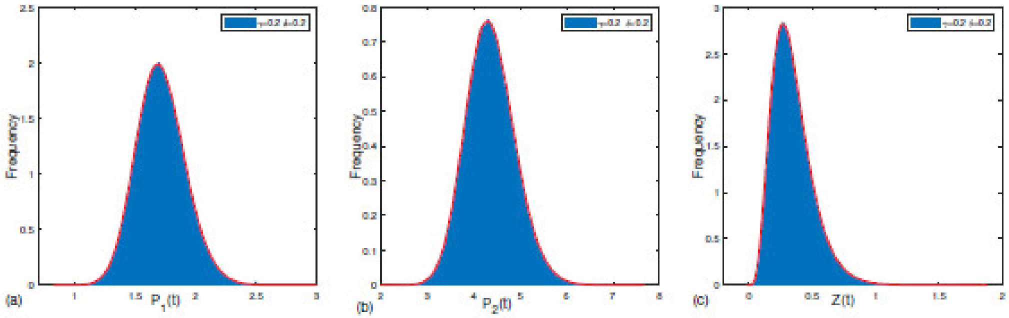

Fig. 4.8 The combined effects of two toxin rates γ and δ on the dynamics of system (4-2). **(a), (b), (c)** The histograms of probability density functions for $P_1(t), P_2(t)$ and $Z(t)$ with $\gamma = 0.2, \delta = 0.2, \sigma_1 = \sigma_2 = \sigma_3 = 0.1$, respectively.

4.4. Conclusions and discussion

It is now well recognized that stochastic population dynamics play a significant role in population dynamics, since environmental fluctuations can affect the growth process of species, such as the growth rate and death rate, which can be described by white noise (May, 1973). And the Gaussian white noise can be theoretically preferred to model environmental fluctuations because of its irregularity and thus a good approximation to the phenomena of rapid fluctuations (Jonsson and Wennergren, 2019). The study of stochastic population dynamics goes back to the pioneering work by Haminskii (1980), who introduced two white noise to stabilize a linear system. After that, lots of attention has been paid on stochastic population dynamics studies

(Mao et al., 2002; Wang et al., 2020; Lee et al., 2020). Mao et al. (2002) pointed out that stochastic noise can suppress potential explosion in population dynamics. Wang et al. (2020) showed that time-periodic forcing can lead to the transitions from a spatially homogeneous stationary state to a periodic oscillation in time. Additionally, lots of stochastic plankton growth systems have been derived by numerous researchers (Zhao et al., 2020; Wang and Liu, 2020; Zhao et al., 2016), and stochastic plankton systems involving toxin-producing phytoplankton have become a hot topic in ecological studies due to harmful phytoplankton can significantly affect the dynamics of plankton systems (Scotti et al., 2015; Banerjee and Venturino, 2011; Khare et al., 2010; Chattopadhyay et al., 2002; Chattopadhyay et al., 2002; Sarkar and Chattopadhyay, 2003a, 2003b; Chattopadhyay, 2004; Sarkar et al., 2005; Yu et al., 2019).

In this thesis, therefore, we first propose a stochastic phytoplankton-zooplankton system with two harmful phytoplankton populations, where the intrinsic growth rates of two harmful phytoplankton populations and the natural death rate of zooplankton are influenced by the environmental noise, and then we study the effects of TPP and white noise on the dynamics of system (4-2) theoretically and numerically. In order to ensure that the system is biologically meaningful, the existence and uniqueness of global positive solutions of system is discussed, and the results demonstrate that for any initial value $\left(P_1(0), P_2(0), Z(0)\right) \in \mathbb{R}_+^3$, the solution will remain in $\mathbb{R}_+^3$ with probability one. Based on this situation, we derive some sufficient conditions for the extinction and persistence in the mean of the system. Obviously, those conditions are great significance to study the extinction and persistence in the mean for the phytoplankton-zooplankton system (Yu et al., 2019; Sarkar and Chattopadhayay, 2003). Significantly, when the system is persistence in the mean, we also investigate the existence and uniqueness of positive recurrent of solution for the system, which implies that the system has a unique stationary distribution under some conditions. Numerical analysis illustrates our theoretical results and further indicates that both two TPP and environmental fluctuations have a significant effect on the controlling of planktonic blooms.

On the one hand, from our dynamical analysis, which follows that, when the low level intensity of white noise satisfies the conditions $\theta_1 > 0, \theta_2 > 0$ and $\Pi > 0$, system (4-2) is persistence in the mean and exists a stationary distribution which is ergodic, indicating the coexistence of all those three species in the random sense for a long time (see Fig. 4.2). However, when we increase the intensity of environmental forcing satisfying the condition of $\theta_2 < 0$ holds, the harmful phytoplankton $P_2(t)$ undergoes extinction although other two species are persistence in the mean, as it is shown in Fig. 4.3. Comparing Figs. 4.2 and 4.3, it can be asserted that white noise can aggravate the emergence of stochastic oscillation and significantly change the dynamics of phytoplankton-zooplankton system. Especially, the strong intensity of white noise can accelerate the extinction of the plankton populations. Consequently, these results may be more realistic than that of in (Sarkar et al., 2005), which implies that the controlling of random environmental fluctuations may be a good way in the termination of planktonic blooms. Therefore, it is great ecological significance to consider environmental noise when studying phytoplankton-zooplankton interaction in the presence of harmful phytoplankton.

On the other hand, it is investigated how the dynamics of system (4-2) strongly depends on TPP. By controlling one toxin liberation rate, the dynamic behaviors of system (4-2) can be changed. That is, when the toxin liberation rate is beyond some a critical value, two harmful phytoplankton can coexist, while zooplankton tends to extinction (see Fig. 4.4). Moreover, when controlling any one of the two TPP, it is obvious to survey from Fig. 4.5 that the increasing value of one toxin liberation rate can reduce the biomass of zooplankton, while increase the survival chance of two phytoplankton populations. In addition, in the presence of both two TPP, it can be seen from Figs. 4.5 and 4.8 that the combined effects of two liberation rates on the changes in plankton populations are stronger than that of controlling any one of the two TPP. Thus, the introduction of two harmful phytoplankton populations is conducive to the persistence of the system (4-2) through the termination of planktonic blooms. Therefore, TPP has a profound impact on the dynamics of phytoplankton-zooplankton systems and may be used as a biological way to control planktonic blooms.

There are some interesting topics waiting for us to further explore. For example, the zooplankton mortality will occur after some time lapse due to the bloom of toxic phytoplankton (Chattopadhyay et al., 2002), it seems to more reasonable to study a stochastic toxic-producing phytoplankton-zooplankton system with time delay. Another problem of interest is to consider impulsive perturbations into the system. We leave those for our future research goals.

Chapter 5 DYNAMICS OF A STOCHASTIC PHYTOPLANKTON –TOXIC PHYTOPLANKTON-ZOOPLANKTON SYSTEM UNDER REGIME SWITCHING

Abstract

In this thesis, a stochastic phytoplankton-toxic phytoplankton-zooplankton system with Beddington-DeAngelis functional response, where both the white noise and regime switching are taken into account, is studied analytically and numerically. The aim of this research is to study the combined effects of the white noise, regime switching and toxin-producing phytoplankton (TPP) on the dynamics of the system. Firstly, the existence and uniqueness of global positive solution of the system is investigated. Then some sufficient conditions for the extinction, persistence in the mean and the existence of a unique ergodic stationary distribution of the system are derived. Significantly, some numerical simulations are carried to verify our analytical results, and show that a high-level intensity of white noise is harmful to the survival of plankton populations, but regime switching can balance the different survival states of plankton populations and thus decrease the risk of extinction. Additionally, it is found that an increase in the toxin liberation rate produced by TPP will increase the survival chance of phytoplankton, while it will reduce the biomass of zooplankton. All these results may provide some insightful understanding on the dynamics of phytoplankton-zooplankton systems in randomly disturbed aquatic environments.

Keyword: Stochastic phytoplankton-toxic phytoplankton-zooplankton system, White noise, Regime switching, Extinction, Stationary distribution

5.1. Introduction

Plankton are the basis of the freshwater and seawater food chains, and their importance for the wealth of aquatic ecosystems and ultimately for the planet itself is nowadays widely recognized (Huppert et al., 2002). Phytoplankton, particularly, can create energy for the aquatic life through photosynthesis and produce large amounts of oxygen by absorbing carbon dioxide from their surroundings. However, the rapid growth of phytoplankton can cause large-scale blooms in one area and the occurrences of harmful phytoplankton blooms have been reported globally with an increasing frequency in the past decades (Hallegraeff, 1993), which are detrimental to the public health, fisheries, tourisms as well as the balance of ecosystems (Anderson et al., 2000). For example, some freshwater lakes in China, such as Lake Taihu, Lake Poyanghu, Lake Chaohu, etc., have suffered varying degrees of toxic cyanobacterial blooms in recent years. In 2011, Lake Erie experienced the largest harmful algal blooms (HABs) in its recorded history, with a peak intensity over three times greater than any previously observed bloom (Michalak et al., 2013). Based on the huge effects of planktonic blooms and the mechanisms behind them are not yet clearly understood, it is necessary and important to understand the dynamics mechanisms of changes in plankton populations.

Many researchers have attempted different approaches to explain the dynamics mechanisms of planktonic blooms in the past decades. The results from experiments suggested that the toxic or noxious chemicals produced by blue-green algae may reduce the grazing pressure of zooplankton population and even cause their mortality for a long time (Fulton and Paerl, 1987; Lampert, 1981), which could be one of the key parameters for planktonic blooms (Krik and Gilbert, 1992). Some experimental evidence demonstrated that the grazing pressure by micro-zooplankton can represent an important factor for the controlling and regulation of HABs (Calbet et al., 2003; Johansson and Coats, 2002). In addition, there is an experiment revealing that under some suitable conditions, the formation of Microcystis blooms is closely related to the presence or absence of zooplankton population and to its selective grazing of the naturally occurring zooplankton (Wang et al., 2010). Another approach the researchers are trying to explain the bloom phenomenon is the role of toxicity. The result that toxicity may be as a strong mediator in the zooplankton feeding rate is found in a field observation (Estep et al., 1990), as well as a laboratory experiment (Huntley et al., 1986). Moreover, their experimental findings and field study revealed that TPP can suppress the grazing pressure of zooplankton and may act as a biological control way for the termination of planktonic blooms (Chattopadhyay et al., 2004; Chattopadhyay et al., 2002). All these results imply that toxin production plays a significant role in the interaction between phytoplankton and zooplankton populations, which may greatly stimulate researchers to explore the way these toxin production affect the coexistence and survival prospect of plankton populations in the presence of non-toxic and toxic phytoplankton.

Due to the complexity and openness of real aquatic ecosystems, establishing mathematical models is now a classical way to study the planktonic blooms (Truscott and Brindley, 1994), which can provide quantitative insights into the dynamic mechanisms of changes in plankton populations. In recent years, many deterministic mathematical models for plankton dynamics, such as delayed nutrient-phytoplankton models (Dai et al., 2016; Guo et al., 2019), a diffusive nutrient-toxic phytoplankton model (Chakraborty et al., 2015), viral infection nutrient-phytoplankton models (Li and Gao, 2016; Chattopadhyay et al., 2003), a phytoplankton-toxin producing phytoplankton-zooplankton model (Chattopadhyay et al., 2004), and so on, have been developed and studied extensively, and many interesting results have been shown. However, plankton populations in the real aquatic environments often fluctuate unpredictably because of the unpredictability of environmental stochasticity, and these deterministic models do not capture random environmental fluctuations which is an important feature of aquatic ecosystems. In fact, some experiments shown that environmental noise has a significantly effect on population systems in ecology (Richardson and Heilmann, 1995; Carpenter et al., 2011). For example, the work of Reichwaldt et al. (2013) demonstrated that the wind can be the most likely driver to control the biomass of cyanobacteria. In addition, the growth rate of toxic Microcystis and environmental biomass rely heavily on the temperature and nutrient concentration (Davis et al., 2009; Fujimoto et al., 2007). Thus, plankton systems with such environmental fluctuations are significantly more reasonable, and the issue of how environmental fluctuations affect plankton systems have attracted increasing attention and great effort has been made towards the study of the dynamics of stochastic plankton systems recently (Yu et al., 2017; Zhao et al., 2020; Sarkar and Chattopadhayay, 2003; Pal et al., 2009; Chen et al., 2020; Chen et al., 2020; Wang and Liu, 2020). But the study of stochastic phytoplankton-toxic phytoplankton-zooplankton system is still in its infancy, especially the dynamics of the phytoplankton-toxin producing-phytoplankton-zooplankton system with white noise and regime switching is currently unclear. Thus, we mainly present the effects of white noise, regime switching and toxic substances produced by TPP on the stochastic phytoplankton-toxic phytoplankton-zooplankton system in this paper.

The rest of this paper is organized as follows. The model is presented in Section 5.2. Section 5.3 introduces some preliminaries firstly, and then we give the main results, including the existence and uniqueness of the global positive solution, extinction and persistence in the mean as well as the stationary distribution and ergodicity of the system. Some numerical simulations are carried out to study the dynamics of the system in Section 5.4. We summarize the results and present our conclusions in Section 5.5.

5.2. Model formation

In this section, we will establish a stochastic two preys-predator model in which the zooplankton feeds on two types of phytoplankton species, including a non-toxic phytoplankton (NTP) and a toxic one. The ecological construction of the stochastic phytoplankton-toxic phytoplankton-zooplankton system is based on the following assumptions:

1. It is assumed that $P_1(t), P_2(t), Z(t)$ are the population densities of NTP, TPP and zooplankton, respectively; m is the natural death rate of zooplankton.
2. It is considered that the growths of NTP and TPP in the absence of the grazer zooplankton are generally considered as logistic type with the intrinsic growth

rates r_1 and r_2 respectively, and their corresponding environmental carrying capacities K_1 and K_2.

3. It is assumed that a_1 and a_2 measure the competitive effects of TPP on NTP and NTP on TPP, respectively. In fact, these competitions have been introduced into ecological systems to explore the properties of plankton dynamics, such as stability (Pal et al., 2009), oscillation and chaos **(**Huisman and Weissing, 1999**)**, etc.
4. Behavior of the entire community is assumed to arise from the coupling of these interacting species. Both groups of phytoplankton exhibit Beddington-DeAngelis functional response to the grazer zooplankton as given by $\alpha Z(t)/\left(1+b_1P_1(t)+b_2P_2(t)\right)$ and $\beta Z(t)/\left(1+b_1P_1(t)+b_2P_2(t)\right)$, where α and β are the attack rates of zooplankton on NTP and TPP, respectively; b_1 and b_2 are the product of attack rate and handling time on NTP and TPP, respectively. In addition, the term $\gamma P_2Z/(1+b_1P_1+b_2P_2)$, which describes the resultant reduction for the growth of zooplankton due to the ingestion of TPP, where γ is the inhibition rate of zooplankton growth, while the term $\delta P_1Z/(1+b_1P_1+b_2P_2)$ can be regarded as the growth form of zooplankton in the present of NTP, where δ is the conversion efficiency. The Beddington-DeAngelis functional response (Beddington, 1975; DeAngelies et al., 1975) here is appropriate in the case of plankton population due to the fact that the predator individuals either search, consume or interfere with each other (Pal et al., 2009).
5. It is assumed that environmental noise exists in the realistic aquatic ecosystems because of the unpredictability of the environmental stochasticity, such as nutrients supply, water temperature, and some other small environmental fluctuations, which may affect population growths of the system. Actually, the work of May (1973) pointed out that these small environmental fluctuations can affect the ecological parameters of a model positively or negatively, which can be described by white noise. Thus, following the idea of (Yu et al., 2017; Zhao et al.,

2020; Chen et al., 2020; Chen et al., 2020), the convenient formulations, which describe the intrinsic growth rates of phytoplankton populations and the death rate of zooplankton population that are influenced by white noise, are taken as $r_i \to r_i + \sigma_i dB_i(t)$ $(i = 1,2)$ and $-d \to -d + \sigma_3 dB_3(t)$, respectively. Here the terms $dB_i(t)$ enote the white noises and $\sigma_i^2(t) > 0$ are their intensities of white noises, $i = 1,2,3$.

6. We further consider the regime switching into the model, where the biomass of plankton often suffer from switch abruptly to a contrasting alternative stable state in the real world due to some kinds of moderate environmental fluctuations, such as environmental pollution (Scheffer and Carpenter, 2003), rain falls (Du et al., 2004; Slatkin, 1978) and biotic exploitation (Scheffer, 2001). The plankton population models in this case can be characterized by the telegraph noise or colored noise (Mao et al., 2003), which may cause the population systems switching from one environmental regime to any other regimes (Mao et al., 2003; Luo and Mao, 2007). In addition, the switching is generally memoryless and the waiting time between two shifts follows exponential distribution. The convenient formulation here is to take $\xi(t), t \geq 0$ as regime switching, which is a continuous-time Markov chain with state space $\mathbb{S} = \{1,2,\cdots,m\}, 1 \leq n < \infty$.
7. It is assumed that the Markov chain $\xi(t)$ and the Brownian motions $dB_i(t), i = 1,2,3$ are defined on a completed probability space $(\Omega, \mathcal{F}_t, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ with a filtration $\{\mathcal{F}_t\}$ satisfying the usual normal conditions, and $\xi(t)$ is independent of $B_i(t), i = 1,2,3$.

Based on above assumptions, a stochastic phytoplankton-toxic producing phytoplankton-zooplankton system under regime switching is presented as follows:

$$\begin{cases} dP_1(t) = P_1(t)\Big[r_1(\xi(t))\Big(1 - \frac{P_1(t)}{K_1(\xi(t))}\Big) - a_1(\xi(t))P_2(t) - \\ \qquad\qquad \frac{\alpha(\xi(t))Z(t)}{1+b_1(\xi(t))P_1(t)+b_2(\xi(t))P_2(t)}\Big]dt + \sigma_1(\xi(t))P_1(t)dB_1(t) \\ dP_2(t) = P_2(t)\Big[r_2(\xi(t))\Big(1 - \frac{P_2(t)}{K_2(\xi(t))}\Big) - a_2(\xi(t))P_1(t) - \\ \qquad\qquad \frac{\beta(\xi(t))Z(t)}{1+b_1(\xi(t))P_1(t)+b_2(\xi(t))P_2(t)}\Big]dt + \sigma_2(\xi(t))P_2(t)dB_2(t) \\ dZ(t) = Z(t)\Big[\frac{\delta(\xi(t))P_1(t)-\gamma(\xi(t))P_2(t)}{1+b_1(\xi(t))P_1(t)+b_2(\xi(t))P_2(t)} - d(\xi(t))\Big]dt \\ \qquad\qquad +\sigma_3(\xi(t))Z(t)dB_3(t) \end{cases} \tag{5-1}$$

where$r_i(k), K_i(k), a_i(k), b_i(k)$ $(i = 1,2), \alpha(k), \beta(k), \delta(k), \gamma(k)and\sigma_j(k)$ $(j = 1,2,3)$ are all positive constants for each $k \in \mathbb{S}$.

5.3. Dynamic analysis

Due to the sudden and rapid changes of the external environment in a real aquatic ecosystem, such as nutrients supply, water temperature and some other small environmental fluctuations (white noise), and even rain falls (Du et al., 2004; Slatkin, 1978), environmental pollution (Scheffer and Carpenter, 2003), biotic exploitation (Meijer et al., 1994) and other kinds of moderate environmental fluctuations (colored noise or telegraph noise) that can be described by regime switching, which may significantly affect the growths of plankton populations. In this section, therefore, we will discuss how these changes (white noise and colored noise or telegraph noise) affect the dynamics of the system. Before it, we firstly investigate and guarantee the existence and uniqueness of global positive solutions of the system, and then discuss the extinction and persistence in the mean, and the positive recurrence and ergodic property of solution of the system under the effect of environmental fluctuations.

5.3.1. Preliminaries

$\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{R}_+^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_i > 0, i = 1,2, \cdots, n\}$, and $|x| = \sqrt{\sum_{i=1}^n x_i^2}$. For convenience, we introduce the following notations. If $\varphi(t)$ is a bounded and integrable function on $\mathbb{R}_+$, we define $\check{\varphi} = \limsup_{t\to\infty}\langle\varphi\rangle$, and $\hat{\varphi} = \liminf_{t\to\infty}\langle\varphi\rangle$, here $\langle\varphi\rangle = \frac{1}{T}\int_0^T \varphi(s)ds, T > 0$.

Let $\xi(t), t \geq 0$ be a right-continuous Markov chain on the probability space $(\Omega, \mathcal{F}_t, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ with $\xi(0) = \xi_0$, taking values in a finite-state space $\mathbb{S} = \{1,2,\cdots,m\}, 1 \leq n < \infty$ with the transition rate $Q = (q_{ij})_{m\times m}$ of $\xi(t)$ given by the following form:

$$\mathcal{P}\{\xi(t+\Delta t) = j | \xi(t) = i\} = \begin{cases} q_{ij}\Delta t + o(\Delta t), & if\ i \neq j, \\ 1 + q_{ii}\Delta t + o(\Delta t), & if\ i = j, \end{cases}$$

where $o(\Delta t)$ is the infinitesimal of higher order, $\Delta t > 0$ and $q_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $q_{ii} = -\sum_{i\neq j} q_{ij}$. Throughout this paper, we always assume that Markov chain $\xi(t), t \geq 0$ is irreductible, which means that the system can switch from any regime to any other regime, indicating that there exist finite number $i_1, i_2, \cdots, i_m \in \mathbb{S}$ such that $q_{i,i_1} q_{i_1,i_2} \cdots q_{i_m,j} > 0$, for any $i, j \in \mathbb{S}$. Under this assumption, the Markov chain $\xi(t), t \geq 0$ has a unique stationary distribution $\pi = (\pi_1, \pi_2, \cdots, \pi_m) \in \mathbb{R}^{1\times m}$, which can be determined by solving the linear equation $\pi Q = 0$ subject to $\sum_{k=1}^{m} \pi_k = 1$ and $\pi_k > 0, \forall\, k \in \mathbb{S}$. For any vector $\phi = \left(\phi(1), \cdots, \phi(m)\right)^T$, we define $\phi^* = \max_{k\in\mathbb{S}}\{\phi(k)\}$ and $\phi_* = \min_{k\in\mathbb{S}}\{\phi(k)\}$.

Now, we introduce some fundamental results on the stationary distribution of stochastic differential equations under regime switching. Let $\left(X(t), \xi(t)\right)$ be the diffusion process described by the following equation:

$$\begin{cases} dX = f\left(X(t), \xi(t)\right)dt + g\left(X(t), \xi(t)\right)dB(t), \\ X(0) = X_0 \in \mathbb{R}^n,\ \ \xi(0) = \xi_0 \in \mathbb{S}, \end{cases} \tag{5-2}$$

where $B(\cdot)$ and $\xi(\cdot)$ are the d-dimentional Brownian motion and right continuous Markov chain in the above discussion, respectively. $f(\cdot,\cdot): \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^n$, and $g(\cdot,\cdot): \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^{n\times d}$ satisfy $g(X,k)g^T(X,k) = \left(d_{ij}(X,k)\right)$, where the superscript T stands for the transpose of a matrix or vector. For each $k \in \mathbb{S}$ and for any twice continuously differentiable function $V(X,k) \in \mathbb{C}^2(\mathbb{R}^n \times \mathbb{S})$that are non-negative, we define a operator L:

$$LV(X,k) = \sum_{i=1}^{n} f_i\,(X,k)\frac{\partial(X,k)}{\partial X_i} + \frac{1}{2}\sum_{i,j=1}^{n} d_{ij}\,(X,k)\frac{\partial^2(X,k)}{\partial X_i \partial X_j} + \Gamma(X,\cdot)(k),$$

where

$$\Gamma(X,\cdot)(k) = \sum_{h=1}^{m} q_{kh} L(X,h) = \sum_{k\neq h, h\in\mathbb{S}} q_{kh}\big(V(X,h) - V(X,k)\big), h \in \mathbb{S}.$$

From Theorems 3.13 (Zhu and Yin, 2007), the following lemma which gives a criterion for the ergodic stationary distribution of system (5-2) can be presented:

Lemma 5.3.1.1. If the following conditions are satisfied:

(i) for $i \neq j, q_{ij} > 0, i,j \in \mathbb{S}$;

(ii) for each $k \in \mathbb{S}$, $\lambda|\mu|^2 \le \mu^T d_{ij}(X,k)\mu \le \lambda^{-1}|\mu|^2$, for all $\mu \in \mathbb{R}^n$, with some constant $\lambda \in (0,1]$ for all $X \in \mathbb{R}^n$;

(iii) there exists a bounded open subset Ξ of $\mathbb{R}^n$ with a regular (i.e. smooth) boundary satisfying that, for each $k \in \aleph$, there exists a nonnegative function $V(\cdot,k)\colon \Xi^c \to \mathbb{R}$ such that $V(\cdot,k)$ is twice continuously differentiable and that for some $\varsigma > 0$, $LV(\cdot,k) \le -\varsigma$, for any $(X,k) \in \Xi^c \times \mathbb{S}$.

Then the diffusion process $\big(X(t),\xi(t)\big)$ for system (5-2) is ergodic and positive recurrent. That is, there exists a unique stationary distribution $\rho(\cdot,\cdot) = (\rho(\cdot,i)\colon i \in \mathbb{S})$, and for any Borel measurable function $\varphi(\cdot,\cdot)\colon \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^n$ such that

$$\sum_{k\in\aleph} \int_{\mathbb{R}^n} |\varphi(X,k)|\rho(X,k)dX < \infty,$$

we have

$$\mathcal{P}\left(\lim_{t\to\infty}\frac{1}{t}\int_0^t \varphi\big(X(s),\xi(s)\big)ds = \sum_{k\in\aleph}\int_{\mathbb{R}^n} \varphi(X,k)\rho(X,k)dX\right) = 1.$$

Lemma 5.3.1.2. Let $Q = \big(q_{ij}\big)$ be irreducible and $\varpi, \eta \in \mathbb{R}^m$. Then the following linear system:

$$Q\varpi = \eta$$

has a solution if and only if $\pi\eta = 0$, where the stationary distribution $\pi = (\pi_1, \pi_2, \cdots, \pi_m) \in \mathbb{R}^{1\times m}$.

5.3.2. Existence and uniqueness of global positive solutions

Before investigating the dynamics of system (5-1), we should first guarantee the existence of global positive solutions according to the biological interpretation. Therefore, in this subsection, we discuss the existence of global positive solutions of system (5-1) in the following.

Actually, from the method of the Theorem 3.15 (Mao and Yuan, 2006), obviously, we can verify that all the coefficients of system (5-1) are locally Lipschitz continuous and system (5-1) admits a unique local solution $\big(P_1(t), P_2(t), Z(t), \xi(t)\big)$ on $t \in [0, \tau_e)$ for any given initial value $\big(P_1(0), P_2(0), Z(0), \xi(0)\big) \in \mathbb{R}_+^3 \times \mathbb{S}$, where τ_e represents the explosion time. In order to illustrate the solution is global, we only need to prove $\tau_e = \infty, a.s.$ Let $n_0 \geq 1$ enough large satisfying $\big(P_1(0), P_2(0), Z(0), \xi(0)\big) \in \left[\frac{1}{n_0}, n_0\right]$. For each integer $n \geq n_0$, we define the following stopping time:

$$\tau_n = \inf\Big\{t \in [0, \tau_e) \colon \max\ \big\{\big(P_1(t), P_2(t), Z(t), \xi(t)\big)\big\} \\ \leq \frac{1}{n} \text{ or } \min\big\{\big(P_1(t), P_2(t), Z(t), \xi(t)\big)\big\} \geq n\Big\},$$

and the set $\inf \emptyset = \infty$ ($\emptyset$ denotes the empty set). Obviously, τ_n is increasing as $n \to \infty$. Let $\tau_\infty = \lim_{n \to +\infty} \tau_n$, then we can obtain that $\tau_\infty \leq \tau_e$ *a.s.* Thus, If we can show that $\tau_\infty = \infty$ a.s. in the following, then $\tau_e = \infty$ and $\big(P_1(0), P_2(0), Z(0), \xi(0)\big) \in \mathbb{R}_+^3 \times \mathbb{S}$ a.s. for all $t \geq 0$. In other words, to complete the proof, we only need to proof $\tau_\infty = \infty$ *a.s.* Otherwise, the statement is false, then there exist two constants $T > 0$ and $\varepsilon \in (0,1)$ such that $\mathcal{P}\{\tau_\infty \leq T\} > \varepsilon$. Hence, for all $n \geq n_1$, there exists an integer $n_1 \geq n_0$ such that $\mathcal{P}\{\tau_\infty \leq T\} \geq \varepsilon$.

Define a C^2-function $\overline{\mathrm{V}} \colon \mathbb{R}_+^3 \to \overline{\mathbb{R}}_+$ by

$$\bar{V}(P_1, P_2, Z, \xi) = \delta^*(P_1 - 1 - \log P_1) + (P_2 - 1 - \log P_2) + \alpha_*(Z - 1 - \log Z).$$

Obviously, the function $\bar{V}(P_1, P_2, Z, \xi)$ is non-negative. By applying the generalized Itô's formula to $\bar{V}(P_1, P_2, Z, \xi)$, we have

$$
\begin{aligned}
&d\bar{V}(P_1, P_2, Z, \xi) \\
&\qquad = L\bar{V}(P_1, P_2, Z)dt + \delta^*\sigma_1\big(\xi(t)\big)(P_1 - 1)dB_1(t) \\
&\qquad + \sigma_2\big(\xi(t)\big)(P_2 - 1)dB_2(t) + \alpha_*\sigma_3\big(\xi(t)\big)(Z - 1)dB_3(t),
\end{aligned}
$$

where $L\bar{V}: \mathbb{R}_+^3 \to \mathbb{R}$ is defined by

$$
\begin{aligned}
&L\bar{V}(P_1, P_2, Z, \xi) \\
&\qquad = \delta^*(P_1 - 1)\left(r_1(\xi)\left(1 - \frac{P_1}{K_1(\xi)}\right) - a_1(\xi)P_2\right. \\
&\qquad \left. - \frac{\alpha(\xi)Z(t)}{1 + b_1(\xi)P_1(t) + b_2(\xi)P_2(t)}\right) + \frac{\delta^*\sigma_1^2(\xi)}{2} \\
&\qquad + (P_2 - 1)\left(r_2(\xi)\left(1 - \frac{P_2}{K_2(\xi)}\right) - a_2(\xi)P_1\right. \\
&\qquad \left. - \frac{\beta(\xi)Z(t)}{1 + b_1(\xi)P_1(t) + b_2(\xi)P_2(t)}\right) + \frac{\sigma_2^2(\xi)}{2} \\
&\qquad + \alpha_*(Z - 1)\left(\frac{\delta(\xi)P_1 - \gamma(\xi)P_2}{1 + b_1(\xi)P_1(t) + b_2(\xi)P_2(t)} - d(\xi)\right) + \frac{\alpha_*\sigma_3^2(\xi)}{2} \\
&\qquad \le \left(\alpha_* d^* + \frac{\alpha_*\gamma^*}{(b_2)_*} - \delta^*(r_1)_* - (r_2)_* + \frac{1}{2}(\delta^*\sigma_1^* + \sigma_2^* + \alpha_*\sigma_3^*)\right) \\
&\qquad + \left(\frac{\delta^* r_1^*}{(K_1)_*} + \delta^* r_1^* + a_2^*\right)P_1 - \frac{\delta^*(r_1)_*}{K_1^*}P_1^2 + \left(\frac{r_2^*}{(K_2)_*} + r_2^* + \delta^* a_1^*\right)P_2 \\
&\qquad - \frac{(r_2)_*}{K_2^*}P_2^2 + |\delta^*\alpha^* + \beta^* - \alpha_* d_*|Z \le M + |\delta^*\alpha^* + \beta^* - \alpha_* d_*|Z,
\end{aligned}
$$

where

$$
\begin{aligned}
M = &\left(\alpha_* d^* + \frac{\alpha_*\gamma^*}{(b_2)_*} - \delta^*(r_1)_* - (r_2)_* + \frac{1}{2}(\delta^*\sigma_1^* + \sigma_2^* + \alpha_*\sigma_3^*)\right) \\
&\qquad + \max_{P_1 \in (0, +\infty)}\left\{\left(\frac{\delta^* r_1^*}{(K_1)_*} + \delta^* r_1^* + a_2^*\right)P_1 - \frac{\delta^*(r_1)_*}{K_1^*}P_1^2\right\} \\
&\qquad + \max_{P_2 \in (0, +\infty)}\left\{\left(\frac{r_2^*}{(K_2)_*} + r_2^* + \delta^* a_1^*\right)P_2 - \frac{(r_2)_*}{K_2^*}P_2^2\right\}.
\end{aligned}
$$

Notice that $Z \le 2(Z - 1 - \log Z) + 2\log 2 \le \frac{2}{\alpha_*}\bar{V}(P_1, P_2, Z, \xi) + 2\log 2$ for all $Z > 0$, then one can obtain that

$$L\bar{V} \le M + 2|\delta^*\alpha^* + \beta^* - \alpha_* d_*|\log 2 + \frac{2}{\alpha_*}|\delta^*\alpha^* + \beta^* - \alpha_* d_*|\bar{V} \le \Upsilon(1 + \bar{V}),$$

where

$$\Upsilon = \max\left\{M + 2\left|\delta^*\alpha^* + \frac{\beta^*}{b_*} - \alpha_* d_*\right|\log 2, \frac{2}{\alpha_*}\left|\delta^*\alpha^* + \frac{\beta^*}{b_*} - \alpha_* d_*\right|\right\}.$$

The reminder of the proof follows that in (Yu et al., 2018), here, we omit it. Therefore, for any given initial value $\big(P_1(0), P_2(0), Z(0), \xi(0)\big) \in \mathbb{R}_+^3 \times \mathbb{S}$, there exists a unique solution $\big(P_1(t), P_2(t), Z(t), \xi(t)\big)$ of system (5-1) on $t \in \mathbb{R}_+$ and the positive solution will remain in $\mathbb{R}_+^3 \times \mathbb{S}$ with probability one, namely, $\big(P_1(t), P_2(t), Z(t), \xi(t)\big) \in \mathbb{R}_+^3 \times \mathbb{S}$ for all $t \ge 0$ almost surely.

5.3.3. Extinction and persistence of plankton

Based on the previous analysis, and the perspective of the study of population dynamics, it is necessary and important to consider whether a population can sustain development or become extinct in the long time under the effect of the environmental fluctuations in the model. Thus, we will discuss the persistence in the mean and extinction of system (5-1) in this subsection, and the following analysis shows that the environmental fluctuations (white noise and regime switching) can significantly affect the persistence in the mean and extinction of plankton populations, which implies that the environmental fluctuations can control the growth of harmful phytoplankton population using modeling analysis. For convenience of discussion in the following, we define

$$A = \sum_{i=1}^{m}\pi_i\left(r_1(i) - \frac{1}{2}\sigma_1^2(i)\right), B = \sum_{i=1}^{m}\pi_i\left(r_2(i) - \frac{1}{2}\sigma_2^2(i)\right), C = \sum_{i=1}^{m}\pi_i\left(d(i) + \frac{1}{2}\sigma_3^2(i)\right).$$

Now, we discuss the statement. Suppose that $(P_1(t), P_2(t), Z(t), \xi(t))$ is the solution of system (5-1) with the initial value $(P_1(0), P_2(0), Z(0), \xi(0)) \in \mathbb{R}_+^3 \times \mathbb{S}$. We firstly consider the toxic phytoplankton species $P_1(t)$. Applying the Itô's formula to system (5-1), one can easily obtain that

$$d\ln P_1(t) = \left[r_1(\xi(t))\left(1 - \frac{P_1(t)}{K_1(\xi(t))}\right) - a_1(\xi(t))P_2(t) - \frac{\alpha(\xi(t))Z(t)}{1 + b_1(\xi(t))P_1(t) + b_2(\xi(t))P_2(t)} - \frac{1}{2}\sigma_1^2(\xi(t))\right]dt + \sigma_1(\xi(t))dB_1(t).$$

Integrating the above from 0 to t and dividing t on both sides yield

$$\frac{1}{t}\frac{\ln P_1(t)}{\ln P_1(0)} = \langle r_1(\xi(t)) - \frac{1}{2}\sigma_1^2(\xi(t))\rangle - \langle \frac{r_1(\xi(t))}{K_1(\xi(t))}P_1(t)\rangle - \langle a_1(\xi(t))P_2(t)\rangle - \langle \frac{\alpha(\xi(t))}{1+b_1(\xi(t))P_1(t)+b_2(\xi(t))P_2(t)}Z(t)\rangle + \frac{M_1(t)}{t} \tag{5-3}$$

where

$$M_1(t) = \int_0^t \sigma_1(\xi(s))dB_1(s)$$

By the strong law of large numbers for martingales (Mao, 1997c) yields

$$\lim_{t\to\infty}\frac{M_1(t)}{t} = 0 \quad a.s. \tag{5-4}$$

According to the ergodic theorem of Markov chain $\xi(t)$ and (5-3), (5-4), it is obvious to find that, if $A < 0$, we have

$$\limsup_{t\to\infty}\frac{1}{t}\frac{\ln P_1(t)}{\ln P_1(0)} \le \lim_{t\to\infty}\langle r_1(\xi(t)) - \frac{1}{2}\sigma_1^2(\xi(t))\rangle = \sum_{i=1}^m \pi_i\left(r_1(i) - \frac{1}{2}\sigma_1^2(i)\right) = A < 0, a.s.$$

which implies $\lim_{t\to\infty} P_1(t) = 0, a.s.$, that is, the toxic phytoplankton $P_1(t)$ is extinct, which implies that the white noise and regime switching can affect the dynamics of the system and provide a biological way to control the growth of harmful phytoplankton.

For other species phytoplankton $P_2(t)$ and zooplankton $Z(t)$, similarly, if the condition $B < 0$ holds, we have

$$\limsup_{t\to\infty} \frac{1}{t}\frac{\ln P_2(t)}{\ln P_2(0)} \le \lim_{t\to\infty} \langle r_2(\xi(t)) - \frac{1}{2}\sigma_2^2(\xi(t))\rangle$$

$$= \sum_{i=1}^{m} \pi_i \left(r_2(i) - \frac{1}{2}\sigma_2^2(i)\right) = B < 0, a.s.$$

and

$$\limsup_{t\to\infty} \frac{1}{\mathrm{t}}\frac{\ln Z(t)}{\ln Z(0)} \le -\lim_{t\to\infty} \langle d(\xi(t)) + \frac{1}{2}\sigma_3^2(\xi(t))\rangle$$

$$= -\sum_{i=1}^{m} \pi_i \left(d(i) + \frac{1}{2}\sigma_3^2(i)\right) < 0, a.s.$$

This imply that $\lim_{t\to\infty} P_2(t) = 0, a.s.$ and $\lim_{t\to\infty} Z(t) = 0, a.s.$, respectively. Namely, phytoplankton $P_2(t)$ and zooplankton $Z(t)$ tend toward extinction provided that $B < 0$. Thus, for any given the initial value $(P_1(0), P_2(0), Z(0), \xi(0)) \in \mathbb{R}_+^3 \times \mathbb{S}$, all the populations will undergo extinction under the conditions of $A < 0$ and $B < 0$.

Interestingly, we find that the toxic phytoplankton $P_1(t)$ and non-toxic phytoplankton $P_2(t)$ can coexist at a stable state, while the zooplankton $Z(t)$ undergoes extinction if the conditions $D_1 = A - \frac{a_1^*(K_2)_*}{r_2^*}B > 0$, $D_2 = B - \frac{a_2^*(K_1)_*}{r_1^*}A > 0$, and $D_3 = \frac{\delta^*(K_1)_*}{r_1^*}A - C < 0$ hold.

Actually, suppose that $(P_1(t), P_2(t), Z(t), \xi(t))$ is the solution of system (5-1) with the initial value $(P_1(0), P_2(0), Z(0), \xi(0)) \in \mathbb{R}_+^3 \times \mathbb{S}$. From the Lemma 4 (Liu et al., 2011; Liu and Bai, 2016), we can obtain

$$\langle \breve{P_1} \rangle \le \frac{(K_1)_*}{r_1^*}A,\ \langle \breve{P_2} \rangle \le \frac{(K_2)_*}{r_2^*}B \tag{5-5}$$

We first consider the zooplankton $Z(t)$. Applying the Itô's formula to system (5-1) and then integrating the above from 0 to t and dividing t, we have

$$\frac{1}{t}\frac{\ln Z(t)}{\ln Z(0)} \le -\langle d(\xi(t)) + \frac{1}{2}\sigma_3^2(\xi(t))\rangle + \langle \frac{\delta(\xi(t))}{1+b_1(\xi(t))P_1(t)+b_2(\xi(t))P_2(t)} P_1(t)\rangle$$

$$-\langle\frac{\gamma(\xi(t))}{1+b_1(\xi(t))P_1(t)+b_2(\xi(t))P_2(t)}P_2(t)\rangle+\frac{M_3(t)}{t} \tag{5-6}$$

Taking upper limit on both sides of (5-6) and using the strong law of large number of local martingale yields

$$\frac{\widetilde{\ln Z(t)}}{t}\leq\frac{\delta^*(K_1)_*}{r_1^*}A-C<0, a.s.$$

if the condition $D_3<0$ holds, which implies that $\lim_{t\to\infty}Z(t)=0\ a.s.$, namely, zooplankton $Z(t)$ is extinct.

Now, we consider the species $P_1(t)$. By (5-5) and integrating (5-3) on the interval $[0,t]$ and making some estimations, one can obtain that

$$\frac{1}{t}\ln P_1(t)\geq\frac{1}{t}\ln P_1(0)+A-\frac{r_1^*}{(K_1)_*}\langle P_1\rangle-\frac{a_1^*(K_2)_*}{r_2^*}B-\alpha^*\tilde{Z}+\frac{M_1(t)}{t}, \tag{5-7}$$

In addition, since the fact that

$$\lim_{\mathrm{t}\to+\infty}\frac{\ln P_1(0)}{t}=\lim_{\mathrm{t}\to+\infty}\frac{M_1(t)}{t}=0$$

and from the definition of A and B, then we can obtain that for arbitrary $\epsilon_2>0$, there exists a constant $T_2>0$ such that

$$\langle P_2\rangle\leq\frac{(K_2)_*}{r_2^*}B+\frac{\epsilon_2}{2a_1^*},\langle Z\rangle\leq\frac{\epsilon_2}{2\alpha^*},\langle r_1(\xi(t))-\frac{1}{2}\sigma_1^2(\xi(t))\rangle\leq A-\frac{\epsilon_2}{3},$$

and

$$\frac{\ln P_1(0)}{\mathrm{t}}\geq-\frac{\epsilon_2}{3},\frac{M_1(t)}{t}\geq-\frac{\epsilon_2}{3}.$$

Substituting the above inequalities into (5-7) and for all $t\geq T_2$, we have

$$\frac{1}{t}\ln\frac{P_1(t)}{P_1(0)}\geq\left(A-\frac{a_1^*(K_2)_*}{r_2^*}B\right)t-\frac{r_1^*}{(K_1)_*}\langle P_1\rangle++\frac{M_1(t)}{t}.$$

Obviously, if $D_1>0$ and by (5-5), one can obtain that

$$\langle\widehat{P_1(t)}\rangle\geq\frac{(K_1)_*}{r_1^*}\left(A-\frac{a_1^*(K_1)_*}{r_1^*}B\right)>0.$$

This implies that toxic phytoplankton $P_1(t)$ is persistence in the mean.

For the species $P_2(t)$, the same analysis to the species $P_1(t)$, we have

$$\langle \widehat{P_2(t)} \rangle \geq \frac{(K_2)_*}{r_2^*}\left(B - \frac{a_2^*(K_2)_*}{r_2^*}A\right) > 0,$$

if the condition $D_2 > 0$ holds. That is, the species $P_2(t)$ is persistence in the mean.

5.3.4. Stationary distribution and ergodic property of plankton

In aquatic ecology, the coexistence of plankton populations is strongly related to the sustainable development of aquatic ecosystem and it is important to study the long time statistical behaviors of plankton populations under the effect of environmental stochasticity in the model, that is, the positive recurrence and the existence of uniqueness of stationary distribution of the system. Thus, by constructing a suitable Lyapunov function and using Khasminskii's method (Khasminskii, 1980), we will discuss and analyze the existence of the stationary distribution and ergodic property of the system in this subsection. The following analysis shows that the regime switching and toxin released rate by toxic phytoplankton can significantly affect the existence of the stationary distribution and ergodic property of plankton populations.

In order to prove our statement, we only need to prove the three conditions in Lemma 5.3.1.1 one by one. Obviously, the condition (i) of Lemma 5.3.1.1 is satisfied by the assumption $q_{ij} > 0$ for $i \neq j, i, j \in \mathbb{S}$ in subsection 5.3.1. On the other hand, it is easy to verify that the diffusion matrix $d_{ij}(X, k) = \text{diag}\,\{\sigma_1^2(\xi), \sigma_2^2(\xi), \sigma_3^2(\xi)\}$ of system (5-1) is positive definite, which implies that the condition (ii) of Lemma 5.3.1.1 holds.

In the following, we prove the condition (iii) of Lemma 5.3.1.1. By constructing a C^2-function $V: \mathbb{R}_+^3 \times \mathbb{S} \to \mathbb{R}$ as follows:

$$\begin{aligned} V(P_1, P_2, Z, \xi) &= (c_1 P_1 + P_2 + c_2 Z) \\ &\quad - (\ln P_1 + 1 + \ln c_1 + \ln P_2 + 1 + \ln Z + 1 + \ln c_2) + (\varpi_\xi + |\varpi|) \\ &= V_1(P_1, P_2, Z, \xi) + V_2(P_1, P_2, Z, \xi) + V_3(\xi), \end{aligned}$$

where c_1 and c_2 are positive constants and $M > 0$ satisfying $-M\Pi \leq -1$, $\varpi = (\varpi_1, \varpi_2, \cdots, \varpi_m)^T$, $|\varpi| = \sqrt{\varpi_1^2 + \varpi_2^2 + \cdots, \varpi_m^2}$ and ϖ_ξ $(\xi \in \mathbb{S})$ will be determined later and the reason for $|\varpi|$ being here is to make $\varpi_\xi + |\varpi|$ non-negative. Obviously, the function $V(P_1, P_2, Z, \xi)$ is non-negative. Applying the Itô's formula to V_1 and V_2, we have

$$
\begin{aligned}
&LV_1(P_1, P_2, Z, \xi)\\
&\qquad = c_1 r_1(\xi)P_1 - \frac{c_1 r_1(\xi)}{K_1(\xi)}P_1^2 - c_1 a_1(\xi)P_1P_2 - \frac{c_1\alpha(\xi)P_1Z}{1 + b_1(\xi)P_1 + b_2(\xi)P_2}\\
&\qquad + r_2(\xi)P_2 - \frac{r_2(\xi)}{K_2(\xi)}P_2^2 - a_2(\xi)P_1P_2 - \frac{\beta(\xi)P_2Z}{1 + b_1(\xi)P_1 + b_2(\xi)P_2}\\
&\qquad + \frac{c_2\delta(\xi)P_1Z}{1 + b_1(\xi)P_1 + b_2(\xi)P_2} - \frac{c_2\gamma(\xi)P_2Z}{1 + b_1(\xi)P_1 + b_2(\xi)P_2} - c_2 d(\xi)Z\\
&\qquad \le -\frac{c_1 r_1(\xi)}{K_1(\xi)}P_1^2 + c_1 r_1(\xi)P_1 - \frac{r_2(\xi)}{K_2(\xi)}P_2^2 + r_2(\xi)P_2 - c_2 d(\xi)Z\\
&\qquad - \frac{[c_1\alpha(\xi) - c_2\delta(\xi)]P_1Z}{1 + b_1(\xi)P_1 + b_2(\xi)P_2},
\end{aligned}
$$

and

$$
\begin{aligned}
&LV_2(P_1, P_2, Z, \xi)\\
&\qquad = -r_1(\xi) + \frac{r_1(\xi)}{K_1(\xi)}P_1 + a_1(\xi)P_2 + \frac{\alpha(\xi)Z}{1 + b_1(\xi)P_1 + b_2(\xi)P_2} + \frac{\sigma_1^2(\xi)}{2}\\
&\qquad - r_2(\xi) + \frac{r_2(\xi)}{K_2(\xi)}P_2 + a_2(\xi)P_1 + \frac{\beta(\xi)Z}{1 + b_1(\xi)P_1 + b_2(\xi)P_2} + \frac{\sigma_2^2(\xi)}{2}\\
&\qquad - \frac{\delta(\xi)P_1 - \gamma(\xi)P_2}{1 + b_1(\xi)P_1 + b_2(\xi)P_2} + d(\xi) + \frac{\sigma_3^2(\xi)}{2}\\
&\qquad \le \left[\frac{r_1(\xi)}{K_1(\xi)} + a_2(\xi)\right]P_1 + \left[\frac{r_2(\xi)}{K_2(\xi)} + a_1(\xi)\right]P_2 + [\alpha(\xi) + \beta(\xi)]Z\\
&\qquad + \left[\frac{\gamma(\xi)}{b_2(\xi)} - r_1(\xi) - r_2(\xi) + d(\xi) + \frac{\sigma_1^2(\xi) + \sigma_2^2(\xi) + \sigma_3^2(\xi)}{2}\right].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&LV_1(P_1,P_2,Z,\xi)+LV_1(P_1,P_2,Z,\xi)\\
&\qquad\le -\frac{c_1 r_1(\xi)}{K_1(\xi)}P_1^2+\left[c_1 r_1(\xi)+\frac{r_1(\xi)}{K_1(\xi)}+a_2(\xi)\right]P_1-\frac{r_2(\xi)}{K_2(\xi)}P_2^2\\
&\qquad+\left[r_2(\xi)+\frac{r_2(\xi)}{K_2(\xi)}+a_1(\xi)\right]P_2+[\alpha(\xi)+\beta(\xi)-c_2 d(\xi)]Z\\
&\qquad-\frac{[c_1\alpha(\xi)-c_2\delta(\xi)]P_1 Z}{1+b_1(\xi)P_1+b_2(\xi)P_2}\\
&\qquad+\left[\frac{\gamma(\xi)}{b_2(\xi)}-r_1(\xi)-r_2(\xi)+d(\xi)+\frac{\sigma_1^2(\xi)+\sigma_2^2(\xi)+\sigma_3^2(\xi)}{2}\right].
\end{aligned}
$$

Choosing $c_1=\frac{\beta^*}{\alpha_*}c_2$ and $c_2=\frac{\alpha^*+\beta^*+1}{d_*}$, one can obtain

$$
\begin{aligned}
&LV_1(P_1,P_2,Z,\xi)+LV_1(P_1,P_2,Z,\xi)\\
&\le -\frac{\beta(\xi)r_1(\xi)(\alpha(\xi)+\beta(\xi)+1)}{\alpha(\xi)d(\xi)K_1(\xi)}\Bigg[P_1\\
&-\frac{K_1(\xi)r_1(\xi)\beta(\xi)(\alpha(\xi)+\beta(\xi)+1)+\alpha(\xi)d(\xi)r_1(\xi)+\alpha(\xi)d(\xi)K_1(\xi)a_2(\xi)}{2[\beta(\xi)r_1(\xi)(\alpha(\xi)+\beta(\xi)+1)]}\Bigg]^2\\
&-\frac{r_2(\xi)}{K_2(\xi)}\left(P_2-\frac{r_2(\xi)+K_2(\xi)\big(r_2(\xi)+a_1(\xi)\big)}{2r_2(\xi)}\right)^2\\
&+\frac{\alpha(\xi)d(\xi)K_1(\xi)\left[\frac{\beta(\xi)(\alpha(\xi)+\beta(\xi)+1)}{\alpha(\xi)d(\xi)}r_1(\xi)+\frac{r_1(\xi)}{K_1(\xi)}+a_2(\xi)\right]^2}{4[\beta(\xi)r_1(\xi)(\alpha(\xi)+\beta(\xi)+1)]}\\
&+\frac{K_2(\xi)\left[r_2(\xi)+\frac{r_2(\xi)}{K_2(\xi)}+a_1(\xi)\right]^2}{4r_2(\xi)}\\
&+\left[\frac{\gamma(\xi)}{b_2(\xi)}-\left(r_1(\xi)-\frac{\sigma_1^2(\xi)}{2}\right)-\left(r_2(\xi)-\frac{\sigma_2^2(\xi)}{2}\right)+\left(d(\xi)+\frac{\sigma_3^2(\xi)}{2}\right)\right].
\end{aligned}
$$

Moreover,

$$LV_3(\xi)=\sum_{h=1}^{m}q_{\xi h}\varpi_h.$$

Note that

$$\sum_{\xi=1}^{m}\pi_\xi=1,\qquad \pi[\Lambda-(\pi\Lambda)I_m]=0,$$

where $\Lambda=(\Lambda_1,\Lambda_2,\cdots,\Lambda_m)^T, I_m=(1,1,\cdots,1)^T\in\mathbb{R}^m$. Using the Lemma 5.3.1.2, we can obtain that the following equation $Q\varpi = \Lambda - (\pi\Lambda)I_m$ has a solution $\varpi = (\varpi_1, \varpi_2, \cdots, \varpi_m)^T \in \mathbb{R}^m$, which implies that

$$-\Lambda_\xi + \sum_{h=1}^{m} q_{\xi h}\varpi_h = -\sum_{h=1}^{m} \pi_h \Lambda_h = -\Pi,$$

Then

$$LV(P_1, P_2, Z, \xi) \leq -\Pi + f_1(P_1) + f_2(P_2),$$

where

$$\begin{aligned}\Pi = \sum_{h=1}^{m} \pi_\xi \Bigg\{ &\left(r_1(\xi) - \frac{\sigma_1^2(\xi)}{2}\right) + \left(r_2(\xi) - \frac{\sigma_2^2(\xi)}{2}\right) - \left(d(\xi) + \frac{\sigma_3^2(\xi)}{2}\right) - \frac{\gamma(\xi)}{b_2(\xi)} \\ &- \frac{\alpha(\xi)d(\xi)K_1(\xi)\left[\frac{\beta(\xi)(\alpha(\xi)+\beta(\xi)+1)}{\alpha(\xi)d(\xi)} r_1(\xi) + \frac{r_1(\xi)}{K_1(\xi)} + a_2(\xi)\right]^2}{4[\beta(\xi)r_1(\xi)(\alpha(\xi)+\beta(\xi)+1)]} \\ &- \frac{K_2(\xi)\left[r_2(\xi) + \frac{r_2(\xi)}{K_2(\xi)} + a_1(\xi)\right]^2}{4r_2(\xi)} \Bigg\},\end{aligned}$$

$$\begin{aligned} &f_1(P_1) \\ &= -\frac{\beta(\xi)r_1(\xi)(\alpha(\xi)+\beta(\xi)+1)}{\alpha(\xi)d(\xi)K_1(\xi)} \Bigg[P_1 \\ &- \frac{K_1(\xi)r_1(\xi)\beta(\xi)(\alpha(\xi)+\beta(\xi)+1) + \alpha(\xi)d(\xi)r_1(\xi) + \alpha(\xi)d(\xi)K_1(\xi)a_2(\xi)}{2[\beta(\xi)r_1(\xi)(\alpha(\xi)+\beta(\xi)+1)]}\Bigg]^2,\end{aligned}$$

and

$$f_2(P_2) = -\frac{r_2(\xi)}{K_2(\xi)}\left(P_2 - \frac{r_2(\xi) + K_2(\xi)\big(r_2(\xi) + a_1(\xi)\big)}{2r_2(\xi)}\right)^2.$$

Thus, we have

$$\begin{aligned} &-\Pi + f_1(P_1) + f_2(P_2) \\ &\leq \begin{cases} -\Pi + f_1(P_1) + f_2^u \to -\infty, & as\ P_1 \to +\infty, \\ -\Pi + f_1(P_1) + f_2^u \leq -\Pi \leq -1, & as\ P_1 \to 0^+, \\ -\Pi + f_1^u + f_2(P_2) \to -\infty, & as\ P_2 \to +\infty, \\ -\Pi + f_1^u + f_2(P_2) \leq -\Pi \leq -1, & as\ P_2 \to 0^+. \end{cases}\end{aligned}$$

if the condition $\Pi = \sum_{h=1}^{m} \pi_\xi \Lambda_\xi > 0$ holds, where

$$
\begin{aligned}
\Lambda_{\xi} = {} & \left(r_1(\xi) - \frac{\sigma_1^2(\xi)}{2}\right) + \left(r_2(\xi) - \frac{\sigma_2^2(\xi)}{2}\right) - \left(d(\xi) + \frac{\sigma_3^2(\xi)}{2}\right) - \frac{\gamma(\xi)}{b_2(\xi)} \\
& - \frac{\alpha(\xi)d(\xi)K_1(\xi)\left[\frac{\beta(\xi)(\alpha(\xi)+\beta(\xi)+1)}{\alpha(\xi)d(\xi)}r_1(\xi) + \frac{r_1(\xi)}{K_1(\xi)} + a_2(\xi)\right]^2}{4[\beta(\xi)r_1(\xi)(\alpha(\xi)+\beta(\xi)+1)]} \\
& - \frac{K_2(\xi)\left[r_2(\xi) + \frac{r_2(\xi)}{K_2(\xi)} + a_1(\xi)\right]^2}{4r_2(\xi)},
\end{aligned}
$$

Therefore, we can take $\varepsilon > 0$ sufficiently small such that for any $(P_1, P_2, Z, \xi) \in F^c \times \mathbb{S}$,

$$LV(P_1, P_2, Z, \xi) \leq -1,$$

where $F = \left(\varepsilon, \frac{1}{\varepsilon}\right) \times \left(\varepsilon, \frac{1}{\varepsilon}\right) \times \left(\varepsilon, \frac{1}{\varepsilon}\right)$. Hence, the condition (iii) of Lemma 5.3.1.1 is verified. It follows from Lemma 5.3.1.1 that system (5-1) admits a unique ergodic stationary distribution for any given initial value $\left(P_1(0), P_2(0), Z(0), \xi(0)\right) \in \mathbb{R}_+^3 \times \mathbb{S}$.

5.4. Experimental simulations

In the previous subsections, we have investigated the effects of environmental fluctuations (i.e. the white noise and regime switching) and TPP on the dynamics of plankton ecosystem and verified the intensity of white noise, regime switching and the liberation rate produced by TPP can lead to the extinction, persistence in the mean and stationary distribution of plankton populations using modeling analysis. In order to further study how the white noise, regime switching and TPP on affect the dynamics mechanisms of the formation and evolution process of planktonic blooms, we perform some experimental simulations for system (5-1) based on Milstein's Higher Order Method (Higham, 2001) in this section. In the following experimental simulations, unless otherwise specified, we always assume that the right-continuous Markov chain $\xi(t)$ takes values on state space $\mathbb{S} = \{1,2\}$ and the values of parameters are listed in table 5.1, the initial condition is $\left(P_1(0), P_2(0), Z(0)\right) = (1,2,1)$ and other parameters are chosen as control variables.

5.4.1. Impact of regime switching on the dynamics of system (5-1)

In order to study how the white noise, regime switching and TPP affect the dynamics of system (5-1), we firstly consider that there is no regime switching in system (5-1). Fixed $\big(\gamma(1),\gamma(2)\big)=(0.2,0.3)$ and choose $\big(\sigma_1(1),\sigma_1(2)\big)=(1.5,1.5)$, $\big(\sigma_2(1),\sigma_2(2)\big)=(1.3,1.4)$, $\big(\sigma_3(1),\sigma_3(2)\big)=(0.7,0.8)$. By direct computation, the sufficient conditions of the extinction for both Subsystem 1 and 2 are easily to verify. Thus, all the species of the Subsystems 1 and 2 tend to extinct (see Fig. 5.1). Furthermore, suppose that the generator Q of the Markov chain $\xi(t)$ is $Q=\begin{pmatrix}-\frac{1}{8} & \frac{1}{8}\\ \frac{1}{12} & -\frac{1}{12}\end{pmatrix}$, by the irreducible property, we can easily to obtain that the stationary distribution of $\xi(t)$ is $\pi=(0.4,0.6)$, which satisfies the conditions of $A<0$, $B<0$, and $C<0$, then all the species of system (5-1) undergo extinction (see Fig. 5.1). This result suggests that the regime switching cannot change the extinction behavior of system (5-1) in this case, that is, system (5-1) is extinct if both two Subsystems die out simultaneously.

Table 5.1 Parameter values

Parameters	Biological meaning	Units	Values
r_1	The intrinsic growth rate of NTP	day^{-1}	$r_1(1)=0.8, r_1(2)=0.85$
K_1	Environmental carrying capacity of NTP	$\mathrm{nos.ml}^{-1}$	$K_1(1)=1.8, K_1(2)=3.8$
r_2	The intrinsic growth rate of TPP	day^{-1}	$r_2(1)=0.65, r_2(2)=0.8$
K_2	Environmental carrying capacity of TPP	$\mathrm{nos.ml}^{-1}$	$K_2(1)=2.5, K_2(2)=3.5$
a_1	The competitive effect of TPP on NTP	ml $\mathrm{nos.}^{-1}\,\mathrm{day}^{-1}$	$a_1(1)=0.01, a_1(2)=0.01$
a_2	The competitive effect of NTP on TPP	ml $\mathrm{nos.}^{-1}\,\mathrm{day}^{-1}$	$a_2(1)=0.28, a_2(2)=0.08$
α	The attack rate of zooplankton on NTP	ml $\mathrm{nos.}^{-1}\,\mathrm{day}^{-1}$	$\alpha(1)=0.85, \alpha(2)=0.95$

β	The attack rate of zooplankton on TPP	ml nos.$^{-1}$ day^{-1}	$\beta(1) = 0.08, \beta(2) = 0.01$
b_1	The mutual interference between individuals of NTP	nos. ml^{-1}	$b_1(1) = 0.2, b_1(2) = 0.1$
b_2	The mutual interference between individuals of TPP	nos. ml^{-1}	$b_2(1) = 0.5, b_2(2) = 0.8$
δ	The conversion efficiency of NTP into zooplankton	ml nos.$^{-1}$ day^{-1}	$\delta(1) = 0.6, \delta(2) = 0.75$
d	The death rate of zooplankton	day^{-1}	$d(1) = 0.28, d(2) = 0.2$

On the other hand, to illustrate the effect of regime switching on the dynamics of system (5-1), we choose $(\sigma_1(1), \sigma_1(2)) = (0.1, 0.05)$, $(\sigma_2(1), \sigma_2(2)) = (0.1, 0.05)$, $(\sigma_3(1), \sigma_3(2)) = (0.1, 1.8)$ and all other parameters remain unchanged. By a simple computation, we can easily verify the conditions of $D_1 > 0$, $D_2 > 0$ and $D_3 < 0$, implying that both species $P_1(t)$ and $P_2(t)$ of system (5-1) are persistence in the mean, while species $Z(t)$ tends to extinction, as shown in Fig. 5.2. From Fig. 5.2, it is clear that Subsystems 1 and 2 have different persistence-extinction behaviors and system (5-1) can switch from one state to another state due to the regime shift, which implies that regime switching can balance the density of the population under different regimes. Significantly, it should be pointed out that the zooplankton of system (5-1) is extinct due to the extinction of zooplankton in Subsystem 1. This indicates that the regime switching may not change persistence-extinction behaviors in this case. So the question may arise: what role does the regime switching play in the dynamics of system (5-1)? Changing the generator Q to $Q = \begin{pmatrix} -\frac{p}{100} & \frac{p}{100} \\ \frac{100-p}{100} & -\frac{100-p}{100} \end{pmatrix}$ by controlling the value of p $(0 \leq p \leq 100)$, it is easy to obtain that stationary distribution of $\xi(t)$ is $\pi = (\pi_1, \pi_2) = \left(\frac{100-p}{100}, \frac{p}{100}\right)$. From Fig. 5.3 (a), one can observe that with the increasing value of p, the dynamical behaviors of species $Z(t)$ change from persistence in the mean to extinction in different areas of I, II and III

and Fig. 5.3 (b) depicts the dynamical behaviors of species $Z(t)$ with respect to Fig. 5.3 (a) for $p = 1, p = 15$ and $p = 25$, respectively. Taking $p = 1$ for example, we can see from the Fig. 5.4 that system (5-1) becomes persistence in the mean, whereas other Subsystems remain unchanged and almost all of the sample trajectories of system (5-1) are in that of Subsystem 2 due to $\pi_1 > \pi_2$. This means that plankton species can choose a better living environmental state to survive due to Markov chain. For the case of $p = 25$, we can obtain that the zooplankton of system (5-1) becomes extinction again (Figures here are not given due to the similarity to Fig. 5.4). Thus, under the effect of the regime switching, we can obtain the result from Figs. 5.3 and 5.4 that even if one population undergoes extinction in one state, it will become persistence in the mean in another state because of its staying longer in a better living environmental state. Therefore, it can be asserted that the regime switching can change the persistence-extinction behaviors of system (5-1) and the distribution of Markov chain $\xi(t)$ is beneficial to the survival of plankton.

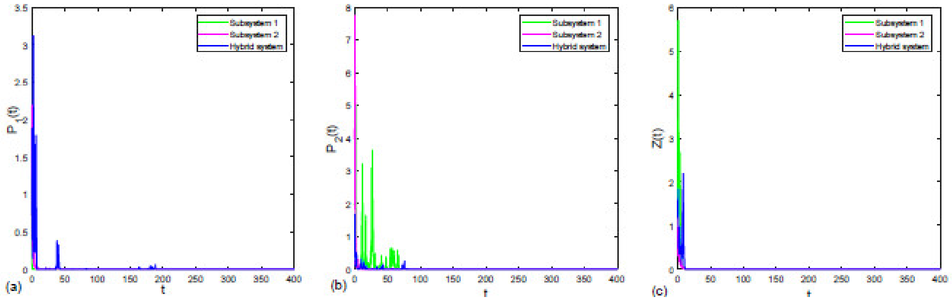

Fig. 5.1 (a), **(b)** and **(c)** denote the solution trajectories of $P_1(t), P_2(t)$ and $Z(t)$ for system (5-1) with $(\gamma(1), \gamma(2)) = (0.2, 0.3)$ and $(\sigma_1(1), \sigma_1(2)) = (1.5, 1.5), (\sigma_2(1), \sigma_2(2)) = (1.3, 1.4), (\sigma_3(1), \sigma_3(2)) = (0.7, 0.8)$, respectively.

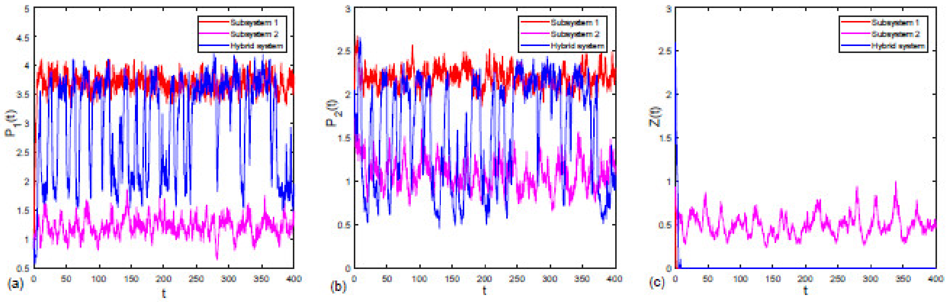

Fig. 5.2 (a), **(b)** and **(c)** denote the solution trajectories of $P_1(t), P_2(t)$ and $Z(t)$ for system (5-1) with $(\gamma(1), \gamma(2)) = (0.2,0.3), (\sigma_1(1), \sigma_1(2)) = (0.1,0.05), (\sigma_2(1), \sigma_2(2)) = (0.1,.05),$ and $(\sigma_3(1), \sigma_3(2)) = (0.1,1.8)$, respectively.

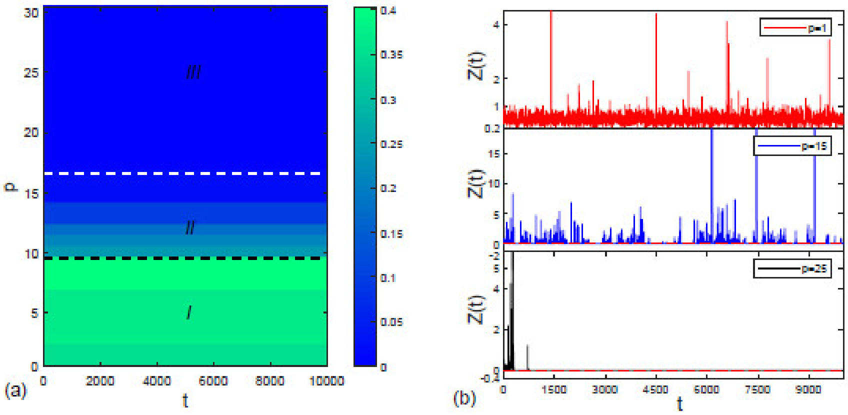

Fig. 5.3 The effect of regime switching on the stochastic behaviors of zooplankton species $Z(t)$ for system (5-1). **(a)** denotes the stochastic behaviors between extinction and persistence in the mean zooplankton species $Z(t)$ for system (5-1) with different values of p in different areas of I, II, III and other parameters as in Fig. 5.2; **(b)** denotes the solution trajectories of zooplankton species $Z(t)$ with respect to Fig. 4.3 (a) for $p = 1, p = 15$ and $p = 25$, respectively.

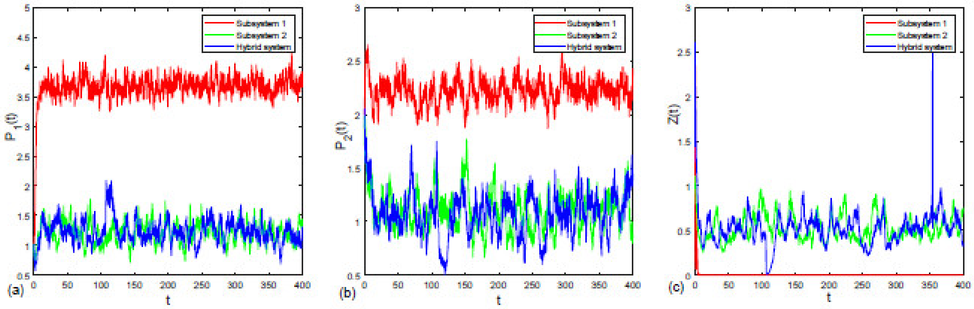

Fig. 5.4 (a), **(b)** and **(c)** denote the solution trajectories of $P_1(t), P_2(t)$ and $Z(t)$ for system (5-1) with $(\gamma(1), \gamma(2)) = (0.2,0.3)$ and $(\sigma_1(1), \sigma_1(2)) = (0.1,0.05)$, $(\sigma_2(1), \sigma_2(2)) = (0.1,.05)$, $(\sigma_3(1), \sigma_3(2)) = (0.1,1.8)$, respectively.

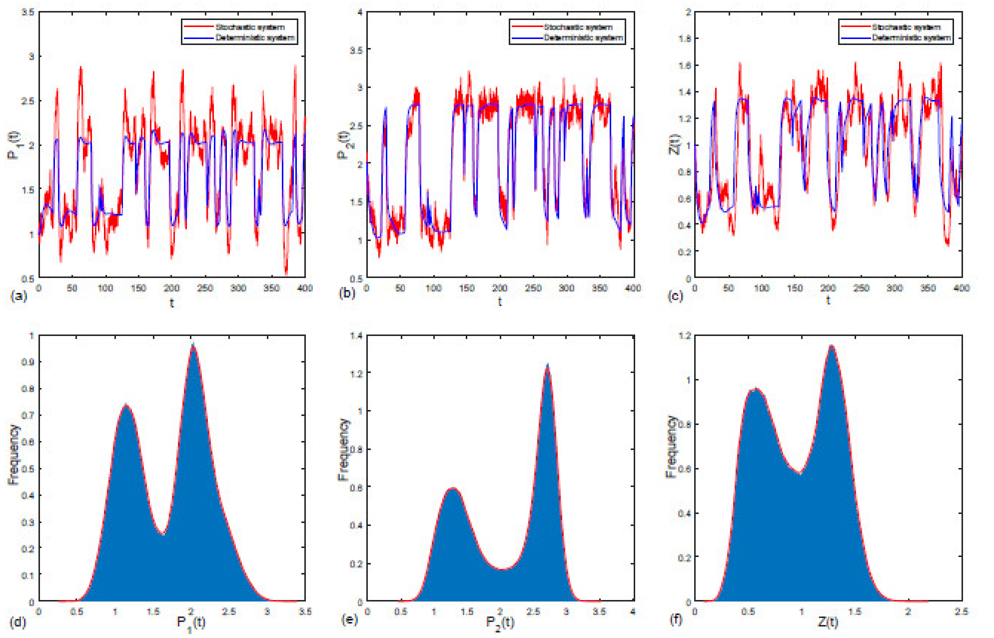

Fig. 5.5 (a), **(b)** and **(c)** denote the solution trajectories of system (5-1) and its corresponding deterministic counterparts, and **(d)**, **(e)** and **(f)** denote the density function diagrams of system (5-1) with $(\gamma(1),\gamma(2)) = (0.2,0.3)$, $(\sigma_1(1),\sigma_1(2)) = (0.1,0.05)$, $(\sigma_2(1),\sigma_2(2)) = (0.1,.05)$, $(\sigma_3(1),\sigma_3(2)) = (0.1,0.05)$ in regimes $\xi = 1$ and $\xi = 2$, respectively.

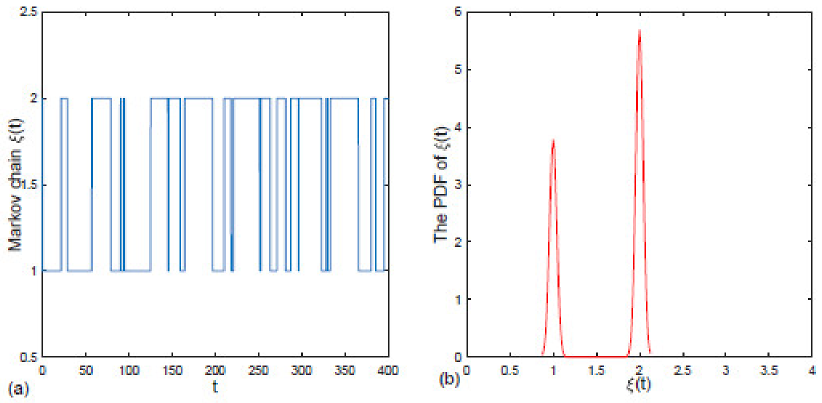

Fig. 5.6 (a) denotes the movement of Markov chain in the state $\mathbb{S} = \{1,2\}$ over time. **(b)** denotes the probability density function (PDF) of $\xi(t)$.

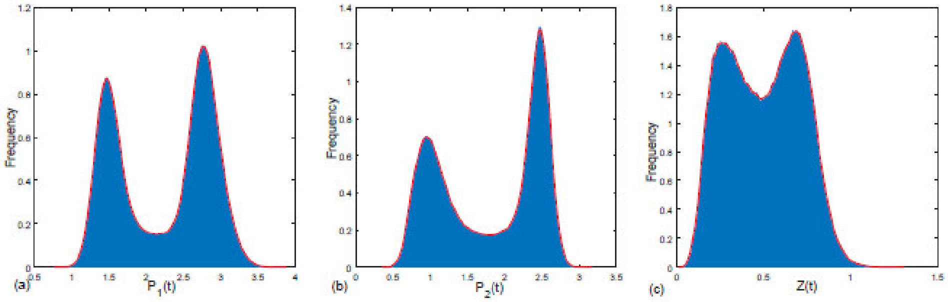

Fig. 5.7 The effect of the toxin released rate γ on the probability density function of system (5-1). **(a)**, **(b)** and **(c)** denote the histograms of probability density function for $P_1(t), P_2(t)$ and $Z(t)$ of system (5-1) with $(\gamma(1), \gamma(2)) = (0.5, 0.55)$, respectively.

5.4.2. Impact of white noise on the dynamics of system (5-1)

Does the white noise affect the survival of plankton populations when the system has one alternative stable state? In the following, we see the impact of the white noise on the persistence-extinction properties of system (5-1). Choosing $\big((\sigma_1(1), \sigma_1(2)) = (0.1, 0.05), (\sigma_2(1), \sigma_2(2)) = (0.1, 0.05),\ (\sigma_3(1), \sigma_3(2)) = (0.1, 0.05)\big)$, it can been seen from Fig. 5.5 that the NTP, TPP and zooplankton populations can coexist at a relatively stable state when the intensities of white noise are comparatively small and all other parameters remain unchanged as in Fig. 5.1. Actually, according to $\Pi > 0$, system (5-1) has a unique ergodic stationary distribution, which are consistent with our experimental simulation. Moreover, it is clear to see from Fig. 5.4 (a)-(c) that the white noise keeps the stochastic processes $P_1(t), P_2(t)$ and $Z(t)$ moving up and down randomly and the solution (the red lines) of system (5-1) fluctuates in a small neighborhood of that (the blue lines) of its corresponding deterministic system. Thus, we can obtain that white noise can affect the distribution of phytoplankton and zooplankton populations. That is, white noise can significantly affect the dynamic evolution mechanism of plankton populations. Significantly, we can observe from Fig. 5.4 (d)-(f) that the probability density functions of NTP, TPP and zooplankton have two wave curves that are corresponding to the two states $\mathbb{S} = \{1, 2\}$ of the Markov switching, respectively. Comparing Figs 5.1, 5.2 and 5.5, it is obvious that a high density of white noise can accelerate the extinction of the plankton populations and be advantageous to the rapid disappearance of planktonic blooms, which may help us control the density of plankton populations in real aquatic ecosystems. Therefore, it can be asserted that the plankton systems incorporating white noise can better simulate planktonic blooms than its corresponding deterministic counterparts. Fig. 5.6 describes that system (5-1) switches from one state $\xi = 1$ to another state $\xi = 2$ by the law of Markov chain $\xi(t)$ over time.

5.4.3. Impact of TPP on the dynamics of system (5-1)

The influence of the toxin liberation rate produced by TPP under the effects of the white noise and regime switching is also studied. Choosing $\left(\gamma(1), \gamma(2)\right) = (0.5, 0.55)$, a simple computation shows that $\Pi > 0$, which means that system (5-1) has a unique stationary distribution and the probability density functions of NTP, TPP and zooplankton populations have two wave curves due to the regime shift (see Fig. 5.7). Comparing Figs. 5-5 and 5-7, we can observe that the peak values of the probability density functions for $P_1(t), P_2(t)$ and $Z(t)$ of system (5-1) are higher than that in the earlier case $\left(\left(\gamma(1), \gamma(2)\right) = (0.2, 0.3)\right)$. In addition, we can also observe that with the increasing value of γ, the mean values of $P_1(t)$ and $P_2(t)$ of system (5-1) are getting larger, while that of $Z(t)$ is becoming smaller. Therefore, it is clear that the introduction of TPP can be benefic to the persistence in the mean of three species through the termination of planktonic blooms and may be used as a controlling agent to control planktoic blooms.

5.5. Conclusions

The occurrences of harmful phytoplankton blooms have been reported globally with an increasing frequency in the past decades (Hallegraeff, 1993), and TPP are among the contributors in these blooms (Hallegraeff, 1993; Philips et al., 2004; Hallam and Luna, 1984). Moreover, plankton populations in real aquatic ecosystems often fluctuate unpredictably because of environmental stochasticity, which plays an important role in the ecosystems (Carpenter, 2011). In order to better understand the effects of environmental fluctuations and TPP on the dynamics of plankton systems, in this thesis, we propose a stochastic phytoplankton-toxic producing phytoplankton-zooplankton system with Beddington-DeAngelis functional response, which incorporates with white noise and regime switching, and study how these factors affect the dynamics of system (5-1) analytically and numerically. We firstly investigate the existence and uniqueness of global positive solutions, and then derive some sufficient conditions for the extinction and persistence in the mean of system (5-1). To prove the existence of the stationary distribution, the theory of Khasminskii (Khasminskii, 1980) for periodic Markov process and a method based on constructing a Lyapunov function are employed. Numerical analysis illustrates our theoretical results and further indicates that the white noise, regime switching and TPP play an important role in controlling planktonic blooms as follows:

- Regime switching plays an important role in the balance of the different survival states of plankton populations. On one hand, Subsystem 1, Subsystem 2 and system (5-1) have the same persistence-extinction behaviors. Actually, the regime switching cannot change the persistence-extinction behaviors of these systems (see Figs. 5.1 and 5.5), which means that system (5-1) still becomes persistence in the mean (or extinction) if Subsystems 1 and 2 becomes persistence in the mean (or extinction). On the other hand, the persistence-extinction behaviors of system (5-1) rely heavily on that of Subsystem 1 and Subsystem 2 due to the role of regime shifts. In the case of Subsystem 1 is persistence in the mean and Subsystem 2 dies out, then system (5-1) will tend to extinction (see Fig. 5.2). However, system (5-1) becomes persistence in the mean although one Subsystem is extinct by controlling the value of p (see Fig. 5.4). Thus, the presence of regime switching in the stochastic system can change the survival of plankton populations and reduce the risk of extinction. Therefore, it can be asserted that whether the regime switching is conducive to the survival of plankton populations or not strongly depends on its staying longer in a 'good' or 'bad' environmental state.

- White noise is adverse to the survival of plankton populations. As the Fig. 5.5 points out, if the white noise densities are relatively small satisfying the conditions of Theorem 5.3.4.1, then system (5-1) has a unique ergodic stationary distribution, which means the NTP, TPP and zooplankton can coexist at a stable state for a long time. However, by enhancing the intensity of white noise on zooplankton only or on all three species simultaneously, the zooplankton of Subsystem 2 or all the three species of every system will go to extinction (see Figs. 5.1 and 5.2). From Figs. 5.1, 5.2 and 5.5, by controlling the intensity of white noise, the dynamic behaviors of system (5-1) can be significantly changed. That is, high intensity of white noise is disadvantageous to the development of plankton and increases the risk of extinction. This is ecologically meaningful as the species deteriorates drastically because of high environmental fluctuations. Thus, it is obliged to be stressed that the controlling of the white noise may be acted as a possible biological way to control planktonic blooms.

- TPP can increase the survival chance of phytoplankton but reduce the biomass of zooplankton. With the increasing value of the toxin liberation rate γ ensuring the condition of $\Pi > 0$, system (5-1) has a unique stationary distribution (see Figs. 5.7), which describes the long time asymptotic behaviors of the system (5-1) from a statistical viewpoint. Additionally, comparing Figs. 5.5 and 5.7, we can conclude that the toxin liberation rate is conducive to the persistence in the mean of phytoplankton but is adverse to the survival of zooplankton population. Therefore, TPP plays an important role in controlling planktonic blooms.

Chapter 6 DYNAMICS OF A STOCHASTIC NON-AUTONOMOUS PHYTOPLANKTON-ZOOPLANKTON SYSTEM INVOLVING TOXIN-PRODUCING PHYTOPLANKTON AND IMPULSIVE CONTROL

Abstract

This thesis describes an analytical and numerical investigation of a stochastic non-autonomous phytoplankton-zooplankton system involving TPP and impulsive perturbations. White noise, impulsive perturbations, and TPP were incorporated into the model to stimulate natural aquatic ecological phenomena. The aim of this thesis was to analyze how these factors affect the dynamics of the system. Mathematical derivations were utilized to investigate some key threshold conditions that ensure the existence and uniqueness of a global positive solution, population extinction, and persistence in the mean. In particular, we determined if there is a positive periodic solution for the system when the toxin liberation rate reaches a critical value. The numerical results indicated that both white noise and the impulsive control parameter can directly influence population extinction and persistence in the mean. Enhancing the toxin liberation rate of TPP increases the possibility of phytoplankton survival but reduce zooplankton biomass. These results improve our understanding of the dynamics of complex of aquatic ecosystems in a fluctuating environment.

Keywords: Stochastic phytoplankton-zooplankton system; Toxin-producing phytoplankton; White noise; Impulsive perturbations; Extinction; Periodic solution

6.1. Introduction

It is widely recognized that plankton populations play an important role in the wealth of aquatic ecosystems and ultimately the planet itself (Huppert et al., 2002). Phytoplankton not only generate organic compounds by absorbing carbon dioxide dissolved from their surroundings, but also carry out photosynthesis, which may have a significant effect on large-scale global processes such as the global carbon cycle, climate change and ocean-atmosphere dynamics (Subhendu et al., 2015). However, phytoplankton blooms, a natural phenomenon involving the rapid increase and almost equally rapid decrease of a certain dominant phytoplankton species in aquatic ecosystems, occur frequently and can persist under certain conditions, disrupting the ecological balance of aquatic ecosystems and becoming detrimental to public health. Harmful algal blooms (HABs), for example, have been widely reported and have become a serious environmental problem worldwide (Anderson, 1997). To understand the mechanisms of planktonic blooms and to regulate the ecological balance of plankton ecosystems, it is crucial to perform a deeper analysis of the function of aquatic ecosystems, and especially the dynamics of plankton populations. Mathematical models can be a powerful tool for revealing plankton dynamics. Indeed, since the pioneering work of Riley et al. (1949), mathematical models have been developed to reveal the dynamic behaviors of plankton populations. In particular, researches on phytoplankton-zooplankton systems have made great progress, in which various biological factors have been taken into account, yielding important results in recent years (Abhijit et al., 2021; Zhao et al., 2016; Han and Dai, 2019; Agnihotri and Kaur, 2019; Zhao et al., 2018).

Toxin-producing phytoplankton (TPP) are a well-known group of phytoplankton that have the ability to release toxic chemicals into aquatic environments during HABs, which may inhibit predation pressure from zooplankton and other predator populations in planktonic systems (Huppert et al., 2002; Falkowski, 1984; Colin and Dam, 2003; Fulton and Paerl, 1987). For example, the results from experimental observations (Colin and Dam, 2003) indicated that the toxic dinoflagellate *Alexandrium fundyense* can negatively affect the growth rate of the copepod *Acartia hudsonica*, and the toxic effects may have profound implications on the ability of grazers to control HABs. Moreover, some experimental evidence has revealed that TPP are one of the contributors to the formation of HABs (Hallegraeff, 1993; Philips et al., 2004), which play a significant role in determining the dynamics of plankton populations. For example, Roy et al. (2007) suggested that the presence of TPP might be a potential reason for the generation of complex interactions between phytoplankton and zooplankton populations, which could lead to the long-term coexistence of plankton populations in a fluctuating biomass. By analyzing a nutrient-plankton model, Jang and Allen (2015) concluded that TPP is conducive to the stability of planktonic interaction and plays a key role in bloom termination. Based on field observation and a model analysis, Chattopadhyay et al. (2002) proposed the following nonlinear coupled ordinary differential equations:

$$\begin{cases} \frac{dP(t)}{dt} = rP(t)\left(1 - \frac{P(t)}{K}\right) - \alpha f\big(P(t)\big)Z(t) \\ \frac{dZ(t)}{dt} = \beta f\big(P(t)\big)Z(t) - dZ(t) - \gamma g\big(P(t)\big)Z(t) \end{cases} \tag{6-1}$$

where $P(t)$ and $Z(t)$ are the population densities of TPP and zooplankton, respectively; r and K are the intrinsic growth rate and environmental carrying capacity of TPP, respectively; d denotes the natural death rate of zooplankton; α is the rate of predation and β is the conversion rate of zooplankton; γ is the TPP toxin liberation rate; $f(P)$ and $g(P)$ represent the predation response function and the distribution of toxic substances, respectively. A previous study (Chattopadhyay et al., 2002), investigated the existence and local stability of positive equilibria, and the existence of the Hopf-bifurcation of the system, by considering different combinations of functional response $f(P)$ and $g(P)$. It was concluded that TPP may act as a biological control for the termination of planktonic blooms. In recent years, many studies have incorporated TPP into phytoplankton-zooplankton systems by considering various factors, including time delay, plankton diffusion, infected phytoplankton, and phytoplankton refuge, yielding interesting results (Wang et al., 2014; Li et al., 2017; Jia et al., 2019; Agnihotri and Kaur, 2019).

It should be noted that most previous studies have focused on continuous systems; however, in the real world, populations undergo inherent discontinuity phenomena due to natural and anthropogenic factors, such as predation, planting, and harvesting, which lead to rapid population decrease or increase over a fixed period of time. Systems with such kinds of discontinuous changes relate to impulsive differential equations, which have attracted research attention over the past decades, since they provide a natural description of the observed evolutionary behavior of certain real-world problems (Zavalishchin and Sesekin, 1997; Samoilenko and Perestyuk, 1995). For example, many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics, and frequency modulation models can exhibit impulsive effects, and considerable research has been conducted into incorporating these impulsive effects into models (Tang and Chen, 2002; Shulgin et al., 1998; Yenicerioglu, 2019; Jatav and Dhar, 2014; Liu et al., 2019). Furthermore, some experimental observations shown that the toxic substances released by TPP is not a constant but change over time (Graneli and Johansson, 2003; Johansson and Graneli, 1999), and seasonal periodicity can either terminate or initiate periodic outbreaks of plankton populations (Mcgillicuddy et al., 2003; Phlips et al., 2004). Therefore, it is meaningful to introduce the impulsive effects and periodicity into the underlying system (6-1). As a result, system (6-1) can be extended into a non-autonomous impulsive system as follows:

$$\begin{cases} \left.\begin{aligned} \frac{dP(t)}{dt} &= r(t)P(t)\left(1-\frac{P(t)}{K(t)}\right) - \alpha(t)P(t)Z(t), \\ \frac{dZ(t)}{dt} &= \beta(t)P(t)Z(t) - d(t)Z(t) - \frac{\gamma(x)P(t)}{a(t)+P(t)}Z(t) \end{aligned}\right\} & t \neq t_k, k \in \mathbb{N}, \\ \left.\begin{aligned} P(t_k^+) &= (1+\lambda_k)P(t_k), \\ Z(t_k^+) &= (1+\mu_k)Z(t_k). \end{aligned}\right\} & t = t_k, k \in \mathbb{N}. \end{cases} \tag{6-2}$$

Note that $f(P)$ and $g(P)$ in system (6-1) adopt a linear and Holling II functional response in system (6-2), respectively, which is presented in the case 4 of a previous study (Chattopadhyay et al., 2002). Moreover, it is assumed that and system (6-2) is subjected to short-term external influences at a fixed time t_k, comprising a sequence of real numbers with $0 = t_0 < t_1 < \cdots < t_k < \cdots$, and $\lim_{k\to+\infty} t_k = +\infty$. From a biological perspective, we impose the following restriction on λ_k and μ_k:

$$1+\lambda_k > 0, \qquad 1+\mu_k > 0, \qquad k \in \mathbb{N}.$$

This means that the impulsive perturbations become a descriptive process of species introductions if $\lambda_k, \mu_k > 0$, while harvesting if $\lambda_k, \mu_k < 0$ (Liu and Wang, 2012).

On the other hand, planktonic systems in nature are inevitably influenced by environmental fluctuations due to the stochasticity and unpredictability of their surroundings. Regarding environmental fluctuations, for example, the availability of necessary nutrients, temperature, artificial eutrophication, and many other physical factors exist in real-world aquatic ecosystems, which may affect the growth rates of plankton populations. For example, some experimental evidence has shown that the growth rate of toxic *Microcystis* spp. and environmental biomass rely heavily on temperature and nutrient concentration (Davis et al., 2009; Fujimoto et al., 1997). The stochastic model analysis of Melbourne and Hastings (2008) demonstrated that the extinction risk in natural populations depends strongly on a combination of factors that contribute to stochasticity. Therefore, increasing attention has been paid to studying the influence of environmental fluctuations on aquatic ecosystems (Abhijit et al., 2021; Sarkar and Chattopadhayay, 2003; Ji et al., 2016; Zhao et al., 2017; Tapaswi and Mukhopadhyay, 1999). Thus, it is reasonable and valuable to further incorporate environmental fluctuations into system (6-2), which may provide a deeper insight of the dynamics of phytoplankton-zooplankton systems in fluctuating environments. Based on previous studies (Ji et al., 2016; Zhao et al., 2017), we assume that the intrinsic growth rate of HAB phytoplankton and the death rate of zooplankton are influenced by environmental factors, and we introduce white noise into the deterministic system (6-2), expressed as follows:

$$\begin{cases} \left.\begin{array}{l} dP(t) = P(t)\left[r(t)\left(1-\frac{P(t)}{K(t)}\right) - \alpha(t)Z(t)\right]dt \\ \qquad\qquad +\sigma_1(t)P(t)dB_1(t) \\ dZ(t) = Z(t)\left[-d(t) + \beta(t)P(t) - \frac{\gamma(t)P(t)}{a(t)+P(t)}\right]dt \\ \qquad\qquad +\sigma_2(t)Z(t)dB_2(t) \end{array}\right\} & t \neq t_k, k \in \mathbb{N}, \\ \left.\begin{array}{l} P(t_k^+) = (1+\lambda_k)P(t_k), \\ Z(t_k^+) = (1+\mu_k)Z(t_k) \end{array}\right\} & t = t_k, k \in \mathbb{N}. \end{cases} \tag{6-3}$$

where $B_i(t)$ and $dB_i(t)\ (i = 1,2)$ are mutually independent one-dimensional standard Brownian motion and standard white noise, respectively, and $\sigma_i^2(t) >$

$0\ (i = 1,2)$ is the white noise intensity.

In recent years, many researchers have investigated the combined effects of stochastic fluctuations and impulsive perturbations on population dynamics (Liu and Wang, 2012, 2013; Zhang et al., 2017; Wu et al., 2015; Zuo and Jiang, 2016; Zhang et al., 2015). For example, Liu and Wang (2012) studied a stochastic logistic system with impulsive perturbations, and obtained some conditions for the extinction, non-persistence in the mean, weak persistence, persistence in the mean and stochastic persistence of the system. Zuo and Jiang (2016) discussed the periodic solutions for a stochastic non-autonomous Holling-Tanner predator-prey system with impulses, and obtained some conditions for positive periodic solution. All these studies have stimulated further research into the dynamics of stochastic impulsive systems. However, studies investigating the effect of environmental fluctuations on the survival of plankton populations in an impulsive control environment with seasonal disturbance are lacking. Therefore, in this study, we investigated the dynamics of a stochastic non-autonomous phytoplankton-zooplankton system involving impulsive perturbations and TPP. The aim of the study was to determine the influence of impulsive perturbations, white noise, and TPP on the dynamics of this system (6-3). Our research questions included: (i) How do environmental noise and the impulsive control parameter affect the dynamics of plankton? (ii) What influences the peak of the cyclic outbreaks of planktonic blooms in an impulsive perturbation and fluctuating aquatic environment?

The rest of this paper is organized as follows: Section 6.2 presents the basic assumptions and useful lemma firstly, and then we investigate the existence and uniqueness of global positive solutions by establishing the equivalent system without impulses, apply Ito's formula to obtain the sufficient conditions for the extinction and persistence in the mean of the system, and the existence of positive τ-periodic solution are obtained by constructing a suitable stochastic Lyapunov function and using the theory of Khasminskii (2012). A series of numerical simulations are carried out to verify the theoretical analysis in section 6.3. In section 6.4, we summarize the results and present our conclusions.

6.2. Dynamic analysis

Environmental stochasticity and impulsive perturbations are the key factors affecting the real aquatic ecosystems, and the growth rates of phytoplankton and zooplankton may be affected by the environmental fluctuations (Tapaswi and Mukhopadhyay, 1999). Thus, it is interesting to study how these factors influence the survival of plankton populations. In this section, therefore, we investigate mainly the existence and uniqueness of global positive solutions, the extinction and persistence in the mean, and the existence of positive periodic solutions of the system (6-3).

6.2.1. Preliminaries

Let $\mathbb{N}, \mathbb{R}$ be the set of positive integers and real numbers, respectively, and denote $\mathbb{R}_+ = [0, +\infty)$. Throughout this paper, unless otherwise indicated, we always assume that $(\Omega, \mathcal{F}_t, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ is a completed probability space with a filtration $\{\mathcal{F}_t\}$ satisfying the usual normal conditions (i.e. it is right continuous and increasing, while $\{\mathcal{F}_0\}$ contains all $\mathcal{P}$-null sets). Moreover, it is assumed that a product equals unity if the number of factors is zero and the following hypotheses are given:

(H1) the functions $r(t), K(t), \alpha(t), \beta(t), d(t), a(t), \gamma(t), \sigma_1(t)$, and $\sigma_2(t)$ are all bounded positive-valued functions on $\mathbb{R}_+ = [0, +\infty)$;

(H2) the functions $r(t), K(t), \alpha(t), \beta(t), d(t), a(t), \gamma(t), \sigma_1(t)$, and $\sigma_2(t)$ are periodic with a common period $\tau > 0$;

(H3) there exists a positive integer p such that $t_{k+p} = t_k + \tau$, $\lambda_{k+p} = \lambda_k$, $\mu_{k+p} = \mu_k$, $k \in \mathbb{N}$ and $[0, \tau) \cap \{t_k, k \in \mathrm{N}\} = \{t_1, t_2, \cdots, t_p\}$.

For convenience, we introduce the following notations. If $\varphi(t)$ is a continuous τ-periodic on $\mathbb{R}_+$, we define $\varphi^u = \max_{t\in\mathbb{R}_+}\varphi$, and $\varphi^l = \min_{t\in\mathbb{R}_+}\varphi$, and if it is an integrable function, we define $\langle\varphi\rangle_t = \frac{1}{t}\int_0^t \varphi(s)ds, t > 0$. Moreover,

$$h_1(t) = \langle r(t) - \frac{\sigma_1^2(t)}{2}\rangle_\tau + \frac{1}{\tau}\sum_{j=1}^{p}\log(1 + \lambda_j)$$

and

$$h_2(t) = \frac{1}{\tau}\sum_{j=1}^{p} \log(1+\mu_j) - \langle d(t) + \frac{\sigma_2^2(t)}{2} \rangle_\tau.$$

Now, we introduce the Itô's formula for general stochastic differential equations and the definitions for the extinction and persistence in the mean of the system. We define the n-dimensional stochastic equation as follows (Zuo and Jiang, 2016):

$$dX = f(t,X)dt + g(t,X)dB(t) \tag{6-4}$$

with the initial value $X(t_0) = X_0$. Here, $f(t,X) = (f_1(t,X), f_2(t,X), \cdots, f_n(t,X))$ is a n-dimensional vector function; $(g(t,X))_{n\times l}$ is a $n \times l$ matrix function and $B(t) = (B_1(t), B_2(t), \cdots, B_l(t))$ is a l-dimensional standard Brownian motion defined on the probability space $(\Omega, \mathcal{F}_t, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$. We define the differential operator L associated with Eq. (6-4)

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i(t,X)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{h=1}^{l} g_{ih}(t,X)g_{jh}(t,X)\frac{\partial^2}{\partial x_i \partial x_j}.$$

Let function $V(t,X) \in \mathbb{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$, resulting in

$$LV(t,X) = \frac{\partial V}{\partial t} + \sum_{i=1}^{n} f_i(t,X)\frac{\partial V}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{h=1}^{l} g_{ih}(t,X)g_{jh}(t,X)\frac{\partial^2 V}{\partial x_i \partial x_j}.$$

Thus, the Itô's formula can be presented.

Lemma 6.2.1.1 (Tang et al., 2015; Mao, 2008) Let $X(t)$ satisfies Eq. (6-4) and the function $V(t,X) \in \mathbb{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$, then

$$dV(t,X) = LV(t,X)dt + V_X(t,X)g(t,X)dB(t).$$

where

$$V_X(t,X) = \left(\frac{\partial V(t,X)}{\partial x_1}, \frac{\partial V(t,X)}{\partial x_2}, \cdots, \frac{\partial V(t,X)}{\partial x_n}\right).$$

Definition 6.2.1.2 (Khasminskii, 2012) System (6-4) is said to be extinct if $\limsup_{t\to\infty} X(t) = 0$ *a.s.*

System (6-4) is said to be persistent in the mean if $\liminf_{t\to\infty} \langle X(t)\rangle_\tau > 0$ *a.s.*

This presents a lemma that describes criteria for the existence of the periodic solution of stochastic differential equations without impulses (Li and Xu, 2013; Jiang et al., 2008).

Lemma 6.2.1.3 Suppose that the coefficients of (6-4) are all continuous τ-periodic function in t and system (6-4) has a global solution, and further suppose that there exists a periodic function $V(t,X)\epsilon C^{1,2}(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$ satisfying the following conditions:

(i) there exists a constant M' such that $\mathcal{L}V(t,X) \leq 0,\ |X| \geq M'$;

(ii) $\inf_{|X|>R} V(t,X) \to \infty$ as $R \to \infty$.

Then the system (6-4) has a periodic solution.

We then consider the following periodic Logistic system:

$$d\phi(t) = \phi(t)\left(r(t) + \frac{1}{\tau}\sum_{j=1}^{p}(1+\lambda_j) - \frac{r(t)}{K(t)}A_1(t)\phi(t)\right)dt$$

$$+\sigma_1(t)A_1(t)\phi(t)dB_1(t) \tag{6-5}$$

with an initial value $\phi(0) = P_1(0) > 0$. By the Lemma 2.2 (Mao and Yuan, 2006) and the Lemma 2.1 (Liu et al., 2011), we can easily obtain that, if $h_1^l > 0$, then one can obtain that

$$\lim_{t\to\infty} \sup\langle\phi(t)\rangle_\tau = \frac{K^u h_1^u}{r^l A_1^l}. \tag{6-6}$$

6.2.2. Existence and uniqueness of global positive solutions

Before investigating the stochastic dynamics of system (6-3), we should first guarantee the existence of global positive solutions. Therefore, we will discuss the existence of global positive solutions in system (6-3) by constructing the equivalent system without impulses in the following.

Actually, suppose that the conditions (H1)-(H3) hold, and consider the following stochastic differential equation without impulses:

$$\begin{cases} dP_1(t) = P_1(t)\Big[r(t) + \frac{1}{\tau}\sum_{j=1}^{p}\log\big(1+\lambda_j\big) - \frac{r(t)}{K(t)}A_1(t)P_1(t) \\ \qquad -\alpha(t)A_2(t)Z_1(t)\Big]dt + \sigma_1(t)P_1(t)dB_1(t) \\ dZ_1(t) = Z_1(t)\Big[-d(t) + \frac{1}{\tau}\sum_{j=1}^{p}\log\big(1+\mu_j\big) + \beta(t)A_1(t)P_1(t) \\ \qquad -\frac{\gamma(t)A_1(t)P_1(t)}{a(t)+A_1(t)P_1(t)}\Big]dt + \sigma_2(t)Z_1(t)dB_2(t) \end{cases} \tag{6-7}$$

with the initial value $\big(P_1(0), Z_1(0)\big) = \big(P(0), Z(0)\big)$, where

$$A_1(t) = \big(\prod_{j=1}^{p}\big(1+\lambda_j\big)\big)^{-\frac{t}{\tau}}\prod_{0\le t_k<t}(1+\lambda_k),$$

and

$$A_2(t) = \big(\prod_{j=1}^{p}\big(1+\mu_j\big)\big)^{-\frac{t}{\tau}}\prod_{0\le t_k<t}(1+\mu_k).$$

The remainder of the proof follows that in the Theorem 2.2 of (Zuo and Jiang, 2016), here, we omit it. Therefore, for any given initial value $\big(P(0), Z(0)\big) \in \mathbb{R}_+^2$, the system (6-3) exists a unique solution $\big(P(t), Z(t)\big)$ on $t \in \mathbb{R}_+$ and positive solution will remain in $\mathbb{R}_+^2$ with probability one, namely $\big(P(t), Z(t)\big) \in \mathbb{R}_+^2$ for all $t > 0$ almost surely.

Remark 6.2.1.2 From the above equivalent in Theorem 6.2.2.1, we can conclude that we only need to consider the asymptotic properties of system (6-7). Then, system (6-3) has similar properties.

6.2.3. Extinction and persistence induced by impulsive control and white noise

From the perspective of population dynamics, it is necessary and important to predict and control the development of populations. Therefore, we are now in position to discuss the properties of extinction and persistence in the mean of system (6-3) in this subsection, and the following modeling analysis indicates that impulsive control and environmental fluctuations can significantly affect the dynamics behaviors of plankton populations, such as the extinction, persistence in the mean, and the existence of periodic solution of the system, indicating harvesting or artificial remove harmful algae is a better biological way to control algal growth in a fluctuating environment.

Actually, suppose that the conditions (H1)-(H3) hold. Applying Itô's formula to the first equation of system (6-7), we have

$$d\log P_1(\mathrm{t}) \leq \left[r(t) + \frac{1}{\tau}\sum_{j=1}^{p}\log(1+\lambda_j) - \frac{\sigma_1^2(t)}{2}\right]dt + \sigma_1(t)dB_1(t).$$

Integrating both sides of the above inequality on the interval $[0, t]$ yields

$$\frac{1}{t}[\log P_1(\mathrm{t}) - \log P_1(0)] \leq \frac{1}{t}\int_0^t \left[r(s) - \frac{\sigma_1^2(s)}{2}\right]ds$$
$$+\frac{1}{\tau}\sum_{j=1}^{p}\log(1+\lambda_j) + \frac{1}{t}\int_0^t \sigma_1(s)dB_1(s).$$

By the strong law of large numbers (Li et al., 2009), one can obtain that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \sigma_1(s)dB_1(s) = 0, a.s.$$

According to the condition (H2), we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \left[r(s) - \frac{\sigma_1^2(s)}{2}\right]ds = \frac{1}{\tau}\int_0^\tau \left[r(t) - \frac{\sigma_1^2(t)}{2}\right]dt.$$

By comparison principle, and if $h_1^u < 0$, we can easily to get

$$\lim_{t\to\infty}\sup\frac{\log P_1(t)}{t} \leq h_1^u < 0,$$

which implies $\lim\limits_{t\to\infty} P_1(t) = 0\ a.s.$, that is, the harmful phytoplankton species undergoes extinction.

Similarly, it can be obtained that

$$\frac{1}{t}[\log Z_1(\mathrm{t}) - \log Z_1(0)] \leq \frac{1}{\tau}\sum_{j=1}^{p}\log(1+\lambda_j) - \frac{1}{t}\int_0^t \left[d(s) + \frac{\sigma_2^2(s)}{2}\right]ds$$
$$+\frac{1}{t}\int_0^t \beta(s)A_1(s)P_1(s)ds + \frac{1}{t}\int_0^t \sigma_2(s)dB_2(s).$$

Since $\lim\limits_{t\to\infty} P_1(t) = 0\ a.s.$, and $\lim_{t\to\infty}\frac{1}{t}\int_0^t \sigma_2(s)dB_2(s) = 0, a.s.$ (Mao and Yuan, 2006), we have

$$\lim_{t\to\infty}\sup\frac{\log Z_1(t)}{t}\le h_2^u<0$$

provided that $h_2^u<0$. In other words, $\lim_{t\to\infty}Z_1(t)=0\ a.s.$

To sum up, for any given initial value $\big(P(0),Z(0)\big)\in\mathbb{R}_+^2$, we can conclude that if $h_1^u<0$ and $h_2^u<0$, both harmful phytoplankton $P(t)$ and zooplankton $Z(t)$ undergo extinction almost surely.

Now, we analyze the case of toxic phytoplankton which is persistence in the mean, while the zooplankton undergoes rapid extinction by artificial activities in a fluctuating environment.

It is obvious to find that the conditions (H1)-(H3) is satisfied. Note that

$$dP_1(t)\le P_1(t)\left(r(t)+\tfrac{1}{\tau}\textstyle\sum_{j=1}^{p}\log\big(1+\lambda_j\big)-\tfrac{r(t)A_1(t)}{K(t)}P_1(t)\right)dt$$

$$+\sigma_1(t)A_1(t)P_1(t)dB_1(t).$$

By the comparison principle of stochastic differential equation and (6-5), we have

$$\lim_{t\to\infty}\sup\langle P_1(t)\rangle_\tau\le\tfrac{K^u h_1^u}{r^l A_1^l}\,a.s. \tag{6-8}$$

Taking Itô's formula to the second equation of system (6-7), we one obtain that

$$\tfrac{1}{t}[\log Z_1-\log Z(0)]\le h_2(t)+\tfrac{1}{t}\textstyle\int_0^t\beta(s)A_1(s)P_1(s)\,ds+\tfrac{1}{t}\int_0^t\sigma_2(s)dB_2(s).$$

Together with (6-8) and $\lim_{t\to\infty}\frac{1}{t}\int_0^t\sigma_2(s)dB_2(s)=0,a.s.$ we have

$$\lim_{t\to\infty}\sup\tfrac{\log Z_1}{t}\le h_2^u+\tfrac{K^u\beta^u h_1^u}{r^l A_1^l}<0$$

provided that the condition $h_2^u+\frac{K^u\beta^u h_1^u}{r^l A_1^l}<0$ holds. That is, $\lim_{t\to\infty}Z_1(t)=0\ a.s.$, which means the species $Z_1(t)$ dies out. Therefore, for any given initial value $\big(P(0),Z(0)\big)\in\mathbb{R}_+^2$, if $h_1^l>0$ and $h_2^u+\frac{K^u\beta^u h_1^u}{r^l A_1^l}<0$, then the species $Z(t)$ will tend to extinction almost surely.

In the previous discussion, we have obtained that the cases of population extinction. Then a certain question may arise and what people usually are interested in is how does all the species of the system coexist in real life under some suitable conditions? We now give a another discussion.

Considering the conditions (H1)-(H3) is satisfied. Applying Itô's formula to the first equation of system (6-7) and (6-5), we have

$$\frac{1}{t}[\log P_1(\mathrm{t}) - \log P(0)] = h_1(t) - \frac{1}{t}\int_0^t \frac{r(s)A_1(s)}{K(s)} P_1(s)\, ds$$
$$- \frac{1}{t}\int_0^t \alpha(s)A_2(s)Z_1(s)\, ds + \frac{1}{t}\int_0^t \sigma_1(s) dB_1(s),$$
$$\frac{1}{t}[\log \Phi(\mathrm{t}) - \log \Phi(0)] = h_1(t) - \frac{1}{t}\int_0^t \frac{r(s)A_1(s)}{K(s)} \Phi(s)\, ds + \frac{1}{t}\int_0^t \sigma_1(s) dB_1(s).$$

Thus, we have

$$0 \geq \frac{1}{t}[\log P_1(\mathrm{t}) - \log \Phi(t)] = \frac{1}{t}\int_0^t \frac{r(s)A_1(s)}{K(s)} \big(\Phi(s) - P_1(s)\big)\, ds$$
$$- \frac{1}{t}\int_0^t \alpha(s)A_2(s)Z_1(s)\, ds.$$

That is,

$$\frac{1}{t}\int_0^t \big(\Phi(s) - P_1(s)\big)\, ds \leq \frac{\alpha^u K^l A_2^u}{r^l A_1^l} \cdot \frac{1}{t}\int_0^t Z_1(s)\, ds. \tag{6-9}$$

Combining (6-9) and applying Itô's formula to the second equation of (6-7), we have

$$\frac{1}{t}\frac{\log Z_1(\mathrm{t})}{\log Z(0)} \geq h_2^l - \gamma^u + \frac{1}{t}\int_0^t \beta(s)A_1(s)\Phi(s)\, d - \frac{1}{t}\int_0^t \beta(s)A_1(s)\big(\Phi(s) - P_1(s)\big)\, ds$$
$$+ \frac{1}{t}\int_0^t \sigma_2(s)\, dB_2(s) \geq h_2^l - \gamma^u + \frac{\beta^l A_1^l}{t}\int_0^t \Phi(s)\, ds$$
$$- \frac{\alpha^u \beta^u K^l A_2^l}{r^u} \cdot \frac{1}{t}\int_0^t Z_1\,(s) ds + \frac{\sigma_2^l B_2(t)}{t}.$$

By the definition of superior limit of (6-6), that is, $\forall\, \delta > 0$, there exists $\tau_0 > 0$ such that

$$\frac{1}{t}\int_0^t \Phi(s)\, ds > \frac{K^u h_1^u}{r^l A_1^l} - \frac{\delta}{\beta^l A_1^l}$$

for $t > \tau_0$. Thus, we have

$$\frac{1}{t}\frac{\log Z_1(\mathrm{t})}{\log Z(0)} \geq h_2^l - \gamma^u + \frac{\beta^l K^u h_1^u}{r^l} - \delta - \frac{\alpha^l \beta^u K^l A_2^l}{r^l} \cdot \frac{1}{t}\int_0^t Z_1\,(s) ds + \frac{\sigma_2^l B_2(t)}{t}.$$

Thus,

$$\log Z_1(t) \geq \left(h_2^l - \gamma^u + \frac{K^u h_1^u}{r^l} - \delta\right) t - \frac{\alpha^l \beta^u K^l A_2^l}{r^u} \int_0^t Z_1(s) ds + \sigma_2^u dB_2(t).$$

According to the Lemma 4 (Li et al. (2009)) and for the arbitrariness of δ, if the condition $\left(h_2^l - \gamma^u\right) r^l + \beta^l K^u h_1^u$ holds, we have

$$\liminf_{t\to\infty} \langle Z_1(t)\rangle_\tau \geq \frac{r^u\left[\left(h_2^l - \gamma^u\right) r^l + \beta^l K^u h_1^u\right]}{\alpha^l \beta^u K^l A_2^l} > 0$$

for any given initial value $\left(P(0), Z(0)\right) \in \mathbb{R}_+^2$, the system (6-3) is persistent in the mean provided that $h_1^l > 0$ and $(h_2^l - \gamma^u) r^l + \beta^l K^u h_1^u > 0$.

6.2.4. Periodic oscillations of plankton density

For the phytoplankton-zooplankton system, the existence of positive periodic solution of the system implies that the populations are persistent. In this subsection, we prove that the system admits a positive τ-periodic solution by constructing a suitable Lyapunov function and using the theory of Khasminskii (Khasminskii, 2012).

It is sufficient to show that a periodic solution exists for the equivalent system (6-7) without impulses. According to the previous discussion, we have obtained that, for any given value $\left(P(0), Z(0)\right) \in \mathbb{R}_+^2$, the system (6-3) exists a unique solution.

Construct a $\mathbb{C}^2$-function $V(t, P_1, Z_1)\colon \mathbb{R}_+^2 \to \mathbb{R}$ defined by

$$\begin{aligned} V(t, P_1, Z_1) = C\left(-\beta^l \ln P_1 - \frac{r^u}{K^l} \ln Z_1 + W_1(t) + \frac{\beta^u \alpha^u A_2^u}{B} Z_1\right) + \\ e^{W_2(t)} \frac{\left(\beta^u A_1^u P_1 + \alpha^l A_2^l Z_1\right)^2}{2} \\ \triangleq V_1(t, P_1, Z_1) + V_2(t, P_1, Z_1), \end{aligned}$$

where $C > 0$ will be determined later and $W_i(t)\ (i = 1,2)$ satisfy

$$\begin{aligned} W'_1(t) = \beta^l \left(r(t) - \frac{\sigma_1^2(t)}{2}\right) - \frac{r^u}{K^l}\left(d(t) - \frac{\sigma_2^2(t)}{2} - \gamma^u\right) \\ - \langle \beta^l \left(r(t) - \frac{\sigma_1^2(t)}{2}\right) - \frac{r^u}{K^l}\left(d(t) + \frac{\sigma_2^2(t)}{2} - \gamma^u\right)\rangle_\tau, \end{aligned}$$

and

$$W'_2(t) = 2d(t) - \sigma_2^2(t) - \langle 2d(t) - \sigma_2^2(t)\rangle_\tau.$$

Obviously, we can verify that $W_i(t)\ (i = 1,2)$ are both τ-periodic functions and

$W'_2(t)$ is a bounded function. Thus, there exists a constant $N > 0$ such that $|W'_2(t)| \leq N$ for all $t \geq 0$. To proof the condition (ii) of the Lemma 6.2.1.3, we only need to prove that

$$\inf_{(t,P_1,Z_1)\in[0,+\infty)\times(\mathbb{R}_+^2 \backslash U_k)} V(t,P_1,Z_1) \to \infty$$

as $k \to \infty$, where $U_k = \left(\frac{1}{k}, k\right) \times \left(\frac{1}{k}, k\right)$, which is clearly established since all the coefficients of the quadratic term in $V(t,P_1,Z_1)$ are positive, and thus, we have proved that the condition (ii) of the Lemma 6.2.1.3 is satisfied.

Next, we will verify that the condition (i) of Lemma 6.2.1.3 is satisfied. Applying Itô's formula to $V_1(t,P_1,Z_1)$ and $V_2(t,P_1,Z_1)$, respectively, one can obtain that

$$\begin{aligned}
LV_1(t,P_1,Z_1) \leq C &\left[-\beta^l\left(r(t) + \frac{1}{\tau}\sum_{j=1}^{p}\log(1+\lambda_j) - \frac{1}{2}\sigma_1^2(t)\right)\right. \\
&\left. + \frac{r^u}{K^l}\left(d(t) - \frac{1}{\tau}\sum_{j=1}^{p}\log(1+\mu_j) - \frac{1}{2}\sigma_2^2(t) + \gamma^u\right) + W'_1(t)\right] \\
&+ C\left[\left(\beta^l\frac{r(t)}{K(t)}A_1(t) - \frac{r^u}{K^l}\beta A_1(t)\right)P_1\right. \\
&\left. + \left(\beta^l\alpha(t)A_2(t) - \frac{\beta^u\alpha^u A_2^u}{B}\left(d(t) - \frac{1}{\tau}\sum_{j=1}^{p}\log(1+\mu_j)\right)\right)Z_1\right] \\
&+ C\frac{\beta^u\alpha^u A_2^u}{B}\beta^u A_1^u P_1 Z_1 \\
\leq C &\left(-\delta_1 + \frac{\alpha^u(\beta^u)^2 A_1^u A_2^u}{B}P_1 Z_1\right),
\end{aligned}$$

and

$$
\begin{aligned}
LV_2(t,P_1,Z_1) = {} & \frac{e^{W_2(t)}}{2} W'_2(t)\big(\beta^u A_1^u P_1 + \alpha^l A_2^l Z_1\big)^2 + e^{W_2(t)}\big(\beta^u A_1^u P_1 + \alpha^l A_2^l Z_1\big) \\
& \cdot \Biggl(\beta^u A_1^u \Biggl(r(t) P_1 + \frac{1}{\tau}\sum_{j=1}^{p} \log\big(1+\lambda_j\big) \cdot P_1 - \frac{r(t)}{K(t)} A_1(t) P_1^2 \\
& - \alpha(t) A_2(t) P_1 Z_1 \Biggr) \\
& + \alpha^l A_2^l \Biggl(-d(t) Z_1 + \frac{1}{\tau}\sum_{j=1}^{p} \log\big(1+\mu_j\big) \cdot Z_1 + \beta(t) A_1(t) P_1 Z_1 \\
& - \frac{\gamma(t) A_1(t) P_1 Z_1}{a(t) + A_1(t) P_1} \Biggr) \Biggr) + \frac{e^{W_2(t)}}{2} \Big((\beta^u A_1^u)^2 \sigma_1^2 P_1^2 + \big(\alpha^l A_2^l\big)^2 \sigma_2^2 Z_1^2 \Big) \\
& \le e^{W_2(t)} \Biggl\{ (\beta^u A_1^u)^2 \Biggl(\frac{W'_2(t)}{2} + r(t) + \frac{1}{\tau}\sum_{j=1}^{p} \log\big(1+\lambda_j\big) + \frac{1}{2}\sigma_1^2 \Biggr) P_1^2 \\
& + \beta^u A_1^u \alpha^l A_2^l \Biggl(W'_2(t) + r(t) - d(t) + \frac{1}{\tau}\sum_{j=1}^{p} \log\big(1+\lambda_j\big) \\
& + \frac{1}{\tau}\sum_{j=1}^{p} \log\big(1+\mu_j\big) \Biggr) P_1 Z_1 \\
& + \big(\alpha^l A_2^l\big)^2 \Biggl(\frac{W'_2(t)}{2} - d(t) + \frac{1}{\tau}\sum_{j=1}^{p} \log\big(1+\mu_j\big) + \frac{1}{2}\sigma_2^2 \Biggr) Z_1^2 \\
& - (\beta^u A_1^u)^2 \frac{r(t)}{K(t)} A_1(t) P_1^3 \Biggr\} \\
& \le e^{|W_2^u|} \Biggl\{ \beta^u A_1^u \alpha^l A_2^l \Biggl(N + r^u + d^u + \frac{1}{\tau}\sum_{j=1}^{p} \log\big(1+\lambda_j\big) \\
& + \frac{1}{\tau}\sum_{j=1}^{p} \log\big(1+\mu_j\big) \Biggr) P_1 Z_1 \\
& + (\beta^u A_1^u)^2 \Biggl(\frac{N}{2} + r^u + \frac{1}{\tau}\sum_{j=1}^{p} \log\big(1+\lambda_j\big) + \frac{(\sigma_1^u)^2}{2} \Biggr) P_1^2 \Biggr\} \\
& - e^{|W_2^l|} \Biggl\{ (\beta^u A_1^u)^2 \frac{r^l A_1^l}{K^u} P_1^3 + \big(\alpha^l A_2^l\big)^2 \delta_2 Z_1^2 \Biggr\}.
\end{aligned}
$$

Then, we can obtain

$$V(t,P_1,Z_1) \leq -C\delta_1 + q_1P_1Z_1 + q_2P_1^2 - e^{|W_2^l|}(\beta^u A_1^u)^2 \frac{r^l A_1^l}{K^u} P_1^3 - e^{|W_2^l|}\left(\alpha^l A_2^l\right)^2 \delta_2 Z_1^2$$

where

$$\begin{aligned} q_1 &= C\frac{\alpha^u(\beta^u)^2 A_1^u A_2^u}{B} \\ &\quad + e^{|W_2^u|}\beta^u A_1^u \alpha^l A_2^l \left(N + r^u + d^u + \frac{1}{\tau}\sum_{j=1}^{p} \log\left(1+\lambda_j\right) \right. \\ &\quad \left. + \frac{1}{\tau}\sum_{j=1}^{p} \log\left(1+\mu_j\right) \right) P_1Z_1, \\ q_2 &= e^{|W_2^u|}(\beta^u A_1^u)^2 \left(\tfrac{N}{2} + r^u + \tfrac{1}{\tau}\textstyle\sum_{j=1}^{p}\log\left(1+\lambda_j\right) + \tfrac{(\sigma_1^u)^2}{2}\right), \\ \delta_1 &= \beta^l h_1(t) + \tfrac{r^u}{K^l}(h_2(t)-\gamma^u), \delta_2 = -h_2(t), \end{aligned}$$

and

$$B = d^l - \tfrac{1}{\tau}\textstyle\sum_{j=1}^{p}\log\left(1+\mu_j\right).$$

Let

$$C = \tfrac{2}{\delta_1}\max_{(P_1,Z_1)\in\mathbb{R}_+^2}\left\{2, q_2P_1^2 - \tfrac{e^{|W_2^l|}}{2}(\beta^u A_1^u)^2\tfrac{r^l A_1^l}{K^u}P_1^3 - \tfrac{e^{|W_2^l|}}{2}\left(\alpha^l A_2^l\right)^2\delta_2 Z_1^2\right\}. \tag{6-10}$$

To confirm the condition (ii) of Lemma 6.2.1.3, we choose a sufficient small $\varepsilon,\ \varepsilon > 0$ such that

$$0 < \varepsilon \leq \tfrac{1}{4q_1}\left\{C\delta_1, 2\tfrac{e^{|W_2^l|}}{2}\left(\alpha^l A_2^l\right)^2\delta_2, 2\tfrac{e^{|W_2^l|}}{2}(\beta^u A_1^u)^2\tfrac{r^l A_1^l}{K^u}\right\}, \tag{6-11}$$

$$-C\delta_1 + M + 1 \leq e^{|W_2^l|}\min\ \left\{\tfrac{(\beta^u A_1^u)^2 r^l A_1^l}{2\varepsilon^3 K^u}, \tfrac{\alpha^l A_2^l \delta_2}{2\varepsilon^3}\right\}, \tag{6-12}$$

where

$$\begin{aligned} M = \max_{(P_1,Z_1)\in\mathbb{R}_+^2} &\left\{\frac{2}{5}q_1(P_1)^{\frac{5}{2}} + q_2P_1^2 - \frac{e^{|W_2^l|}}{2}(\beta^u A_1^u)^2\frac{r^l A_1^l}{K^u}P_1^3 + \frac{3}{5}q_1Z_1 \right. \\ &\left. - \frac{e^{|W_2^l|}}{2}\left(\alpha^l A_2^l\right)^2\delta_2 Z_1^2\right\}. \end{aligned}$$

Define the following bounded open set

$$D_\varepsilon = \left\{ (P_1, Z_1) \middle| \varepsilon < P_1 < \tfrac{1}{\varepsilon}, \varepsilon < Z_1 < \tfrac{1}{\varepsilon} \right\}.$$

and consider

$$D_{\mathcal{E}}^1 = \{(P_1, Z_1) \in \mathbb{R}_+^2 | 0 < P_1 \leq \varepsilon \}, D_{\mathcal{E}}^2 = \{(P_1, Z_1) \in \mathbb{R}_+^2 | 0 < Z_1 \leq \varepsilon \},$$

$$D_{\mathcal{E}}^3 = \left\{ (P_1, Z_1) \in \mathbb{R}_+^2 \middle| P_1 \geq \tfrac{1}{\varepsilon} \right\}, D_{\mathcal{E}}^4 = \left\{ (P_1, Z_1) \in \mathbb{R}_+^2 \middle| Z_1 \geq \tfrac{1}{\varepsilon} \right\}.$$

Obviously, we can obtain that

$$D_{\mathcal{E}}^C = D_{\mathcal{E}}^1 \cup D_{\mathcal{E}}^2 \cup D_{\mathcal{E}}^3 \cup D_{\mathcal{E}}^4.$$

In the following, let the conditions of $\delta_1 > 0, \delta_2 > 0,$ and $B > 0$ hold, we need to prove that $LV(t, P_1, Z_1) \leq -1$ on $\mathbb{R}_+ \times D_{\mathcal{E}}^C$, respectively.

Case 1 If $(t, P_1, Z_1) \in \mathbb{R}_+^2 \times D_{\mathcal{E}}^1$, then $P_1 Z_1 \leq \varepsilon Z_1 \leq \varepsilon(1 + Z_1^2)$ and we have

$$\begin{aligned} V(t, P_1, Z_1) \leq & -\frac{C\delta_1}{4} + \left(-\frac{C\delta_1}{4} + q_1\varepsilon\right) - \frac{e^{|W_2^l|}}{2}(\beta^u A_1^u)^2 \frac{r^l A_1^l}{K^u} P_1^3 \\ & + \left(q_1\varepsilon - \frac{e^{|W_2^l|}}{2}\left(\alpha^l A_2^l\right)^2 \delta_2\right) Z_1^2 \\ & + \left(-\frac{C\delta_1}{2} + q_2 P_1^2 - \frac{e^{|W_2^l|}}{2}(\beta^u A_1^u)^2 \frac{r^l A_1^l}{K^u} P_1^3 - \frac{e^{|W_2^l|}}{2}\left(\alpha^l A_2^l\right)^2 \delta_2 Z_1^2\right). \end{aligned}$$

According to (6-10) and (6-11), we have

$$V(t, P_1, Z_1) \leq -\tfrac{C\delta_1}{4} \leq -1.$$

Case 2 If $(t, P_1, Z_1) \in \mathbb{R}_+^2 \times D_{\mathcal{E}}^2$, then $P_1 Z_1 \leq \varepsilon P_1 \leq \varepsilon(1 + P_1^3)$ and we have

$$\begin{aligned} V(t, P_1, Z_1) \leq & -\frac{C\delta_1}{4} + \left(-\frac{C\delta_1}{4} + q_1\varepsilon\right) + \left(q_1\varepsilon - \frac{e^{|W_2^l|}}{2}(\beta^u A_1^u)^2 \frac{r^l A_1^l}{K^u}\right) P_1^3 \\ & - \frac{e^{|W_2^l|}}{2}\left(\alpha^l A_2^l\right)^2 \delta_2 Z_1^2 \\ & + \left(-\frac{C\delta_1}{2} + q_2 P_1^2 - \frac{e^{|W_2^l|}}{2}(\beta^u A_1^u)^2 \frac{r^l A_1^l}{K^u} P_1^3 - \frac{e^{|W_2^l|}}{2}\left(\alpha^l A_2^l\right)^2 \delta_2 Z_1^2 \right). \end{aligned}$$

Similarly, we have

$$V(t,P_1,Z_1) \le -\frac{C\delta_1}{4} \le -1.$$

On the other hand, by Young inequality, we have

$$P_1 Z_1 \le \frac{2}{5} P_1^{\frac{5}{2}} + \frac{3}{5} Z_1^{\frac{5}{3}}.$$

Thus,

$$V(t,P_1,Z_1) \le -C\delta_1 + M - \frac{e^{|W_2^l|}}{2} (\beta^u A_1^u)^2 \frac{r^l A_1^l}{K^u} P_1^3 - \frac{e^{|W_2^l|}}{2} \left(\alpha^l A_2^l\right)^2 \delta_2 Z_1^2.$$

Case 3 If $(t,P_1,Z_1) \in \mathbb{R}_+^2 \times D_\varepsilon^3$ and by (6-12) and (6-13), then we have

$$\begin{aligned} V(t,P_1,Z_1) &\le -C\delta_1 + M - \frac{e^{|W_2^l|}}{2} \frac{(\beta^u A_1^u)^2 r^l A_1^l}{K^u} P_1^3 \\ &\le -C\delta_1 + M - e^{|W_2^l|} \frac{(\beta^u A_1^u)^2 r^l A_1^l}{K^u 2\varepsilon^3} \le -1. \end{aligned}$$

Case 4 If $(t,P_1,Z_1) \in \mathbb{R}_+^2 \times D_\varepsilon^4$ and from (6-12) and (6-13), then we have

$$\begin{aligned} V(t,P_1,Z_1) &\le -C\delta_1 + M - \frac{e^{|W_2^l|}}{2} \left(\alpha^l A_2^l\right)^2 \delta_2 Z_1^2 \le -C\delta_1 + M - \frac{e^{|W_2^l|}}{2\varepsilon^2} \left(\alpha^l A_2^l\right)^2 \delta_2 \\ &\le -1. \end{aligned}$$

Thus, we obtain that $V(t,P_1,Z_1) \le -1$, for all $(t,P_1,Z_1) \in \mathbb{R}_+^2 \times D_\varepsilon^C$, namely the condition (ii) of Lemma 6.2.1.3 is satisfied.

To sum up, we can obtain that, for given initial value $\big(P(0),Z(0)\big) \in \mathbb{R}_+^2$ and suppose the conditions (H1)-(H3) hold, and further if the conditions of $\delta_1 > 0, \delta_2 > 0$, and $B > 0$ hold, then system (6-3) has a positive τ-periodic solution.

6.3. Experimental simulations

In the previous section, we have studied the effects of environmental fluctuations, impulsive perturbations and TPP in a periodic environment on the dynamics of system (6-3) using modeling analysis, including the existence and uniqueness of global positive solutions, the extinction and persistence in the mean and periodic solutions of the system. To find the treatment strategy of planktonic blooms, and the effect of artificial interventions on the growth of plankton populations, the white noise, artificial interventions and TPP are chosen as control parameters and other parameters are listed as follows: $r(t) = 0.75 + 0.01\sin\frac{\pi t}{40}, K(t) = 0.75 + 0.05\sin\frac{\pi t}{40}, \alpha(t) = 0.85 + 0.0001\sin\frac{\pi t}{40}$, $\beta(t) = 0.65 + 0.0001\sin\frac{\pi t}{40}$, $a(t) = 0.05 + 0.008\sin\frac{\pi t}{40}$, $d(t) = 0.2 + 0.005\sin\frac{\pi t}{4}$, $\gamma(t) = 0.12 + 0.06\sin\frac{\pi t}{4}$, $\lambda_k = 0.05 + 0.005\sin\frac{\pi t}{40}$ $\mu_k = 0.001 + 0.95\, sin\frac{\pi t}{40}$, $\tau = 80$, and the initial condition is $(P(0), Z(0)) = (0.5, 0.5)$. In this section, based on Milstein's method mentioned (Higham, 2001) by supplementing impulsive perturbations into it, some numerical simulations are carried out to further investigate how the factors influence the stochastic dynamics of system (6-3).

6.3.1. Impact of white noise on the dynamics of system (6-3)

In order to study how white noise, impulsive control, and TPP affect the dynamics of system (6-3), we firstly consider that system (6-3) does not experience the white noise $(\sigma_1(t) = \sigma_2(t) = 0)$, that is, the stochastic system (6-3) becomes its corresponding deterministic system (6-2). As mentioned above, the conditions (H1)-(H3) can be easily verified. Thus, by simple calculations, we can obtain that both phytoplankton and zooplankton populations in system (6-2) are persistent and the system has a periodic solution provided that the value of impulsive control parameter is small ($p = 2$), depicting the coexistence of plankton populations at a stable state (see Fig. 6.1).

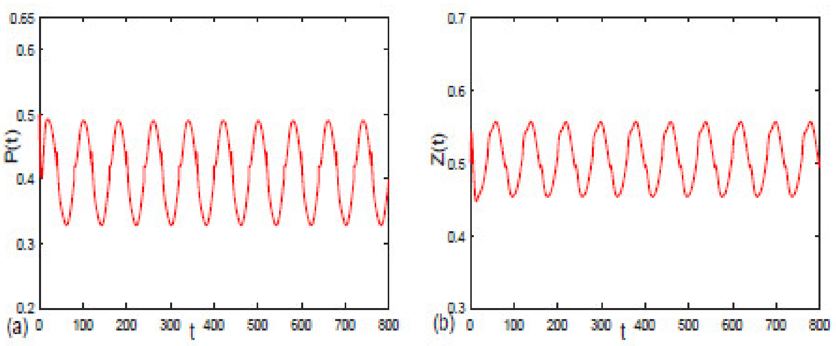

Fig. 6.1 The dynamics of system (6-2) with impulsive control parameter $p = 2$. **(a)-(b)**: The

periodic solutions of phytoplankton population $P(t)$ and zooplankton population $Z(t)$ of deterministic system (6-2), respectively.

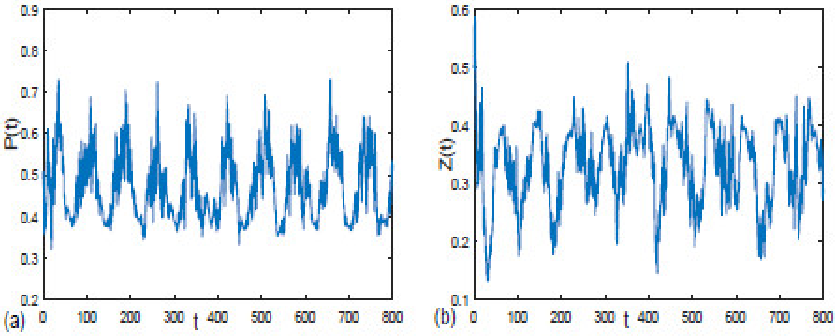

Fig. 6.2 The stochastic dynamics of system (6-3) with $\sigma_1(t) = \sigma_2(t) = 0.005 + 0.005\sin\frac{\pi t}{40}$. **(a)-(b)**: The periodic solutions of phytoplankton population $P(t)$ and zooplankton population $Z(t)$ of system (6-3), respectively.

Now we explore the impact of white noise on the persistence-extinction properties of system (6-3). Fig. 6.2 shows that phytoplankton and zooplankton populations can coexist in a relatively stable state when the white noise intensity is comparatively weak $\left(\sigma_1(t) = \sigma_2(t) = 0.005 + 0.005\sin\frac{\pi t}{40}\right)$. Actually, according to dynamical analysis, it is not difficult to find that system (6-3) is persistence in the mean and has a positive periodic solution with a period $\tau = 80$, which are consistent with our numerical analysis. Comparing Figs. 6.1 and 6.2, it is evident that white noise can affect the distribution of phytoplankton and zooplankton populations. That is, white noise can significantly affect the dynamic evolution mechanism of plankton populations. Furthermore, when $\sigma_1(t)$, the effect of white noise intensity on phytoplankton population varies within some critical level, and the amplitude of random oscillation for the phytoplankton population increases with an increasing value of $\sigma_1(t)$, while that of zooplankton population decreases (figures here are not given because of the similarity to Fig. 6.2). However, when we change the white noise $\sigma_1(t)$ to $\sigma_1(t) = 1.8 + 0.005\sin\frac{\pi t}{40}$, satisfying the conditions of $h_1^u < 0$ and $h_2^u < 0$, as a result, both phytoplankton and zooplankton population of system (6-3) undergo rapid extinction, whereas the corresponding deterministic system still presents obvious periodicity, which is a different phenomenon from that of its deterministic system (see Fig. 6.3). Comparing Figs. 6.2 and 6.3, it can be seen that white noise has a significant effect on the dynamics of system (6-3), and a large white noise intensity can accelerate the extinction of plankton and is advantageous for the termination of planktonic blooms. Moreover, it should be emphasized that white noise can not only aggravate the emergence of random oscillation, but also change the periodicity of plankton density. Therefore, it is worth pointing out that the results from Figs 6.2 and 6.3 can support the notion that planktonic systems incorporating white noise can better simulate plankton blooms than the corresponding deterministic counterparts.

6.3.2. Impact of impulsive perturbations on the dynamics of system (6-3)

In order to study how the impulsive perturbations on the dynamical behaviors of the system, we control some a parameter values satisfying the conditions of $h_1^l > 0$ and $h_2^u + \frac{K^u \beta^u h_1^u}{r^l A_1^l} < 0$, the zooplankton population of system (6-3) will tend towards extinction rapidly, but the phytoplankton population generates periodic oscillation in the sense of joint distribution after a period of time and the corresponding deterministic system (6-2) shares similar properties, shown in Fig. 6.4. Moreover, comparing Figs. 6.2 and 6.4, the dynamical behaviors of system (6-3) are significantly altered by controlling the impulsive control parameter value of p $(p = 40)$. Thus, it should be stressed that some critical parameters have a profound impact on the persistence and extinction of plankton populations in fluctuating environments. These results suggest that the impulsive control parameter value plays a restrictive role in the survival of species $Z(t)$, which verifies that the system has potential to be applied in real-world biological control situations.

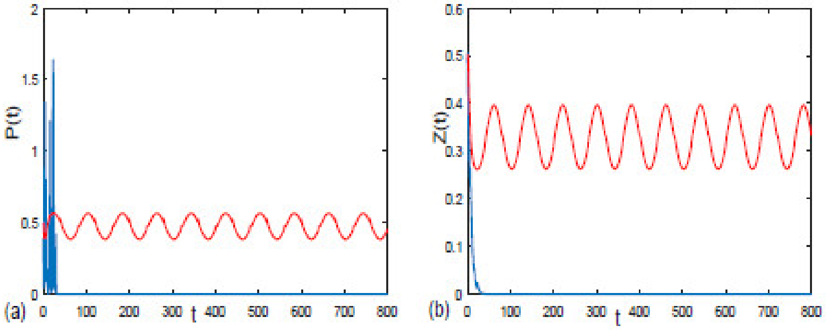

Fig. 6.3 The stochastic dynamics behaviors of system (6-3) with $\sigma_1(t) = 1.8 + 0.005\sin\frac{\pi t}{40}$ and its corresponding deterministic counterparts with initial value $(P(0), Z(0)) = (0.5,0.5)$, respectively. **(a)-(b)**: Blue curves are the extinction of phytoplankton population $P(t)$ and zooplankton population $Z(t)$ of stochastic system (6-3), respectively, and red curves represent the persistence of phytoplankton and zooplankton populations for its corresponding deterministic counterparts, respectively.

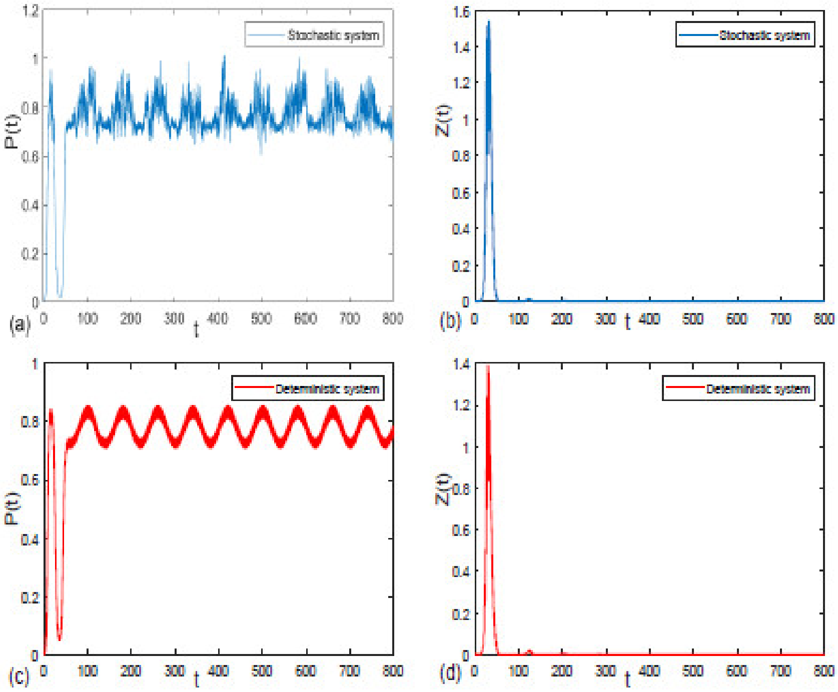

Fig. 6.4 The stochastic dynamic behaviors of system (6-3) with $p = 40$ and its corresponding deterministic counterparts with initial value $(P(0), Z(0)) = (0.5,0.5)$. **(a)-(b)**: the persistence in the mean of phytoplankton population $P(t)$ and the extinction of zooplankton population $Z(t)$

of stochastic system (6-3), respectively; (c)-(d): the persistence of phytoplankton population $P(t)$ and the extinction of zooplankton population $Z(t)$ of its corresponding deterministic counterparts, respectively.

6.3.3. Impact of TPP on the dynamics of system (6-3)

Finally, the influence of TPP in a periodic environment under impulsive perturbations and environmental fluctuations on planktonic blooms was studied. Fig. 6.5 depicts how the TPP toxin liberation rate $\gamma(t)$ in one period affects the stochastic dynamic behaviors of system (6-3), where $0 \leq m \leq 1$, and the colorbars represent the biomass of the corresponding species. Fig. 6.5 clearly shows that the population densities of the system (6-3) change periodically and the density of species $P(t)$ increases with the value of m, while that of species $Z(t)$ will rapidly decline and finally undergo extinction when m reaches a critical value $(m \approx 0.2971)$. In particular, when we select three different values of m and the other parameters remain unchanged, as in Fig. 6.2, system (6-3) exists as periodic oscillation solutions, and the phytoplankton and zooplankton populations can coexist in a stable state when $m = 0.06, 0.18$. For these two cases, one can observe that periodic solutions of system (6-3) can fluctuate in a small neighborhood of the periodic solutions of its corresponding deterministic counterparts, whereas the overall trend still presents obvious periodic oscillatory while considering the effects of impulsive perturbations and environmental fluctuations. However, when we continue to increase the value of m $(m = 0.5)$, the zooplankton population will rapidly decrease to extinction, as shown in Fig. 6.6. Moreover, Fig. 6.6 shows that the maximum random amplitude of the phytoplankton population will increase, while that of zooplankton will decrease, and finally declining to zero when m increases. This means that an increase in the toxic phytoplankton population leads to massive die-off of zooplankton, depicting an unstable situation for some critical values of the system parameters. However, the decrease of the toxin liberation rate can reduce the peak of the cyclic breakouts of planktonic blooms, but increase the survival of zooplankton. Thus, TPP plays an important role in controlling planktonic blooms.

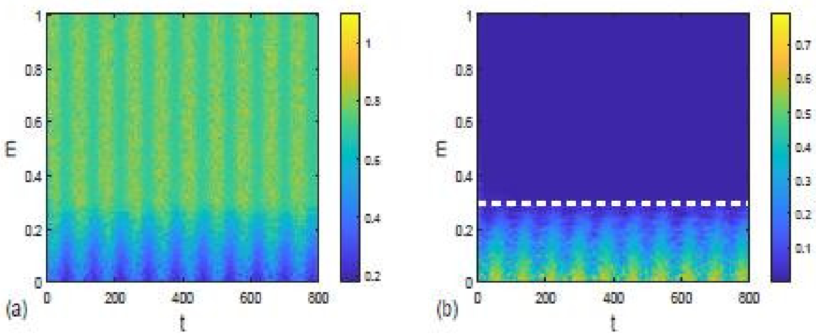

Fig. 6.5 The effect of TPP on the stochastic dynamic behaviors of system (6-3) with $\gamma(t)=m+0.06\sin\frac{\pi t}{40}$. **(a)** The persistence in the mean of population $P(t)$; **(b)** The persistence in the mean and extinction of population $Z(t)$.

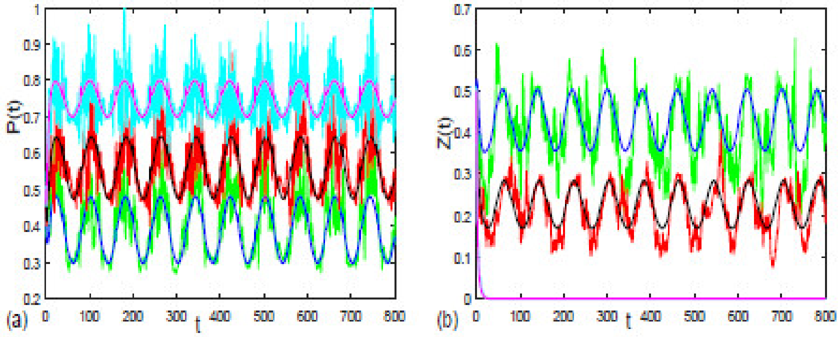

Fig. 6.6 The effect of TPP on the stochastic dynamic behaviors of system (6-3). Panels **(a)** and **(b)** denote the sample path of phytoplankton $P(t)$ and zooplankton $Z(t)$ of the stochastic system (6-3) and the periodic solutions of its corresponding deterministic system (6-2) with different $\gamma(t)$, respectively. $\gamma(t)=0.06+0.06\sin\frac{\pi t}{40}$ (green curves), $\gamma(t)=0.18+0.06\sin\frac{\pi t}{40}$ (red curves) and $\gamma(t)=0.5+0.06\sin\frac{\pi t}{40}$ (cyan curves), and their corresponding deterministic counterparts (blue, black and magenta curves). Here, $p=2$.

6.4. Conclusions

Understanding the dynamic mechanisms of phytoplankton-zooplankton systems involving TPP can help guide the control of planktonic blooms, and impulsive perturbations and environmental stochasticity can significantly affect real-world ecosystems (Liu and Wang, 2012; Carpenter et al., 2011). Taking these factors into account, we proposed a stochastic non-autonomous phytoplankton-zooplankton system involving TPP and impulsive perturbations, and studied the dynamics of system (6-3) analytically and numerically. By establishing an equivalent system without impulses, we initially investigated the existence and uniqueness of global positive solutions, and then obtained the sufficient conditions for the extinction and persistence in the mean of the system by the comparison principle and Ito's formula. In order to prove the existence of positive periodic solutions, the theory of Khasminskii and a method based on Lyapunov function construction were employed. Numerical analysis verified our theoretical results and further demonstrated that the dynamics of the system are intimately associated with white noise, impulsive perturbations, and TPP.

(i) When the densities of white noise are relatively weak, fulfilling the conditions of $(h_2^l - \gamma^u)r^l + \beta^l K^u h_1^u > 0$, $\delta_1 > 0, \delta_2 > 0$, and $B > 0$, system (6-3) is persistence in the mean and has a positive periodic solution, namely, the species phytoplankton and zooplankton can coexist in a stable state (see Fig. 6.2). According to Figs. 6.1 and 6.2, white noise can affect the distribution of plankton populations. Moreover, by enhancing the intensity of white noise, we observed that both phytoplankton and zooplankton populations tend towards rapid extinction (see Fig. 6.3), which are consistent with the analysis of the conditions of $h_1^u < 0$ and $h_2^u < 0$. Comparing Figs. 6.2 and 6.3, a high intensity of white noise is disadvantageous to the development of plankton populations and increases the risk of population extinction, which implies that adjusting the white noise intensity can help control plankton population densities. Therefore, controlling white noise is a key factor in the termination of planktonic blooms, which is consistent with the results obtained by the experimental and field observations (Sarkar and Chattopadhayay, 2003a, 2003b).

(ii) As impulsive control parameter value of p increases over time, it can be seen from Fig. 6.4 that the zooplankton population will undergo rapid extinction, but the phytoplankton population can generate periodic oscillation in the sense of joint distribution. The impulsive control parameter can change the stochastic dynamic behaviors of the system significantly (see Figs. 6.2 and 6.4). Therefore, this suggests that the impulsive parameter value p may play a restrictive role in zooplankton survival and has potential application in planktonic bloom control.

(iii) With increasing toxin liberation rate, the conditions of $h_1^l > 0$ and $h_2^u + \frac{K^u \beta^u h_1^u}{r^l A_1^l} < 0$ become increasingly difficult to be met. Thus, the zooplankton population will tend towards extinction, but the phytoplankton population can generate periodic oscillation in the sense of joint distribution (see Fig. 6.5). This means that an increase in the toxic phytoplankton population leads to massive die-off of zooplankton, depicting an unstable situation for some critical values of the system parameters. However, a low toxin liberation rate can reduce the peak of the cyclic breakouts of planktonic blooms, but increase the survival changes of zooplankton (Fig. 6.6). Therefore, TPP may be used as a biological way to control planktonic blooms.

To sum up, we presents an investigation on the effects of environmental fluctuations, impulsive perturbations and TPP on a phytoplankton-zooplankton system. Studies on the dynamic mechanisms of phytoplankton-zooplankton interactions have been a key topic in theoretical ecology, and can enhance our general understanding of real-world aquatic ecosystems. Thus, it would be interesting to further incorporate some real-world factors into our proposed model, such as the time-lag effect (Caperon, 1969), prey refuge (Mullin et al., 1975), and cell size (Hart and Bychek, 2011), and to investigate how these factors affect the dynamics of the complex models. We recommend this as the focus of future research.

Chapter 7 CONCLUSIONS AND RECOMMENDATIONS

7.1. Conclusions

Based on the knowledge and methods of plankton dynamics models, impulsive control dynamics and stochastic dynamics, the dissertation adapts the techniques of dynamics modeling, dynamics analysis and experimental experiments to study the nonlinear dynamics of the growth and evolution of plankton, and discuss the effects of some key factors such as population diffusion, impulsive control, environmental fluctuations (including white noise and regime switching) and toxins produced by TPP on the dynamics mechanisms of the formation and disappearance of the planktonic blooms, and predict the dynamics evolutionary processes of plankton growth. The main results are as follows:

(i) To discuss the prevention and control strategies on planktonic blooms, an impulsive reaction-diffusion hybrid system was developed. On the one hand, the dynamic analysis showed that impulsive control can significantly influence the dynamics of the system, including the ultimate boundedness, extinction, permanence, and the existence and uniqueness of positive periodic solution of the system. On the other hand, some experimental simulations were preformed to reveal that impulsive control can lead to the extinction and permanence of population directly. More prexisely, the prey and intermediate predator populations can coexist at any time and location of their inhabited domain, while the top predator population undergone extinction when the impulsive control parameter exceeds some a critical value, which can provide some key arguments to control population survival by means of some reaction-diffusion impulsive hybrid systems in the real life. Additionally, a heterogeneous environment can influence the spatial distribution of plankton density and change the temporal-spatial oscillation of plankton distribution. All results are expected to be helpful in the study of dynamic complex of ecosystems.

(ii) A stochastic phytoplankton-zooplankton system with toxic phytoplankton was proposed and the effects of environmental stochasticity and TPP on the dynamics mechanisms of the termination of planktonic blooms were discussed. The research illustrated that white noise intensity can aggravate the stochastic oscillation of plankton populations density and a high-level intensity of white noise can accelerate the extinction of plankton and may be advantageous for the disappearance of harmful phytoplankton, which imply that the white noise can help control the biomass of plankton and provide a guide to the termination of planktonic blooms. Additionally, some experimental simulations were carried out to further reveal that the increasing toxin liberation rate can increase the survival chance of phytoplankton and reduce the biomass of zooplankton, but the combined effects of those two liberation rates produced by TPP on the changes in plankton are stronger than that of controlling any one of the two TPP. All results suggest that both white noise and TPP play an important role in controlling planktonic blooms.

(iii) We developed a stochastic phytoplankton-toxic producing phytoplankton-zooplankton under regime switching and investigated how the white noise, regime switching and TPP affect the dynamics mechanisms of planktonic blooms. The dynamical analysis indicated that both high level of white noise intensity and toxin liberation rate released by TPP are disadvantageous to the development of plankton and may increase the risk of plankton extinction. Also, a series of experimental simulations were performed to verify the correctness of dynamical analysis and further reveal the effects of the white noise, regime switching and TPP on the dynamics mechanisms of the termination of planktonic blooms. On the one hand, the numerical study revealed that the system can switch from one state to another due to regime shift, and further indicated that the regime switching can balance the different survival states of plankton and decrease the risk of plankton extinction when the density of white noise are particularly weak. On the other hand, an increase in the toxin liberation rate can increase the survival chance of the phytoplankton but reduce the biomass of zooplankton, which implies that the presence of toxic phytoplankton may have a positive effect on the termination of planktonic blooms. These results may provide some insightful understanding on the dynamics of phytoplankton-zooplankton systems in randomly disturbed aquatic environments.

(iv) A stochastic non-autonomous phytoplankton-zooplankton system involving TPP and impulsive perturbations was studied, where the white noise, impulsive perturbations and TPP are incorporated into the system to simulate the natural aquatic ecological phenomena. The dynamical analysis revealed some key threshold conditions that ensure the existence and uniqueness of a global positive solution, plankton extinction and persistence in the mean. In particular, we determined if there is a positive periodic solution for the system when the toxin liberation rate reaches a critical value. The results from experimental simulations revealed that both white noise and impulsive control parameter can directly influence the plankton extinction and persistence in the mean. Significantly, enhancing the toxin liberation rate of TPP increases the possibility of phytoplankton survival but reduces zooplankton biomass. All these results can improve our understanding of the dynamics of complex of aquatic ecosystems in a fluctuating environment.

All these results suggest that population diffusion, impulsive control, environmental stochasticity and toxins produced by TPP can significantly affect the dynamics of plankton growth, which play a key role in the dynamics mechanisms of the formation and disappearance of planktonic blooms. Therefore, it is necessary and important to consider the population diffusion, artificial interventions, environmental uncertainty and toxic phytoplankton into plankton dynamics models when studying and predicting the growth and evolution mechanisms of plankton, which is conducive to the better understanding of the interactions and dynamics mechanisms of the formation and disappearance of planktonic blooms, and provide theoretical basis for the prediction and control strategies of algal blooms.

7.2. Limitations and Future research

In this dissertation, we have obtained some interesting results, but considering the complexity of stochastic modeling and dynamical behaviors analysis of plankton under the effect of environmental fluctuations, there are still some further works in the future to discuss.

- Considering the discrete time-delay or distributed time-delay into the model

Since the time delay exists in the growth response of some phytoplankton populations (Caperon, 1969) and is indispensable for zooplankton population to digest phytoplankton population, thus, the introduction of time delay can significantly affect the dynamical behaviors of the original plankton systems, which may be a major part of the controlling in phytoplankton blooms. For example, the interaction process of allelopathic effect. For the stochastic plankton model with discrete or distributed delay, whether can we apply some related theories such as the ergodicity theory given by Hasminskii, Markovian semi-group theory and Hasminskii's periodic solution theory, to study the stationary distribution, and the uniqueness and existence of periodic solution of the plankton systems with time delay? which will be an important content in our future research.

- Establishing a stochastic plankton dynamics model with Levy noise

Although the model (5-1) has achieved some excellent results, it cannot describe the sudden and devastating disasters that the plankton populations may suffer in the nature, such as tsunamis, floods, hurricanes, etc. These disasters can significantly destroy the habitat of plankton populations, which may lead to the rapid extinction of plankton. To describe the impact of these huge, sudden and devastating disasters, therefore, it seems to be more realistic to combine the environmental noise (Levy noise) on the basis of the model in this dissertation and study how the environmental fluctuations affects the dynamics mechanisms of planktonic blooms.

- The dynamics of a stochastic nutrient-phytoplankton-zooplankton model under the influence of environmental fluctuations

In addition to recognizing the availability of nutrient (nitrogen, phosphorous, etc.) and nutrient input are important factors affecting phytoplankton growth (Huppert et al., 2002), the litter fall and decomposition processes of dead phytoplankton and other litters are also dominant pathways to accomplish nutrients to the aquatic environments.

Thus, nutrient input act as a key role in the quick appearance of algal blooms or limits their growth. Based on bottom-up and top-down control mechanisms, and considering the nutrient recycle and nutrient input in the real aquatic ecosystems, a stochastic nutrient-phytoplankton-zooplankton model with environmental fluctuations and TPP can be proposed, trying to investigate the effects of nutrient input and TPP on the dynamics mechanisms of planktonic blooms in a fluctuating environment using modeling analysis and experimental simulations.

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Appendix A

Lemma A1 (Walter, 1997) Suppose that the vector-functions $v(t,x) = \big(v_1(t,x),\cdots,v_m(t,x)\big)$ and $w(t,x) = \big(w_1(t,x),\cdots,w_m(t,x)\big), m \geq 1$, satisfy the following conditions:

(i) they are of class C^2 in $x, x \in \Omega$ and class of C^1 in $(t,x) \in [a \times b\,] \times \bar{\Omega}$, where $\Omega \in R^n$ is a bounded domain with smooth boundary;

(ii) $v_t - \mu\Delta v - g(t,x,v) \leq w_t - \mu\Delta w - g(t,x,w)$, where $(t,x) \in [a \times b\,] \times \Omega$, $\mu = (\mu_1,\cdots,\mu_m)$, and vector-function $g(t,x,u) = \big(g_1(t,x,u),\cdots,g_m(t,x,u)\big)$ is continuously differentiable and quasi-monotonically increasing with respect to $u = (u_1,\cdots,u_m) : \geq 0, i,j = 1,\cdots,m, i \neq j$;

(iii) $\frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0, (t,x) \in [a,b] \times \partial\Omega$,

then $v(t,x) \leq w(t,x)$ for $(t,x) \in [a \times b\,] \times \bar{\Omega}$.

Lemma A2 (Smith, 1999) Suppose that T and d are positive numbers, a function $u(t,x)$ is continuous on $[0,T\,] \times \bar{\Omega}$, continuously differentiable in $x \in \bar{\Omega}$,, with continuous derivatives $\partial^2 u/\partial x_i x_j$ and $\partial u/\partial t$ on $(0,T] \times \Omega$, and $u(t,x)$ satisfies the following inequalities:

$$\frac{\partial u}{\partial t} - d\Delta u + c(t,x)u \geq 0, (t,x) \in (0,T] \times \Omega,$$

$$\frac{\partial u}{\partial n} \geq 0, (t,x) \in (0,T] \times \bar{\Omega},$$

and

$$u(0,x) \geq 0, x \in \Omega,$$

where $c(t,x)$ is bounded on $(0,T] \times \Omega$, then $u(t,x) \geq 0$ on $(0,T] \times \bar{\Omega}$, and $u(t,x)$ is strictly positive on $(0,T] \times \bar{\Omega}$ if $u(0,x)$ is not identically zero.

Appendix B

In the following, we consider the following impulsive logistic differential equations:

$$\begin{cases} \dfrac{du}{dt} = az(b-z), \quad t \neq t_k \\ z(t_k^+) = \lambda_k\big(z(t_k)\big) \end{cases} \tag{B1}$$

where $z \in R_+$, a and b are positive constants, strictly increasing sequence t_k satisfies condition (C4), and λ_k $k \in N$, are continuous positive-valued functions such that $\lambda_{k+p}(z) = \lambda_k(z)$ for all $z \in R_+$, $k \in N$. From condition (C4), we deduce that $t_{k+1} - t_k \geq \theta = \min_{i=0,1,2,\cdots,\mathrm{p}}(t_{i+1} - t_i), k \geq 1$. Denote $A = b/\big(1 - e^{-ab\theta}\big)$, $B = A \max_{i=0,1,2,\cdots,p} \max_{z\in[0,A]} \lambda_k(z)$, and $C = \max(z_0, A, B)$, where z_0 is given below. Then we have the following useful result.

Lemma B Every solution $z(t) = z(t, 0, z_0), z_0 = z(0) = z(0^+) > 0$ of the system (B1) satisfies $0 < z(t) < C$ for all $t \geq 0$.

Proof For $t \in [0, t_1]$, we can obtain that

$$z(t) = \frac{bz_0}{z_0(1 - e^{-abt}) + be^{-abt}}.$$

Then, it is easy to find that the solution is positive-valued and no larger than $\max\{z_0, b\}$ on the interval. If $\theta \leq t \leq t_1$, then

$$z(t) = \frac{bz_0}{z_0(1 - e^{-abt}) + be^{-abt}} \leq \frac{b}{1 - e^{-abt}} = A. \tag{B2}$$

In particular, $0 < z(t_1) \leq A$, hence, $0 < z(t_1^+) = z(t_1)\lambda_1\big(z(t_1)\big) \leq B$. Therefore, $0 < z(t) \leq \max(A, B) \leq C$ if $t \in [t_1, t_2]$, and similar to (B2), we can verify that $0 < z(t_2) \leq A$. In the same manner we can show that $0 < z(t) < C$ if $t \in (t_k, t_{k+1}], k = 1,2,3,\cdots$. This completes the proof of the lemma.

Appendix C

Let $C^{m+\alpha}(\Omega)$ is the space of m-times continuously differentiable functions $f:\Omega \to R$, where m is a positive integer and $0 < \alpha < 1$, which have m-order derivatives satisfying the Holder condition with exponent α, and $w = (w_1, w_2, w_3) \in L_p(\Omega) \times L_p(\Omega) \times L_p(\Omega), p > n$ is a positive integer. For some $\varepsilon > 0$, let

$$A_1 = \begin{pmatrix} \mu_1\Delta - \varepsilon & 0 & 0 \\ 0 & \mu_2\Delta - \varepsilon & 0 \\ 0 & 0 & \mu_3\Delta - \varepsilon \end{pmatrix},$$

$$F(t,w) = \begin{pmatrix} u_1\left(a_1(t,x) - b_1(t,x)u_1 - \dfrac{c_1(t,x)u_1u_2}{1 + r_1(t,x)u_1 + r_2(t,x)u_2 + r_1(t,x)r_2(t,x)u_1u_2} \right) + \varepsilon u_1 \\ u_2\left(a_2(t,x) - b_2(t,x)u_2 + \dfrac{c'_1(t,x)u_1u_2}{1 + r_1(t,x)u_1 + r_2(t,x)u_2 + r_1(t,x)r_2(t,x)u_1u_2} \right. \\ \left. - \dfrac{c_2(t,x)u_2u_3}{1 + r_3(t,x)u_2 + r_4(t,x)u_3 + r_3(t,x)r_4(t,x)u_2u_3} \right) + \varepsilon u_2 \\ u_3\left(-a_3(t,x) + \dfrac{c_2(t,x)u_2u_3}{1 + r_3(t,x)u_2 + r_4(t,x)u_3 + r_3(t,x)r_4(t,x)u_2u_3} \right) + \varepsilon u_3 \end{pmatrix},$$

and

$$G_i(w(t_i)) = \begin{pmatrix} u_1(t_i,x)f_i(x, u_1(t_i,x), u_2(t_i,x), u_3(t_i,x)) - u_1(t_i,x) \\ u_2(t_i,x)f_i(x, u_1(t_i,x), u_2(t_i,x), u_3(t_i,x)) - u_2(t_i,x) \\ u_3(t_i,x)f_i(x, u_1(t_i,x), u_2(t_i,x), u_3(t_i,x)) - u_3(t_i,x) \end{pmatrix}.$$

Thus, we can rewrite (3-1)-(3-3) in the form

$$\frac{dw}{dt} = A_1 w^T + F(t,w), t \neq t_i \qquad (C3)$$

$$w(t_i^+) = w(t_i) + G_i(w(t_i)), i \in N. \qquad (C4)$$

The operator A_1 has the domain $D(A_1) = \left\{\varphi : \varphi \in W^{2,p}(\Omega), \frac{\partial\varphi}{\partial n}|\partial\Omega = 0\right\}$, where $W^{2,p}(\Omega)$ is the Sobolev space of functions from $L_p(\Omega)$ that have two generalized derivatives. Functions $F(t, w(t))$ satisfy $\sup_t \|F(t, w(t))\| < \infty$, $G_i(w)$ is p-period in i . Denote the spectrum of A_1 by $\sigma(A_1)$, then it follows from (Henry, 1981) that the sectorial operator A_1 satisfies $\mathrm{Re}\sigma(A_1) \leq -\delta$. For any $\alpha > 0$, the fractional power $A^{-\alpha}$ is

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int e^{-sA_1} s^{\alpha-1} ds,$$

where Γ is the gamma function. The operators $A_1^{-\alpha}$ are bounded and bijective, and we define A_1^{α} as $(A_1^{-\alpha})^{-1}$, and $D(A_1^{\alpha}) = R(A_1^{-\alpha})$, where $R(\cdot)$ is the range of an operator, and A_1^0 is the identity operator in X. For $0 \leq \alpha \leq 1$, we introduce the space $X^{\alpha} = D(A_1^{\alpha})$ with norm $\|x\|_{\alpha} = \|A_1^{\alpha} x\|$, where, $\|\cdot\|$ is the norm in the space $X = L_p(\Omega) \times L_p(\Omega) \times L_p(\Omega)$.

From Theorem 9 of (Akhmet, et al., 2006), we have the following lemma.

Lemma C Assume that the functions G_i are continuously differentiable and there exists a positive-valued function $\eta(M)$ such that

$$\sup_{\|w\|_{\alpha} \leq M} \|G_k(w)\|_{\alpha} \leq \eta(M), k \in N,$$

for some $\alpha \in \left(\frac{1}{2} + \frac{n}{2p}, 1\right)$. Let $w(t, w_0), w = (w_{10}, w_{20}, w_{30}) \in X^{\alpha}$ be a bounded solution of (C3) and (C4), i.e.

$$\|w(t, w_0)\|_C \leq M', t > 0.$$

Then the set $\{w(t, w_0): t > 0\}$ is relatively compact in $C^{1+v}(\bar{\Omega}, R^3)$ for $0 < v < 2\alpha - 1 - \frac{n}{p}$.