ON IRRESPONSIBLE HOMOMORPHISMS AND STRONG DUALITY

by

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Abstract

This thesis looks at algebras with positive primitively defined binary relations that are almost reflexive, anti-symmetric, and transitive and provides new machinery for determining when these algebras are not strongly dualizable.

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Dedication

This thesis is dedicated to my parents, Lynn and Jennifer Kelly, without whom I would never have had the opportunity to attend the University of Northern British Columbia nor the support to pursue mathematics.

Acknowledgement

I would like to thank my original supervisor Jennifer Hyndman. I would also like to thank Matt Reid for stepping in to take on supervisory duties when Dr Hyndman was away, a massive undertaking which required him to step outside of his usual area of research in physics. I would like to thank David Casperson for his valuable insight into the research herein, and his incredibly helpful discussions. I thank David Casperson and Brian Schaan for their help with LaTeX. I appreciate my friends, and my family Lynn, Jennifer, Robyn, and Jade Kelly for their support throughout.

Chapter 1

The Context of this Work

In this document we look at algebras which have a positive primitively defined binary relation that is almost reflexive, transitive, and antisymmetric. We choose these algebras with pp-formulae partially as it has been shown that a particular kind of pp-formula prevents algebras from having enough algebraic operations [7] but also because we can further develop the work in [1] to prove that irresponsibility with respect to $\preceq$ forces an algebra to not be strongly dualizable by any alter ego. The main result in this thesis, Theorem 4.2.1, provides a refinement of the main theorem in [1] to a much more usable form by eliminating the most complicated condition.

Dualizable algebras allow us to consider problems in the language of some alter ego rather than that of the original algebra and this process tends to reduce the complexity of our original problem. If an algebra is strongly dualizable then this principle applies to the alter ego as well, that is we may view problems of the topological alter ego in the language of the original algebra rather than in terms of topology. Early examples of natural duality are documented in the work of Priestley (of distributive lattices), Stone (of Boolean algebras), and Pontryagin (of Abelian groups). This is similar in nature to Galois Theory, wherein a lattice of field extensions is dualized by a lattice of automorphism groups which can be used to gain insight into the nature of the field generating the original lattice. Of note, the Galois dual allows, for example, to prove the insolubility by radicals of 5th degree and higher polynomials. Thus it is of great interest which algebras are dualizable and which of these algebras are strongly dualizable and developing simpler tools to determine the structures which cannot be (strongly) dualized is important.

The authors of [8] use an escalator algebra to provide an example of an algebra which is dualizable but not fully dualizable (and thus not strongly dualizable), an example which we revisit in Chapter 5. In addition, the author of [12] found an example of an algebra which is fully dualizable but not strongly dualizable. These two examples motivate the desire to determine conditions to prove strong duality or, as in this document, simple tools for determining when an algebra is not strongly dualizable.

On strong dualities, we briefly discuss rank in Chapter 2. Rank was one of the first tools used to prove an algebra is strongly dualizable. The authors of [11] developed an improved tool, height, for determining strong duality which we explore in Chapter 3. We then use height in Chapter 4 to develop our new mechanisms. In Chapter 5 we will focus our view on unary algebras and provide two examples where we can guarantee the existence of an irresponsible homomorphism and immediately apply Theorem 4.2.1 to prove that these algebras are, in fact, not strongly dualizable.

Chapter 2

Notation and Background

This chapter lays out the necessary background in algebra, topology, and duality theory used in this thesis. Most of the material on algebra comes from [3], the material on topology and duality theory are from [5] and where appropriate the notation is updated to that of [11].

2.1 Algebra

We begin with a review of products of sets, projection maps, and operations. From here we will lead into the definition of algebras, unary algebras, product algebras, and the rows of an algebra.

2.1.1 Product Sets and Operations

Given a collection of non-empty sets A_k for $1 \leq k \leq n$ we define the **product**, $\prod_{k=1}^{n} A_k$, to be the set

$$\prod_{k=1}^{n} A_k = \{(a_1, a_2, \ldots, a_n) \mid a_k \in A_k \text{ for } 1 \leq k \leq n\}$$

The elements of $\prod_{k=1}^{n} A_k$ are called n-tuples, and occasionally we choose to write $a_1 a_2 \ldots a_n$ instead of $(a_1, a_2, \ldots, a_n)$. For each i, $1 \leq i \leq n$ we define the **projection map**

$$\pi_i : \prod_{k=1}^{n} A_k \to A_i$$

to be

$$\pi_i(a_1, a_2, \ldots, a_n) = a_i$$

for any n-tuple in $\prod_{k=1}^{n} A_k$. In the case that $A_i = A$ for each $1 \leq i \leq n$ then we simply write A^n instead of $\prod_{i=1}^{n} A_i$, and we define A^0 to be the canonical one-element set. A map $f : A^n \to A$ is called an n-ary **operation** on A, and the integer n is called the **arity** of f. If the arity of f is 0 then it is called **nullary** and if the arity is 1 then f is called **unary**. If f is a unary operation that sends each element of A to a particular element of the range then f is called a **constant map**, for example $f : A \to A$ defined by $f(x) = 0$ for each x in A is a constant map. The map $id : A \to A$ defined by $id(x) = x$ for each x in A is called the **identity map**. Finally, if $h : A^n \to A$ is any n-ary operation and a is an element of $(A^n)^I$, for some indexing set I, we define $h(a)$ to be the element of $(A^n)^I$ such that for each $i \in I$ we have

$$h(a)(i) = h(a(i)).$$

2.1.2 Algebras and Subalgebras, Unary Algebras and Products

A collection of operation symbols, $\mathscr{F}$, is called a **language** or **type** if for each operation symbol f in $\mathscr{F}$ there is a nonnegative integer n associated with it called the **arity** of f. An algebra, $\mathbf{A}$, is a pair $\langle A, \mathscr{F} \rangle$ where A is a nonempty set, called the **universe** of $\mathbf{A}$ and each operation symbol $f \in \mathscr{F}$ of arity n has an associated n-ary operation $f^{\mathbf{A}} : A^n \to A$. We sometimes use F to denote the set of all operations on the particular universe A, and so sometimes we will write $\langle A, F \rangle$ instead of $\langle A, \mathscr{F} \rangle$ for the algebra $\mathbf{A}$. The notation $f^{\mathbf{A}}$ is used to make it clear that the operation is on the algebra $\mathbf{A}$. When there is no risk of ambiguity we will drop the superscript and simply write f. A **term** function, $t^{\mathbf{A}}$, is any finite composition of the operations in the language of $\mathbf{A}$ using at most finitely many variables $x_1, x_2, \ldots, x_n$. Again, if there is no risk of ambiguity we drop the superscript and simply write t. Given an algebra $\mathbf{A} = \langle A, F \rangle$ and a set, G, of operations on A, we define the algebra $\mathbf{A}' = \langle A, F \cup G \rangle$ to be an **extension** of $\mathbf{A}$ and the algebra $\mathbf{A}$ is called a **reduct** of $\mathbf{A}'$.

Let $\mathbf{A} = \langle A, \mathscr{F}_1 \rangle$ and $\mathbf{B} = \langle B, \mathscr{F}_2 \rangle$ be two algebras. They are said to be of the same type if $\mathscr{F}_1$ and $\mathscr{F}_2$ are the same. Furthermore, if B is contained in A and for each operation on B we have $f^{\mathbf{B}} = f^{\mathbf{A}} \restriction_B$ then $\mathbf{B}$ is called a **subalgebra** of $\mathbf{A}$, denoted $\mathbf{B} \leq \mathbf{A}$. A **subuniverse** of $\mathbf{A}$ is any (possibly empty) subset of A that is closed under the operations of the algebra $\mathbf{A}$. Therefore, whenever $\mathbf{B}$ is a subalgebra of $\mathbf{A}$, B is also a subuniverse of $\mathbf{A}$.

Note that operations on products of algebras of the same type are defined coordinate-wise as above. So if $\{A_i\}_{i \in I}$ is a collection of algebras of the same type, each $f^{\mathbf{A}_i} : (A_i)^n \to A_i$ is an n-ary operation, and $\mathbf{A} = \prod_{i \in I} \mathbf{A}_i$ then the n-ary operation $f^{\mathbf{A}} : \mathbf{A}^n \to \mathbf{A}$ is defined by

$$f^{\mathbf{A}}(a)(i) = f^{\mathbf{A}_i}(a_i).$$

Furthermore, operations on subalgebras are simply restrictions of the operations on the original algebra. This means that any formula in the language $\mathscr{F}$ can be interpreted in any algebra of the same type, we will revisit this in Chapter 3 when we discuss positive primitive formulae in detail.

When each operation in F has arity 1, we call the algebra a **unary algebra**, and when the range of each operation symbol is contained in $\{0,1\}$ we call the algebra $\{0,1\}$**-valued**. If $|A|$ is finite, then the algebra $\mathbf{A}$ is called **finite**. Furthermore, if $|\mathscr{F}|$ is finite we say the algebra $\mathbf{A}$ is of **finite type**.

Example 2.1.1. *Let $\mathbf{M}_{2.1.1} = \langle \{0,1,2,3\}, f_1, f_2, f_3 \rangle$ be the algebra defined in Table 2.1. Since each operation is unary the algebra is a unary algebra. In addition, each operation maps each element of $\{0,1,2,3\}$ to either 0 or 1, so the algebra is $\{0,1\}$-valued. Thus, $\mathbf{M}_{2.1.1}$ is a $\{0,1\}$-valued unary algebra. Also, since $f_1(f_1(a)) = 0$ for each $a \in M_{2.1.1}$ this algebra has the constant operation $g(x) = 0$ as a term function. Furthermore, we notice that the sets $\{0,1\}$, $\{0,1,2\}$, and $\{0,1,3\}$ are*

	f_1	f_2	f_3
0	0	0	0
1	0	1	1
2	1	1	0
3	0	1	0

Table 2.1: $\mathbf{M}_{2.1.1} = \langle \{0,1,2,3\}, f_1, f_2, f_3 \rangle$

closed under the operations of $\mathbf{M}_{2.1.1}$ and so they all form subalgebras.

Given a collection of algebras of type $\mathscr{F}$, $\mathbf{A}_i$ for $1 \leq i \leq n$ then the product $\mathbf{A} = \prod_{i=1}^{n} \mathbf{A}_i$ is also an algebra of type $\mathscr{F}$, called the **product algebra** of the $\mathbf{A}_i$. For each n-ary operation symbol $f \in \mathscr{F}$ and $a_1, a_2, \ldots, a_n$ in A we define

$$f^{\mathbf{A}}(a_1, a_2, \ldots, a_n)(i) = f^{\mathbf{A}_i}(a_1(i), a_2(i), \ldots, a_n(i))$$

2.1.3 Homomorphisms and the Rows of an Algebra

Let **A** and **B** be two algebras of the same type, $\mathscr{F}$. A map $h : A \to B$ is a **homomorphism** if for each n-ary operation symbol in $\mathscr{F}$ we have

$$h(f^{\mathbf{A}}(a_1, a_2, \ldots, a_n)) = f^{\mathbf{B}}(h(a_1), h(a_2), \ldots, h(a_n)).$$

This property in particular is called **operation preserving**. If in addition h is bijective, then we call it an **isomorphism**. If there is an isomorphism between the algebras **A** and **B** we say they are **isomorphic** and denote this $\mathbf{A} \cong \mathbf{B}$. If there is a homomorphism $h : \mathbf{A} \to \mathbf{B}$ that is onto, then we call **B** a **homomorphic image** of **A**. A one-to-one homomorphism is called an **embedding**.

The authors of [4] define the rows of an algebra as follows. Let $\mathbf{M} = \langle M, F \rangle$ be a finite unary algebra of finite type. Let $F' = \{f_1, f_2, \ldots, f_\nu\}$ be a fixed enumeration of the non-constant, non identity operations in F. For each $a \in M$ we define the **row at** a, denoted $\operatorname{row}(a)$, to be the ν-tuple

$$\operatorname{row}(a) = (f_1(a), f_2(a), \ldots, f_\nu(a)).$$

We then define the ν-ary relation, $\operatorname{Rows}(\mathbf{M})$ as

$$\begin{aligned}\operatorname{Rows}(\mathbf{M}) &= \{\operatorname{row}(a) \mid a \in M\} \\ &= \{(f_1(a), f_2(a), \ldots, f_\nu(a)) \mid a \in M\}.\end{aligned}$$

If **M** is a finite unary algebra of finite type and there is an element $0 \in M$ such that $\operatorname{row}(0) = (0, 0, \ldots, 0)$ then **M** is a **unary algebra with 0**. If there are elements $a, b \in M$ such that $a \neq b$ and $\operatorname{row}(a) = \operatorname{row}(b)$ then we say the algebra **M** has a **repeated row**, or sometimes that **M** has **non-unique rows**. If **M** does not have a repeated row, then we say the algebra has **unique rows**. If **M** has unique rows then we say that the rows of $\mathbf{M}$ are **uniquely witnessed**.

2.1.4 Lattices and Semilattices

The following definitions of lattices and semilattices come from [3].

A binary operation $\leq$ defined on a set A is a **partial order** if for every a, b, and c in A the following are true

1. $a \leq a$ ($\leq$ is reflexive),
2. $a \leq b$ and $b \leq a$ implies $a = a$ ($\leq$ is antisymmetric), and
3. $a \leq b$ and $b \leq c$ implies $a \leq c$ ($\leq$ is transitive).

A set with a partial order is called a **partially ordered set** or simply **poset** and we often denote this $\langle A, \leq \rangle$. If for every pair of elements $a, b \in A$ either $a \leq b$ or $b \leq a$ then $\leq$ is called a **total** order.

An algebra $\mathbf{L} = \langle L, \wedge \rangle$ is a **semilattice** provided $\wedge : L^2 \to L$ satisfies the following properties

1. $x \wedge y \approx y \wedge x$ ($\wedge$ is commutative),
2. $(x \wedge y) \wedge z \approx x \wedge (y \wedge z)$ ($\wedge$ is associative), and
3. $x \wedge x \approx x$ ($\wedge$ is idempotent).

The algebra $\mathbf{L} = \langle L, \wedge, \vee \rangle$, with $\wedge, \vee$ both binary is a **lattice** if it satisfies the following,

1. $\langle L, \wedge \rangle$ is a semilattice,

2. $\langle L, \vee \rangle$ is a semilattice, and

3. $x \wedge (x \vee y) \approx x$ and $x \vee (x \wedge y) \approx x$ ($\wedge$ absorbs $\vee$ and $\vee$ absorbs $\wedge$).

The operations $\wedge$ and $\vee$ are called **meet** and **join** respectively and a semilattice $\langle L, \wedge \rangle$ or $\langle L, \vee \rangle$ is often called **meet semilattice** and **join semilattice** respectively.

If $\mathbf{L}$ is a lattice then we can define a partial order on L by $a \leq b$ whenever $a = a \wedge b$.

If $\mathbf{A}$ is a unary algebra we say that $\mathbf{A}$ has a **meet homomorphism** if there is a homomorphism $\wedge : A^2 \to A$ such that $\langle A, \wedge \rangle$ is a semilattice.

2.2 Topology, Quasi Varieties, and Topological Quasi Varieties

Our discussion of topology will be limited to the absolute basics since we will never be doing any topology explicitly. Once we have what we need we will discuss the categories of (algebraic) quasi varieties and topological quasi varieties.

Given a set X, a collection of subsets of X, $\mathscr{T}$, is a **topology** provided:

1. $\emptyset$ and X are in $\mathscr{T}$,

2. If A and B are in $\mathscr{T}$, then $A \cap B \in \mathscr{T}$, and

3. If $A_i \in \mathscr{T}$ for i in some indexing set I, then $\bigcup_{i \in I} A_i \in \mathscr{T}$.

If the only sets in the topology are the empty set and X then it is called **trivial**. If instead every subset of X is in the topology then it is called **discrete**.

Let $\mathbf{M}$ be a finite algebra. A homomorphism $g : \mathbf{M}^n \to \mathbf{M}$ for $n \in \omega$ is called a **(total) algebraic operation**. If $\mathbf{A} \leq \mathbf{M}^n$ for $n \in \omega$, a homomorphism $h : \mathbf{A} \to \mathbf{M}$ is called a **partial algebraic operation**. Lastly, a subset $r \subseteq M^n$ that forms a subuniverse of $\mathbf{M}^n$ for $n \in \omega$ is called an **algebraic relation**. The integer n is called the arity of the relation.

Let G, H, R be sets of algebraic operations, partial operations, and relations on $\mathbf{M}$ respectively and let $\mathscr{T}$ be the discrete topology on M. The topological structure $\mathbb{M} = \langle M, G, H, R, \mathscr{T} \rangle$ is called an **alter ego** of $\mathbf{M}$. The set $G \cup H \cup R$ is called the **type** of $\mathbb{M}$, and if $|G \cup H \cup R|$ is finite then $\mathbb{M}$ is said to be of **finite type**. The (algebraic) **quasi variety** generated by $\mathbf{M}$ is the class of all isomorphic copies of subalgebras of products of the finite algebra $\mathbf{M}$, and is denoted $\mathbb{ISP}(\mathbf{M})$. Note that the operators $\mathbb{ISP}(\mathbf{M})$ is closed under $\mathbb{S}$ and $\mathbb{P}$, that is $\mathbb{S}(\mathbb{ISP}(\mathbf{M})) = \mathbb{ISP}(\mathbf{M}) = \mathbb{P}(\mathbb{ISP}(\mathbf{M}))$. The **topological quasi variety** generated by $\mathbb{M}$ is the class of all isomorphic copies of closed substructures of positive powers of the alter ego $\mathbb{M}$, and is denoted $\mathbb{IS}_c\mathbb{P}^+(\mathbb{M})$. Similarly, $\mathbb{IS}_c\mathbb{P}^+(\mathbb{M})$ is also closed under the operators $\mathbb{S}_c$ and $\mathbb{P}^+$

In our investigation of algebras we reviewed how isomorphism, subalgebras, and products related to the algebraic structure of a given algebra. But how do isomorphic copies, closed substructures, and positive powers work with the topological structure of the alter ego? We will answer this question from the "inside out," first with positive products, then closed substructures, and finally isomorphic copies.

Suppose $\mathbb{M}$ is an alter ego of the finite algebra $\mathbf{M}$. Let $g : M^n \to M$ be an algebraic operation in G, for some $n \in \omega$ and let S be a non-empty set. We define the algebraic operation $g^{\mathbb{M}^S} : (M^S)^n \to M^S$ by

$$g^{\mathbb{M}^S}(m_1, m_2, \ldots, m_n)(s) = g(m_1(s), m_2(s), \ldots, m_n(s))$$

for all $m_1, m_2, \ldots, m_n \in M^S$ and $s \in S$.

Let $h : \mathrm{dom}(h) \to M$ be an n-ary partial algebraic operation in H, with $\mathrm{dom}(h) \subseteq M^n$ for some $n \in \omega$. The domain of the partial algebraic operation $h^{\mathbb{M}^S}$ is defined as

$$\mathrm{dom}\left(h^{\mathbb{M}^S}\right) = \{(a_1, a_2, \ldots, a_n) \in (M^S)^n \ : \ (a_1(s), a_2(s), \ldots, a_n(s)) \in \mathrm{dom}(h)\, \forall s \in S\}$$

and the partial algebraic operation $h^{\mathbb{M}^S} : \mathrm{dom}\left(h^{\mathbb{M}^S}\right) \to M^S$ is defined by

$$h^{\mathbb{M}^S}(m_1, m_2, \ldots, m_n)(s) = h(m_1(s), m_2(s), \ldots, m_n(s))$$

for all $(m_1, m_2, \ldots, m_n) \in \mathrm{dom}\left(h^{\mathbb{M}^S}\right)$ and $s \in S$.

Given an n-ary relation r in R, for $n \in \omega$, we define the relation $r^{\mathbb{M}^S}$ in the same manner:

$$r^{\mathbb{M}^S} = \{(a_1, a_2, \ldots, a_n) \in (M^S)^n \ : \ (a_1(s), a_2(s), \ldots, a_n(s)) \in r, \forall s \in S\}.$$

Finally, the topology on $\mathbb{M}^S$ is derived from the topology on $\mathbb{M}$ by the standard product topology. This topology is determined by the clopen subbasis

$$U_{s,m} = \{a \in M^S \ : \ a(s) = m\}.$$

Notice that the topology on $\mathbb{M}^S$ is always compact and Hausdorff, and if S is finite then it is discrete, definitions of which can be found in [10].

Each closed substructure $\mathbb{Y}$ of $\mathbb{X}$ is closed in the topology of $\mathbb{X}$ and under the algebraic total and partial operations in $G \cup H$, that is for g an n-ary total operation in G then whenever $(a_1, a_2, \ldots, a_n) \in Y$ we have $g(a_1, a_2, \ldots, a_n) \in Y$. If h is an n-ary partial operation in H then whenever $(a_1, a_2, \ldots, a_n) \in Y$ and $(a_1, a_2, \ldots, a_n) \in \mathrm{dom}(h)$ we have $h(a_1, a_2, \ldots, a_n) \in Y$.

Given two structures $\mathbb{X}$ and $\mathbb{Y}$ of the same type, the map $\alpha : \mathbb{X} \to \mathbb{Y}$ is a **morphism** if α is continuous, and for each n-ary operation in G

$$\alpha(g^{\mathbb{X}}(x_1, x_2, \ldots, x_n)) = g^{\mathbb{Y}}(\alpha(x_1), \alpha(x_2), \ldots, \alpha(x_n)),$$

for each n-ary partial operation in H, whenever $(x_1, x_2, \ldots, x_n) \in \mathrm{dom}\big(h^{\mathbb{X}}\big)$ we have $(\alpha(x_1), \alpha(x_2), \ldots, \alpha(x_n)) \in \mathrm{dom}\big(h^{\mathbb{Y}}\big)$ and

$$\alpha(h^{\mathbb{X}}(x_1, x_2, \ldots, x_n)) = h^{\mathbb{Y}}(\alpha(x_1), \alpha(x_2), \ldots, \alpha(x_n)),$$

and for each n-ary relation r in R, if $(x_1, x_2, \ldots, x_n) \in r^{\mathbb{X}}$ then $(\alpha(x_1), \alpha(x_2), \ldots, \alpha(x_n)) \in r^{\mathbb{Y}}$. If a bijective morphism exists between the structures $\mathbb{X}$ and $\mathbb{Y}$ we borrow from algebra and call them isomorphic, and call the morphism an isomorphism. A one-to-one morphism is called an embedding.

2.3 Natural, Full, and Strong Dualities

In this section we lay out the groundwork for developing dualities, full dualities, and strong dualities. While this thesis does not prove dualizability, it is a necessary condition for strong duality and so is detailed here.

2.3.1 Natural Dualities

Let $\mathbf{M}$ be a finite algebra and $\mathbb{M}$ an alter ego of the algebra $\mathbf{M}$. First we define the contravariant functors D and E between the categories $\mathbb{ISP}(\mathbf{M})$ and $\mathbb{IS}_c\mathbb{P}^+(\mathbb{M})$. For an algebra $\mathbf{A} \in \mathbb{ISP}(\mathbf{M})$ the set $\hom(\mathbf{A},\mathbf{M})$ is the set of all homomorphisms with domain $\mathbf{A}$ and codomain $\mathbf{M}$. We define the **dual** of $\mathbf{A}$ in $\mathbb{IS}_c\mathbb{P}^+(\mathbb{M})$ by

$$D(\mathbf{A}) = \hom(\mathbf{A},\mathbf{M}).$$

For a structure $\mathbb{X} \in \mathbb{IS}_c\mathbb{P}^+(\mathbb{M})$ the set $\hom(\mathbb{X},\mathbb{M})$ is the set of all morphisms with domain $\mathbb{X}$ and

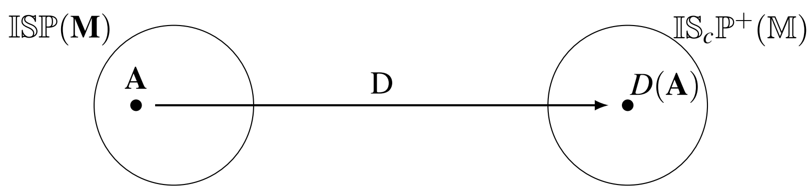

Figure 2.1: The dual of $\mathbf{A}$ is in $\mathbb{IS}_c\mathbb{P}^+(\mathbb{M})$

codomain $\mathbb{M}$. We define the **dual** of $\mathbb{X}$ in $\mathbb{IS}_c\mathbb{P}^+(\mathbb{M})$ by

$$E(\mathbb{X}) = \hom(\mathbb{X},\mathbb{M}).$$

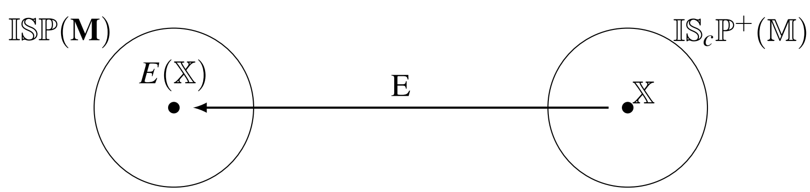

Figure 2.2: The dual of $\mathbb{X}$ is in $\mathbb{ISP}(\mathbf{M})$

For each homomorphism $\varphi : \mathbf{A} \to \mathbf{B}$, with $\mathbf{A}, \mathbf{B} \in \mathbb{ISP}(\mathbf{M})$ the morphism $D(\varphi) : D(\mathbf{B}) \to D(\mathbf{A})$ is defined by

$$D(\varphi)(x) = x \circ \varphi$$

for each $x \in \hom(\mathbf{B}, \mathbf{M})$. For each morphism $\psi : \mathbb{X} \to \mathbb{Y}$, with $\mathbb{X}, \mathbb{Y} \in \mathbb{IS}_c\mathbb{P}^+(\mathbb{M})$ the homomorphism $E(\psi) : E(\mathbb{Y}) \to E(\mathbb{X})$ is defined by

$$E(\psi)(\alpha) = \alpha \circ \psi$$

for each $\alpha \in \hom(\mathbb{Y}, \mathbb{M})$.

Let $\mathbf{A} \in \mathbb{ISP}(\mathbf{M})$, the **evaluation map** $e_{\mathbf{A}} : \mathbf{A} \to ED(\mathbf{A})$ is given by

$$e_{\mathbf{A}}(a)(x) = x(a)$$

for $a \in A$ and $x \in \hom(\mathbf{A}, \mathbf{M})$. If $\mathbf{A} \cong ED(\mathbf{A})$ via the evaluation map then we say that $\mathbb{M}$ **yields a duality** on $\mathbf{A}$. If $\mathbb{M}$ yields a duality on every $\mathbf{A} \in \mathbb{ISP}(\mathbf{M})$ then we say that $\mathbb{M}$ **yields a duality on** $\mathbb{ISP}(\mathbf{M})$, or that $\mathbb{M}$ **dualizes M**. If there is an alter ego that yields a duality on $\mathbf{M}$ we say that $\mathbf{M}$ is **dualizable**.

2.3.2 Full Dualities

There is also a natural embedding for the topological quasi variety, for $\mathbb{X} \in \mathbb{IS}_c\mathbb{P}^+(\mathbb{M})$ the evaluation map $\varepsilon_{\mathbb{X}} : \mathbb{X} \to DE(\mathbb{X})$, given by

$$\varepsilon_{\mathbb{X}}(x)(\alpha) = \alpha(x)$$

for $x \in X$ and $\alpha \in \hom(\mathbb{X},\mathbb{M})$. If $\mathbb{M}$ yields a duality on $\mathbf{M}$ and $\varepsilon_{\mathbb{X}}$ is an isomorphism for each $\mathbb{X} \in \mathbb{IS}_c\mathbb{P}^+(\mathbb{M})$ then we say $\mathbb{M}$ **yields a full duality on** $\mathbb{ISP}(\mathbf{M})$, or that $\mathbb{M}$ **fully dualizes M**. If there is an alter ego that yields a full duality on $\mathbf{M}$ we say that $\mathbf{M}$ is **fully dualizable**.

2.3.3 Strong Dualities

For a pair of maps $x : A \to B$ and $y : A \to B$, the **equalizer** of x and y, is given by

$$\mathrm{eq}(x,y) = \{a \in A \ : \ x(a) = y(a)\}$$

A subset X of M^S, for a non-empty set S is **term closed** if for each $y \in M^S \setminus X$ there are S-ary term functions t_1, t_2 such that t_1 and t_2 agree on X but differ at y. That is, X is term closed if

$$X = \bigcap\{\mathrm{eq}(t_1,t_2) \mid t_1,t_2 \ S\text{-ary and } t_1{\restriction_X} = t_2{\restriction_X}\}$$

If $\mathbb{M}$ yields a duality on $\mathbf{M}$ and each closed substructure of each non-zero power of $\mathbb{M}$ is term closed then $\mathbb{M}$ **yields a strong duality** on $\mathbf{M}$, or $\mathbf{M}$ is **strongly dualized** by $\mathbb{M}$. If there is an alter ego that yields a strong duality on $\mathbf{M}$ then we say that $\mathbf{M}$ is **strongly dualizable**.

The following is the definition of rank as it appears in [6]. Let $\mathbf{M}$ be a finite algebra, n a positive integer, $\mathbf{B}$ a subalgebra of $\mathbf{M}^n$, and $h : \mathbf{B} \to \mathbf{M}$ a homomorphism. We write $\mathbf{B} \Rrightarrow_\sigma \mathbf{B}'$ to denote that $\mathbf{B}'$ is a subalgebra of $\mathbf{M}^{n+k}$ for some finite integer k, σ embeds $\mathbf{B}$ in $\mathbf{B}'$ by repetition of some coordinates and $\mathbf{B} \cong \mathbf{B}'$. Let $h' = \sigma^{-1} \circ h$ be the natural extension of h to $\mathbf{B}'$. Let $\mathbf{B}' \leq \mathbf{C} \leq \mathbf{D} \leq \mathbf{M}^{n+k}$ and assume there exists a homomorphism $h^+ : \mathbf{D} \to \mathbf{M}$ such that $h' = h^+{\restriction_{\mathbf{B}'}}$. When we refer to the commuting diagram in Figure 2.3 we assume the above setup holds.

Let $Y \subseteq \hom(\mathbf{D},\mathbf{M})$, $\mathbf{D}/Y$ is defined as the algebra $\mathbf{D}/\bigcap\{\ker(g) \mid g \in Y\}$ and $\mathbf{C}/Y$ is the algebra

$$\begin{tikzcd} \mathbf{B} \arrow[r, phantom, "\Rrightarrow_\sigma"] \arrow[drr, "h"'] & \mathbf{B}' \arrow[r, phantom, "\leq"] \arrow[dr, "h'"] & \mathbf{C} \arrow[r, phantom, "\leq"] & \mathbf{D} \arrow[dl, dotted, "h^+"] \\ & & \mathbf{M} & \end{tikzcd}$$

Figure 2.3: A commuting diagram for the rank setup

$\mathbf{C}/\bigcap\{\ker(g\restriction_{\mathbf{C}} \mid g \in Y\}$. The set Y **separates $\mathbf{B}'$** if for every pair of elements $a,b \in B'$ there is a homomorphism $g \in Y$ such that $g\restriction_{\mathbf{B}'}(a) \neq g\restriction_{\mathbf{B}'}(b)$. The homomorphism h' **lifts to $\mathbf{C}/Y$** if Y separates $\mathbf{B}'$ and there exists a homomorphism μ such that Figure 2.4 commutes.

$$\begin{tikzcd} \mathbf{B}' \arrow[r, phantom, "\leq"] \arrow[d, "h'"'] \arrow[dr, hook] & \mathbf{C} \arrow[d, twoheadrightarrow] \\ \mathbf{M} & \mathbf{C}/Y \arrow[l, dotted, "\mu"] \end{tikzcd}$$

Figure 2.4: h' lifts to $\mathbf{C}/Y$

Given a homomorphism $h : \mathbf{B} \to \mathbf{M}$, we define the **rank** of h, denoted $\operatorname{rank}(h)$, as follows: $\operatorname{rank}(h) \leq 0$ if and only if h is a projection. $\operatorname{rank}(h) \leq \alpha$ if and only if there exists a finite N such that for all nonnegative integers k, for all subalgebras $\mathbf{D}$ of $\mathbf{M}^{n+k}$, and for all commuting diagrams like Figure 2.3, where h' lifts to $\mathbf{D}$, there exists $Y \subseteq \hom(\mathbf{D},\mathbf{M})$ such that:

- $|Y| \leq N$,
- h' lifts to $\mathbf{C}/Y$, and
- $\operatorname{rank}(g\restriction_{\mathbf{C}}) < \alpha$ for all $g \in Y$.

Further, $\operatorname{rank}(h) = \alpha$ if $\operatorname{rank}(h) \leq \alpha$ and $\operatorname{rank}(h) \not< \alpha$. The **rank** of the algebra $\mathbf{M}$, denoted rank($\mathbf{M}$), is the least α such that rank$(h) \leq \alpha$ for every homomorphism $h \in \hom(\mathbf{A}, \mathbf{M})$ where $\mathbf{A} \leq \mathbf{M}^n$ for some $n \in \omega$. If rank($\mathbf{M}$) is finite then we say $\mathbf{M}$ has **finite rank**.

Now we present the definition of enough algebraic operations, as found in [9]. For each subset Y of $\hom(\mathbf{M}^n, \mathbf{M})$ the **natural product homomorphism**, $\sqcap Y : \mathbf{M}^n \to \mathbf{M}^Y$ is the map

$$\sqcap Y(a)(y) = y(a)$$

for each $a \in M^n$ and $y \in Y$. We say that $\mathbf{M}$ has **enough algebraic operations** if there is a map $f : \omega \to \omega$ such that for all nonnegative integers n, algebras $\mathbf{B} \leq \mathbf{A} \leq \mathbf{M}^n$, and all homomorphisms $h : \mathbf{A} \to \mathbf{M}$ there is a set $Y \subseteq \hom(\mathbf{M}^n, \mathbf{M})$ with $|Y| \leq f(|B|)$ and a homomorphism $h' : \sqcap Y(\mathbf{A}) \to \mathbf{M}$ such that $h' \circ \sqcap Y\upharpoonright_{\mathbf{B}} = h\upharpoonright_{\mathbf{B}}$. That is, the diagram in Figure 2.5 commutes.

$$\begin{tikzcd}
\mathbf{B} \arrow[r, hook] \arrow[d, "h\upharpoonright_{\mathbf{B}}"'] & \mathbf{A} \arrow[r, hook] \arrow[d, "\sqcap Y\upharpoonright_{\mathbf{A}}"] & \mathbf{M}^n \\
\mathbf{M} & \sqcap Y(\mathbf{A}) \arrow[l, dotted, "h'"]
\end{tikzcd}$$

Figure 2.5: A commuting diagram for Enough Algebraic Operations

There is another concept, called height, which is related to rank and enough algebraic operations, but we will reserve our discussion of height to the next chapter of this document.

2.4 Theorems About Strong Duality

To prove a given algebra **M** is strongly dualizable we need to not only find an alter ego $\mathbb{M}$ that dualizes the algebra, but we need to prove that each closed substructure $\mathbb{X} \in \mathbb{IS}_c\mathbb{P}^+(\mathbb{M})$ is actually term closed. In order to prove that a dualizable algebra **M** is *not* strongly dualizable we need to show that for any alter ego $\mathbb{M}$ that dualizes the algebra there is some closed subtructure $\mathbb{X} \in \mathbb{IS}_c\mathbb{P}^+(\mathbb{M})$ that is not term closed. In this section we detail some theorems that allow us to prove that an algebra is strongly dualizable or that it is not strongly dualizable without needing to investigate every possible alter ego. We first make a small definition and note its importance. Throughout this section **M** denotes a finite algebra.

Let H_ω be the set of all algebraic partial operations on **M** with finite arity. The **strong brute-force alter ego** of **M**, defined in [11], is the structure

$$\mathbb{M}_\Omega = \langle M, H_\omega, \mathscr{T} \rangle.$$

This alter ego is special in that if some alter ego strongly dualizes **M**, then so does $\mathbb{M}_\Omega$. Since we are only concerned with whether our algebra *is* strongly dualizable, not finding a useful structure that strongly dualizes **M**, unless we specify the alter ego we will be working with respect to $\mathbb{M}_\Omega$.

We defined rank in the previous section, the following theorem allows us to determine the strong dualizability of **M** based on its rank.

Theorem 2.4.1. *[13] Let* **M** *be a finite algebra. If* **M** *is dualizable and has finite rank, then* **M** *is strongly dualizable.*

With rank in hand we note the stronger, but easier to manage, concept of enough algebraic operations and its significance.

Theorem 2.4.2. *[9], [11] Let* $\mathbf{M}$ *be a finite algebra. If* $\mathbf{M}$ *has enough algebraic operations, then* $\mathbf{M}$ *has rank at most 2.*

Since enough algebraic operations is sufficient to show finite rank, we notice the following

Theorem 2.4.3. *[9], [11] Let* $\mathbf{M}$ *be a finite algebra. If* $\mathbf{M}$ *is dualizable and has enough algebraic operations, then* $\mathbf{M}$ *is strongly dualizable.*

Rank and enough algebraic operations are useful for showing an algebra is strongly dualizable, but if $\mathbf{M}$ does not have enough algebraic operations or does not have finite rank we cannot determine if the algebra is strongly dualizable. Fortunately, in [11] a condition, height, that is equivalent to strong duality is developed.

Theorem 2.4.4. *[11] Let* $\mathbf{M}$ *be a finite algebra.* $\mathbf{M}$ *is strongly dualizable if and only if it is dualizable and has a height.*

In the following chapter we will discuss the concept of height, as well as present the main findings from [1] that inspired the research involved in this thesis.

Chapter 3

Height, Irresponsibility, and Strong Duality

In this chapter we outline the concepts of height and irresponsibility that will form the backbone of the proofs supplied in Chapter 4. The conclusion of this chapter will be the main theorem from [1] that inspires the main results of this document. We utilize the fact that a dualizable algebra does not have a height if and only if it is not strongly dualizable in order to prove particular algebras are not strongly dualizable. Throughout the rest of this thesis we assume that $\mathbf{M}$ is a finite algebra, and whenever we refer to the dual of an algebra in $\mathbb{ISP}(\mathbf{M})$ we take it with respect to the strong brute-force alter ego. All of the definitions in this chapter are found in [1] and [7]

3.1 Height

Let $\mathbf{A}$ be an algebra in $\mathbb{SP}(\mathbf{M})$, the category of all subalgebras of products of the algebra $\mathbf{M}$. For a subalgebra $\mathbf{C}$ of $\mathbf{A}$ we use the notation $\mathbf{C} \ll \mathbf{A}$ to denote that $\mathbf{C}$ is finite. For $\mathbf{C} \ll \mathbf{A}$ a homomorphism $h : \mathbf{C} \to \mathbf{M}$ is a $D(\mathbf{A})$**-fragment** if there is a homomorphism $h^+ : \mathbf{A} \to \mathbf{M}$ such that $h = h^+ {\restriction}_{\mathbf{C}}$ [1]. We denote the set of all $D(\mathbf{A})$-fragments by $\mathrm{Frag}_{\mathrm{D}(\mathbf{A})}$. Finally, if $\mathbf{B}$ is a subalgebra of $\mathbf{A}$, $h : \mathbf{B} \to \mathbf{M}$ and $Y \subseteq \hom(\mathbf{A}, \mathbf{M})$ we say that h **lifts through** Y if there is a homomorphism $p : \sqcap Y(\mathbf{A}) \to \mathbf{M}$ such that the diagram in Figure 3.1 commutes [1].

$$
\begin{tikzcd}
\mathbf{B} \arrow[r, hook] \arrow[d, "h"'] & \mathbf{A} \arrow[d, two heads, "\sqcap Y\restriction_{\mathbf{A}}"] & \\
\mathbf{M} & \sqcap Y(\mathbf{A}) \arrow[l, dotted, "p"] \arrow[r, hook] & \mathbf{M}^Y
\end{tikzcd}
$$

Figure 3.1: h lifts through Y

In [11] the authors describe the height of any homomorphism by beginning with a strong duality. Taking $\mathbb{X}$ as the smallest closed substructure of $\hom(\mathbf{A}, \mathbf{M})$ that contains the projections we have by strong duality that $\mathbb{X}$ is also term closed. Since $\mathbb{X}$ is term closed and contains the projections we have that $\mathbb{X} = \hom(\mathbf{A}, \mathbf{M})$ [11]. So by beginning with the projections and alternately closing under the algebraic operations and closing under topology we will eventually reach all of $D(\mathbf{A})$. The number of steps necessary to construct the homomorphism h is the height of h, and the total number of steps to construct all of $D(\mathbf{A})$ is the height of the algebra $\mathbf{A}$. Instead of using the process of [11] we use a simpler definition found in [1].

The following is the definition of height as it appears in [1]. Let $\mathbf{A}$ be a subalgebra of $\mathbf{M}^I$ for some set I and let $h : \mathbf{B} \to \mathbf{M}$ be a $D(\mathbf{A})$-fragment. We define the height of h in $\mathrm{Frag}_{\mathrm{D}(\mathbf{A})}$ by transfinite induction as follows. The homomorphism h has **height at most** 0 in $\mathrm{Frag}_{\mathrm{D}(\mathbf{A})}$ if it is a coordinate projection. For every ordinal $\alpha > 0$, we say that h has **height at most** α in $\mathrm{Frag}_{\mathrm{D}(\mathbf{A})}$ if there exists a nonnegative integer r such that for all algebras $\mathbf{C}$ with $\mathbf{B} \leq \mathbf{C} \ll \mathbf{A}$ there exists an ordinal $\beta < \alpha$ and a collection of $D(\mathbf{A})$-fragments $Y \subseteq \hom(\mathbf{C}, \mathbf{M})$ such that

1. $|Y| \leq r$,

2. every homomorphism in Y has height at most β in $\mathrm{Frag}_{\mathrm{D}(\mathbf{A})}$, and

3. h lifts through Y.

The **height** of h in $\mathrm{Frag}_{\mathrm{D}(\mathbf{A})}$ is the least ordinal α such that h has height at most α in $\mathrm{Frag}_{\mathrm{D}(\mathbf{A})}$. If no such α exists then we say that h **does not have a height** in $\mathrm{Frag}_{\mathrm{D}(\mathbf{A})}$.

We say that **M does not have a height** if there is an algebra $\mathbf{A} \in \mathbb{SP}(\mathbf{M})$ and a $D(\mathbf{A})$-fragment h that does not have a height in $\mathrm{Frag}_{\mathrm{D}(\mathbf{A})}$. Next we define when a collection of $D(\mathbf{A})$-fragments is dense in $\mathrm{Frag}_{\mathrm{D}(\mathbf{A})}$, a condition that aids in determining when $\mathbf{M}$ does not have a height. We note that if a set is not dense in $\mathrm{Frag}_{\mathrm{D}(\mathbf{A})}$ then the corresponding algebra may still have a height.

Let $\mathbf{A} \in \mathbb{SP}(\mathbf{M})$ and $h : \mathbf{B} \to \mathbf{M}$ with $\mathbf{B} \ll \mathbf{A}$. A collection U of $D(\mathbf{A})$-fragments is **dense at** h if for all positive integers r there is an algebra $\mathbf{C}$ with $\mathbf{B} \leq \mathbf{C} \ll \mathbf{A}$ such that for all nonempty collections of $D(\mathbf{A})$-fragments $Y \subseteq \hom(\mathbf{C}, \mathbf{M})$ for which we have that $|Y| \leq r$ and h lifts through Y implies that $Y \cap U$ is nonempty. A collection U of $D(\mathbf{A})$-fragments is **dense** in $\mathrm{Frag}_{\mathrm{D}(\mathbf{A})}$ if it is dense at each of its members [1]. Note:

Theorem 3.1.1. *[1] Let* $\mathbf{M}$ *be a finite algebra and* $\mathbf{A} \in \mathbb{SP}(\mathbf{M})$. *If* U *is dense in* $\mathrm{Frag}_{\mathrm{D}(\mathbf{A})}$ *and does not contain any projections or constant homomorphisms, then* h *does not have a height in* $\mathrm{Frag}_{\mathrm{D}(\mathbf{A})}$ *for every* $h \in U$.

And so an immediate consequence of Theorem 3.1.1 is:

Corollary 3.1.2. *[1] Let* $\mathbf{M}$ *be a finite algebra and* $\mathbf{A} \in \mathbb{SP}(\mathbf{M})$. *If* U *is a nonempty, dense subset of* $\mathrm{Frag}_{\mathrm{D}(\mathbf{A})}$ *that does not contain any projections or constant homomorphisms, then* $\mathbf{M}$ *is not strongly dualizable.*

3.2 PP-Formulæ and Irresponsibility

A **primitive positive formula**, abbreviated **pp-formula**, in the language of an algebra $\mathbf{M}$ is an existentially quantified conjunction of equations in the language of $\mathbf{M}$ [7]. For a unary algebra such an equation is of the form $f(x) \approx g(y)$ for term operations f and g, either of which may be the identity operation and variables x and y which need not be distinct. For instance, a pp-formula may take the form

$$\Phi : \exists w_1, w_2[f(x) \approx w_2 \mathbin{\&} y \approx g(w_1) \mathbin{\&} x \approx h(x)].$$

Here w_1 and w_2 are existentially quantified variables, x and y are free variables, and f, g, and h are unary term operations. We denote this pp-formula as $\Phi(x,y)$, or simply Φ. Note that we exclude any existentially quantified variables from the argument. In general a pp-formula can have any number of existentially quantified variables.

Given a pp-formula $\Phi(x_1, x_2, \ldots, x_n)$ with existentially quantified variables $w_1, w_2, \ldots, w_m$ and $a_1, a_2, \ldots, a_n$ in M, we say that $\Phi(a_1, a_2, \ldots, a_n)$ **holds in M**, or that **M satisfies** $\Phi(a_1, a_2, \ldots, a_n)$ denoted $\mathbf{M} \models \Phi(a_1, a_2, \ldots, a_n)$, if substituting each x_i with the corresponding a_i in Φ results in a true statement [7]. The value(s) of the tuples $(w_1, w_2, \ldots, w_m)$ such that $\Phi(a_1, a_2, \ldots, a_n)$ holds in $\mathbf{M}$ are called **witnesses** of $\Phi(a_1, a_2, \ldots, a_n)$. For an algebra $\mathbf{A} \in \mathbb{SP}(\mathbf{M})$ and elements $a_1, a_2, \ldots, a_n$ in A, $\mathbf{A} \models \Phi(a_1, a_2, \ldots, a_n)$ if there are witnesses $(w_1, w_2, \ldots, w_m)$ in A such that $\Phi(a_1, a_2, \ldots, a_n)$ is true in $\mathbf{A}$.

Example 3.2.1. *Let* $\mathbf{M}_{3.2.1} = \langle \{0,1,2,3\}, f_0, f_1, f_2, f_3 \rangle$ *be the algebra defined in Table 3.1. Define the pp-formula* $\Phi(x,y)$ *by*

$$\Phi : \exists w[x \approx f_2(w) \mathbin{\&} y \approx f_3(w)].$$

We can see that $\mathbf{M}_{3.2.1}$ *satisfies* $\Phi(0,0), \Phi(0,1)$, *and* $\Phi(1,1)$ *witnessed by* 0 *and* 2, 1, *and* 3, *respectively. Notice that we can define the usual* $\leq$ *relation on* $\{0,1\}$ *by* $a \leq b$ *if* $\mathbf{M} \models \Phi(a,b)$.

	f_0	f_1	f_2	f_3
0	0	0	0	0
1	0	0	0	1
2	0	1	0	0
3	0	0	1	1

Table 3.1: $\mathbf{M}_{3.2.1} = \langle \{0,1,2,3\}, f_0, f_1, f_2, f_3 \rangle$

Given a pp-formula Φ in two free variables, we can define a binary relation $\preceq$ on $\mathbf{M}$ like we did in Example 3.2.1 by

$$a \preceq b \text{ if and only if } a,b \in M \text{ and } \mathbf{M} \models \Phi(a,b).$$

In general, we can define the relation $\preceq$ for any algebra $\mathbf{A}$ in the quasi variety $\mathbb{ISP}(\mathbf{M})$ in a similar way. So we say,

$$a \preceq b \text{ in } \mathbf{A} \quad \text{ if and only if } \quad a,b \in A \text{ and } \mathbf{A} \models \Phi(a,b).$$

Recall that by the definition of operations on product algebras and subalgebras the formula Φ described in the language of $\mathbf{M}$ can in fact be interpreted in the language of any algebra $\mathbf{A}$ in $\mathbb{ISP}(\mathbf{M})$ by using the appropriate term function, $f^{\mathbf{A}}$. Finally we note that for $a,b \in \mathbf{M}^I$ we have that $a \preceq b$ in $\mathbf{M}^I$ if and only if $a(i) \preceq b(i)$ in $\mathbf{M}$ for every $i \in I$.

Note that due to the definition of a pp-formula, and $a_i \preceq b_i$ in $\mathbf{M}$ for each i in some indexing set I then the product of witnesses $w_j = (w_{i,j})_{i\in I}$ is the jth witness of $a = (a_i)_{i\in I} \preceq (b_i)_{i\in I} = b$. Furthermore, if $h : \mathbf{M}^I \to \mathbf{M}$ is a homomorphism, and $a \preceq b$ in $\mathbf{M}^I$, then $h(a) \preceq h(b)$ since if $w_{i,j}$ is the jth witness for $a(i) \preceq b(i)$ in $\mathbf{M}$, then $w_j = (w_{i,j})_{i\in I}$ is the jth witness for $a \preceq b$ in $\mathbf{M}^I$. Since h is a homomorphism it preserves the operations in the type of $\mathbf{M}$, and so $h(w_j)$ is the jth witness for $h(a) \preceq h(b)$. However, if $\mathbf{A}$ is a subalgebra of $\mathbf{M}^I$ and $a,b \in A$ with $a \preceq b$ in $\mathbf{M}^I$ and $g : \mathbf{A} \to \mathbf{M}$, then $g(a) \preceq g(b)$ in $\mathbf{M}$ need not be true since the witnesses w_j of $a \preceq b$ in $\mathbf{M}^I$ may not be an element of A.

With this in mind, for $\mathbf{A}$ a subalgebra of $\mathbf{M}^I$ and $h : \mathbf{A} \to \mathbf{M}$ we say that h is **responsible with respect to** $\preceq$ if for all $a, b \in A$ whenever $a \preceq b$ in $\mathbf{M}^I$ we have $h(a) \preceq h(b)$ in $\mathbf{M}$. Otherwise, we say that h is **irresponsible with respect to** $\preceq$, or simply that h is **irresponsible**. If we are concerned with a particular pair that demonstrates that h is irresponsible, then we may say that h is **irresponsible with respect to** $a \preceq b$.

3.3 Irresponsibility and Height

A given binary relation is called **almost reflexive** if whenever (a, b) is in the relation so are (a, a) and (b, b) [1]. Let $\mathbf{M}$ be a finite algebra with a pp-definable, transitive, antisymmetric, almost reflexive binary relation $\preceq$ and assume there is an irresponsible homomorphism $h' : \mathbf{B}_0 \to \mathbf{M}$ with $\mathbf{B}_0 \leq \mathbf{M}^n$ and a', b' distinct elements of B_0 such that h' is irresponsible with respect to $a' \preceq b'$. We can take $\mathbf{B}_0 = Sg_{\mathbf{M}^n}(\{a', b'\})$, the subalgebra of $\mathbf{M}^n$ generated by the elements a', and b'. The following construction builds toward a result that guarantees a set that is dense in $\mathrm{Frag}_{\mathrm{D}(\mathbf{A})}$. For $a \in M$, the notation $\overline{a}$ will denote the tuple in M^J with value a at each coordinate.

Construction 3.3.1. [1] Let $I = [0,1) \cup \{1, \ldots, n\}$. Let $\mathbf{B} \leq \mathbf{M}^I = \mathbf{M}^{[0,1)} \times \mathbf{M}^n$ be an isomorphic copy of $\mathbf{B}_0$ obtained by the following. Since $a' \preceq b'$ in $\mathbf{M}^n$ and $a' \neq b'$, we have that for all j, $1 \leq j \leq n$ $a'(j) \preceq b'(j)$, and for some $\imath$, $a'(\imath) \neq b'(\imath)$. Define the embedding $\sigma : \mathbf{M}^n \to \mathbf{M}^I$ by

$$\sigma(c) = \left(\overline{c(\imath)}, c\right).$$

Fix $a = \sigma(a')$ and $b = \sigma(b')$, and define $\mathbf{B} = \sigma(\mathbf{B}_0)$. Note that $\sigma : \mathbf{B}_0 \to \mathbf{B}$ is an isomorphism. Finally, let $h = h' \circ \sigma^{-1}\restriction_{\mathbf{B}}$. We call h the **associate** of h'.

The next construction allows us to build the desired algebra $\mathbf{A}$ in $\mathbb{SP}(\mathbf{M})$ to form the dense in $\mathrm{Frag}_{\mathrm{D}(\mathbf{A})}$ set.

Construction 3.3.2. [1] Fix m_0 an even integer with $m_0 > |M|$. Set $k_0 = 0$ and for each integer $r \geq 1$, set $k_r = rm_0 + 2$. Note that each k_r is even. For integers r and i, $r \geq 0$ and $0 \leq i \leq 1 + k_r$ set $q_i^r = \frac{i}{1+k_r}$ and define $c_i^r \in M^I$ as:

$$c_0^r = a$$

$$c_i^r(x) = \begin{cases} a(x), & \text{for } x \in [0, 1-q_i^r), \\ b(x), & \text{for } x \in [1-q_i^r, 1), \\ a(x), & \text{for } x \geq 1, \end{cases} \qquad \text{for } 1 \leq i \leq k_r, \text{ and}$$

$$c_{1+k_r}^r = b.$$

Note that, for $1 \leq i \leq k_r$ the element c_i^r has value $a'(\imath)$ on $[0, 1-q_i^r)$ and value $b'(\imath)$ on $[1-q_i^r, 1)$ as in the following figure.

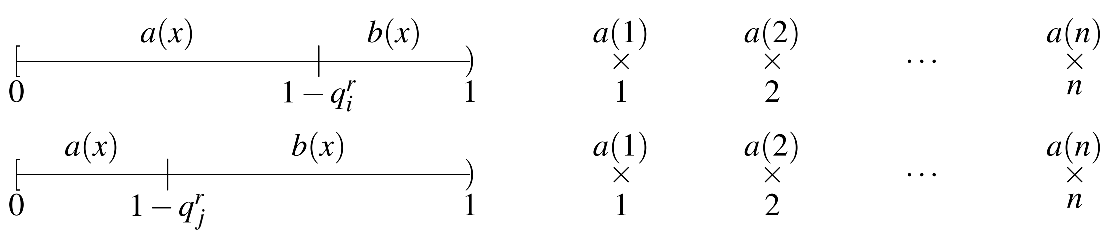

Figure 3.2: Elements c_i^r and c_j^r with $i < j$.

As we can see in the figure, it would appear that $c_i^r \preceq c_{i+1}^r$ in $\mathbf{M}^I$ for each i with $0 \leq i \leq k_r$. We wish to build an algebra $\mathbf{A}$ so that $c_i^r \preceq c_j^r$ is witnessed in $\mathbf{A}$ for particular choices of i and j, namely when $j - i$ is even. To do so we construct our choices of witnesses.

Let Γ be an indexing set for the witnesses of the pp-formula that defines $\preceq$ that is, the pp-formula Φ defining $\preceq$ has $|\Gamma|$ existentially quantified variables and each $\gamma \in \Gamma$ corresponds to one of these variables. For $\gamma \in \Gamma$ let ${}^{\gamma}w_{a',a'}, {}^{\gamma}w_{a',b'}$, and ${}^{\gamma}w_{b',b'}$ be a fixed γ^{th} witness for $a' \preceq a', a' \preceq b'$, and $b' \preceq b'$ in $\mathbf{M}^n$ respectively. Let ${}^{\gamma}w_{a,a} = \sigma({}^{\gamma}w_{a',a'}), {}^{\gamma}w_{a,b} = \sigma({}^{\gamma}w_{a',b'})$, and ${}^{\gamma}w_{b,b} = \sigma({}^{\gamma}w_{b',b'})$. For $\gamma \in \Gamma$ and $0 \leq i < j \leq k_r$ with $j - i$ even, define ${}^{\gamma}w^r_{i,j} \in M^I$ by

$$
{}^{\gamma}w^r_{i,j}(x) = \begin{cases} {}^{\gamma}w_{a,a}(x), & \text{for } x \in [0, 1 - q^r_j), \\ {}^{\gamma}w_{a,b}(x), & \text{for } x \in [1 - q^r_j, 1 - q^r_i), \\ {}^{\gamma}w_{b,b}(x), & \text{for } x \in [1 - q^r_i, 1), \\ {}^{\gamma}w_{a,a}(x), & \text{for } x \geq 1, \end{cases}
$$

and for i odd, with $0 < i < k_r$, we define ${}^{\gamma}w_{i,1+k_r} \in M^I$ by

$$
{}^{\gamma}w^r_{i,1+k_r}(x) = \begin{cases} {}^{\gamma}w_{a,b}(x), & \text{for } x \in [0, 1 - q^r_i), \\ {}^{\gamma}w_{b,b}(x), & \text{for } x \in [1 - q^r_i, 1), \\ {}^{\gamma}w_{a,b}(x), & \text{for } x \geq 1. \end{cases}
$$

Notice that for $0 \leq i < j \leq 1 + k_r$ and $j - i$ even, the element ${}^{\gamma}w^r_{i,j}$ witnesses $c^r_i \preceq c^r_j$ in $\mathbf{M}^I$.

For $r \geq 0$, let

$$
\begin{aligned}
C'_r &= \{c^r_i \mid 0 \leq i \leq 1 + k_r\}, \\
W_r &= \{{}^{\gamma}w^r_{i,j} \mid \gamma \in \Gamma,\ 0 \leq i < j \leq 1 + k_r \text{ with } j - i \text{ even}\}, \\
\mathbf{C}_r &= Sg_{\mathbf{M}^I}(C'_r \cup W_r), \\
\mathbf{A} &= Sg_{\mathbf{M}^I}\left(\bigcup\nolimits_{r \geq 0} C_r\right),
\end{aligned}
$$

completing the construction.

Now, $\mathbf{C}_0 = \mathbf{B} \leq \mathbf{C}_r$, since W_0 is empty. Also, $c_0^r \preceq c_1^r \preceq \cdots \preceq c_{1+k_r}^r$ in $\mathbf{M}^I$ and by choice of witnesses $c_i^r \preceq c_j^r$ in $\mathbf{C}_r$ when $i < j$ and $j - i$ is even.

Let $r > 0$, and let P_r be the set of irresponsible $D(\mathbf{A})$-fragments with domain $\mathbf{C}_r$. We define the set U_h to be $U_h = \{h\} \cup \bigcup_{r>0} P_r$. Now we present the main theorem of [1].

Theorem 3.3.1. *[1] Let $\mathbf{M}$ be a finite algebra. Assume there is a binary relation $\preceq$ on $\mathbf{M}$ that is pp-definable, antisymmetric, transitive, and almost reflexive. Further assume that*

1. *For some $\mathbf{B}_0 \leq \mathbf{M}^n$, there exists a homomorphism $h' : \mathbf{B}_0 \to \mathbf{M}$ and there exist distinct elements a', b' in B_0 with $a' \preceq b'$ in $\mathbf{M}^n$, such that $h'(a') \not\preceq h'(b')$, and such that $\mathbf{B}_0 = Sg_{\mathbf{M}^n}(\{a', b'\})$.*
2. *The associate $h : \mathbf{B} \to \mathbf{M}$ of h' is a $D(\mathbf{A})$-fragment, where $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}_r$ with $r \geq 0$ are defined as in Construction 3.3.1 and Construction 3.3.2.*
3. *For all $r > 0$ and $u, v \in C_r$ with $u \preceq v$ in $\mathbf{M}^I$ such that there is a $D(\mathbf{A})$-fragment $g : \mathbf{C}_r \to \mathbf{M}$ with $g(u) \not\preceq g(v)$, there exist integers i, j with $0 \leq i < j \leq 1 + k_r$ such that $u = c_i^r$ and $v = c_j^r$.*

Then the set U_h is dense in $\mathrm{Frag}_{\mathbf{D(A)}}$, and therefore $\mathbf{M}$ is not strongly dualizable.

There are two other results that will find some use in the following chapters, they are

Lemma 3.3.2. *[1] Fix $r_1, r_2 > 0$. If $c_{i_1}^{r_1} \in C'_{r_1}$ and $c_{i_2}^{r_2} \in C'_{r_2}$ with $c_{i_1}^{r_1} = c_{i_2}^{r_2}$, then $i_2 - i_1$ is even.*

If there exists a term τ such that for all $\gamma \in \Gamma$, $r \geq 1$, and i and l with $0 \leq i < l \leq 1 + k_r$ and $l - i$ even, we have $\tau({}^{\gamma}w_{i,l}^r) \in \{c_i^r, c_l^r\}$, then, whenever ${}^{\gamma}w_{i_1,j_1}^{r_1}$ and ${}^{\gamma}w_{i_2,j_2}^{r_2}$ are defined and equal, $i_2 - i_1$ is even.

and

Lemma 3.3.3. *[1] Fix a non-negative integer* ν *and a positive integer* s. *There exist integers* μ *and* t *with* t *odd,* $t > s$, *and* $\mu > \nu$ *such that* $\mathbf{C}_\nu \leq \mathbf{C}_\mu$ *and such that for all integers* i *with* $0 \leq i \leq 1 + k_\nu$ *we have* $c_i^\nu = c_{ti}^\mu$.

Lemma 3.3.3 allows us to create superalgebras that "fill in" the gaps between the c_i^rs, to guarantee density at each irresponsible homomorphism with domain C_r.

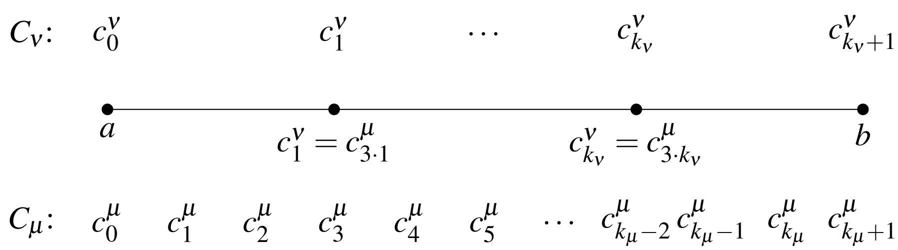

Figure 3.3: Two example algebras $\mathbf{C}_\nu$ and $\mathbf{C}_\mu$ from Lemma 3.3.3

Chapter 4

Height of Algebras and Irresponsibility

In this chapter we construct a different dense-in-$\mathrm{Frag}_{\mathrm{D}(\mathbf{A})}$ set using tools outlined in the previous chapter. While this construction applies to all algebras, regardless of type, in the next chapter we apply the result to unary algebras. Throughout this chapter, we assume that $\mathbf{M}$ is an algebra with a positive primitively defined binary relation, $\preceq$ that is almost reflexive, anti-symmetric, and transitive.

4.1 On Irresponsible and Responsible Fragments

Let $\mathbf{A} \leq \mathbf{M}^I$. Recall that $h : \mathbf{A} \to \mathbf{M}$ is irresponsible if there exists $a, b \in A$ such that $a \preceq b$ in $\mathbf{M}^I$ but $h(a) \not\preceq h(b)$ in $\mathbf{M}$. Otherwise, the homomorphism is responsible. Thus, if a given homomorphism $h : \mathbf{A} \to \mathbf{M}$ extends to $\mathbf{M}^I$, then it must be responsible. Note that projections and constant homomorphisms are responsible.

Given a set X with a binary relation $\preceq$ a subset C of X is called **convex** if,

$$a \preceq c \ \& \ c \preceq b \text{ implies } c \in C \text{ for all } a, b \in C \text{ and } c \in X$$

The following Lemma from [1] provides the maximum number of convex sets the kernel of $\sqcap Y$ can partition an ordered set into, provided each y in Y is *responsible*.

Lemma 4.1.1. *[1] Let* $\mathbf{M}$ *be a finite algebra and let* $\preceq$ *be an antisymmetric and transitive pp-definable binary relation on* $\mathbf{M}$*. Let* $\mathbf{C} \ll \mathbf{A} \leq \mathbf{M}^I$ *with* $d_1, \ldots, d_k \in C$ *for some* k*, and assume that*

$$d_1 \preceq d_2 \preceq \cdots \preceq d_k \textit{ in } \mathbf{M}^I$$

Let $Y \subseteq \hom(\mathbf{C}, \mathbf{M})$ *be a finite collection of responsible* $D(\mathbf{A})$*-fragments. Then the kernel of* $\sqcap Y$ *partitions* $\{d_1, \ldots, d_k\}$ *into at most* $|Y|(|M|-1)+1$ *convex sets with respect to* $\preceq$ *on* $\mathbf{M}^I$.

Lemma 4.1.2 is a subtle modification to this result. In Lemma 4.1.2 we take Y to be a collection of $D(\mathbf{A})$-fragments that only need to be responsible with respect to $d_i \preceq d_j$ for $i < j$ rather than fragments which are simply responsible.

Lemma 4.1.2. *Let* $\mathbf{M}$ *be a finite algebra and let* $\preceq$ *be an antisymmetric and transitive pp-defined binary relation on* $\mathbf{M}$*. Let* $\mathbf{C} \ll \mathbf{A} \leq \mathbf{M}^I$ *with* $d_1, \ldots, d_k \in C$ *for some* k*, and*

$$d_1 \preceq d_2 \preceq \cdots \preceq d_k \textit{ in } \mathbf{M}^I$$

Let $Y \subseteq \hom(\mathbf{C}, \mathbf{M})$ *be a collection of* $D(\mathbf{A})$*-fragments responsible with respect to* $d_i \preceq d_j$ *for* $1 \leq i < j \leq k$*. Then the kernel of* $\sqcap Y$ *partitions* $\{d_1, \ldots, d_k\}$ *into at most* $|Y|(|M|-1)+1$ *convex sets with respect to* $\preceq$ *on* $\mathbf{M}^I$.

Proof. Since each $y \in Y$ is responsible with respect to $d_1 \preceq \cdots \preceq d_k$, we have that $y(d_i) \preceq y(d_j)$ for each $i < j$. Since $\preceq$ is antisymmetric and transitive on $\mathbf{M}$ it is antisymmetric and transitive on $\mathbf{M}^I$. If $y(d_i) = y(d_j)$, then $y(d_i) \preceq y(d_t) \preceq y(d_j) = y(d_i)$ for each $i \leq t \leq j$, and so $y(d_i) = y(d_t) = y(d_j)$ for each $i \leq t \leq j$. So each y partitions $\{d_1, \ldots, d_k\}$ into at most $|M|$ convex sets. Now, if $\sqcap Y(d_i) \neq \sqcap Y(d_j)$ then there is some y such that $y(d_i) \neq y(d_j)$. There are at most $|M| - 1$ values i such that $y(d_i) \neq y(d_{i+1})$, so there are at most $|Y|(|M| - 1)$ values i such that $\sqcap Y(d_i) \neq \sqcap Y(d_{i+1})$. Therefore the kernel of $\sqcap Y$ partitions $\{d_1, \ldots, d_k\}$ into at most $|Y|(|M| - 1) + 1$ convex sets with respect to $\preceq$. □

Using Constructions 3.3.1, 3.3.2, and Lemma 4.1.2, we prove that irresponsible $D(\mathbf{A})$-fragments have no height, and therefore certain algebras are not strongly dualizable. This result sets us up to provide our alternative dense-in-$\mathrm{Frag}_{\mathrm{D}(\mathbf{A})}$ set.

4.2 The New Dense Set

We utilize the same constructions provided in [1], as presented in Chapter 3. We restate those constructions here.

Construction 3.3.1. [1] Suppose there is a pair of elements $a', b' \in \mathbf{M}^n$ such that there is a homomorphism $h' : \mathbf{B}_0 \to \mathbf{M}$ such that $a' \preceq b'$ in $\mathbf{M}^n$ but $h'(a') \neq h'(b')$, where $\mathbf{B}_0 = Sg_{\mathbf{M}^n}(\{a', b'\})$. Let $I = [0,1) \cup \{1, \ldots, n\}$. Let $\mathbf{B} \leq \mathbf{M}^I = \mathbf{M}^{[0,1)} \times \mathbf{M}^n$ be an isomorphic copy of $\mathbf{B}_0$ obtained by the following. Since $a' \preceq b'$ in $\mathbf{M}^n$ and $a' \neq b'$, we have that for all j, $1 \leq j \leq n$ $a'(j) \preceq b'(j)$, and for some $\imath$, $a'(\imath) \neq b'(\imath)$. Define the embedding $\sigma : \mathbf{M}^n \to \mathbf{M}^I$ by

$$\sigma(c) = \left(\overline{c(\imath)}, c\right).$$

Fix $a = \sigma(a')$ and $b = \sigma(b')$, and define $\mathbf{B} = \sigma(\mathbf{B}_0)$. Finally, let $h = h' \circ \sigma^{-1} \restriction_{\mathbf{B}}$.

Construction 3.3.2. [1] Fix m_0 an even integer with $m_0 > |M|$. Set $k_0 = 0$ and for each integer $r \geq 1$, set $k_r = rm_0 + 2$, so each k_r is even. For integers r and i, $r \geq 0$ and $0 \leq i \leq 1 + k_r$ set $q_i^r = \frac{i}{1+k_r}$ and define $c_i^r \in M^I$ as:

$$c_0^r = a$$

$$c_i^r(x) = \begin{cases} a(x), & \text{for } x \in [0, 1 - q_i^r), \\ b(x), & \text{for } x \in [1 - q_i^r, 1), \\ a(x), & \text{for } x \geq 1, \end{cases} \qquad \text{for } 1 \leq i \leq k_r, \text{ and}$$

$$c_{1+k_r}^r = b.$$

Note that, for $1 \leq i \leq k_r$ the element c_i^r has value $a'(\iota)$ on $[0, 1 - q_i^r)$ and value $b'(\iota)$ on $[1 - q_i^r, 1)$.

Let Γ be an indexing set for the witnesses of the pp-formula that defines $\preceq$, for $\gamma \in \Gamma$ let ${}^{\gamma}w_{a',a'}, {}^{\gamma}w_{a',b'}$, and ${}^{\gamma}w_{b',b'}$ be the γ^{th} witnesses for $a' \preceq a', a' \preceq b'$, and $b' \preceq b'$ in $\mathbf{M}^n$ respectively. Let ${}^{\gamma}w_{a,a} = \sigma({}^{\gamma}w_{a',a'}), {}^{\gamma}w_{a,b} = \sigma({}^{\gamma}w_{a',b'})$, and ${}^{\gamma}w_{b,b} = \sigma({}^{\gamma}w_{b',b'})$. For $\gamma \in \Gamma$ and $0 \leq i < j \leq k_r$ with $j - i$ even, define ${}^{\gamma}w_{i,j}^r \in M^I$ by

$${}^{\gamma}w_{i,j}^r(x) = \begin{cases} {}^{\gamma}w_{a,a}(x), & \text{for } x \in [0, 1 - q_j^r), \\ {}^{\gamma}w_{a,b}(x), & \text{for } x \in [1 - q_j^r, 1 - q_i^r), \\ {}^{\gamma}w_{b,b}(x), & \text{for } x \in [1 - q_i^r, 1), \\ {}^{\gamma}w_{a,a}(x), & \text{for } x \geq 1, \end{cases}$$

and for i odd, with $0 < i < k_r$, we define ${}^{\gamma}w_{i,1+k_r} \in M^I$ by

$$
{}^{\gamma}w^r_{i,1+k_r}(x) = \begin{cases} {}^{\gamma}w_{a,b}(x), & \text{for } x \in [0, 1-q^r_i), \\ {}^{\gamma}w_{b,b}(x), & \text{for } x \in [1-q^r_i, 1), \\ {}^{\gamma}w_{a,b}(x), & \text{for } x \geq 1. \end{cases}
$$

Notice that for $0 \leq i < j \leq 1 + k_r$ and $j - i$ even, the element ${}^{\gamma}w^r_{i,j}$ witnesses $c^r_i \preceq c^r_j$ in $\mathbf{M}^I$.

For $r \geq 0$, let

$$
\begin{aligned}
C'_r &= \{c^r_i \mid 0 \leq i \leq 1 + k_r\}, \\
W_r &= \{{}^{\gamma}w^r_{i,j} \mid \gamma \in \Gamma,\ 0 \leq i < j \leq 1 + k_r \text{ with } j - i \text{ even}\}, \\
\mathbf{C}_r &= Sg_{\mathbf{M}^I}(C'_r \cup W_r), \\
\mathbf{A} &= Sg_{\mathbf{M}^I}\left(\bigcup\nolimits_{r \geq 0} C_r\right).
\end{aligned}
$$

In [1] the authors constructed their dense set by taking every homomorphism with domain C_r for $r \geq 0$. Lemma 4.1.2 allows us to build the new dense set using only those homomorphisms with domain C_r that are irresponsible themselves. Let $P'_r = \{g : \mathbf{C}_r \to \mathbf{M} \,:\, g(c^r_i) \not\preceq g(c^r_j) \text{ for some } i < j\}$. Then, define the set V_h by

$$
V_h = \{h\} \cup (\bigcup_{r \geq 0} P'_r).
$$

With this construction in hand and Lemmas 3.3.2 and 3.3.3 we have the tools to present our main theorem: the set V_h is dense-in-$\mathrm{Frag}_{\mathrm{D}(\mathbf{A})}$.

Theorem 4.2.1. *Let* $\mathbf{M}$ *be a finite algebra. Suppose that* $\mathbf{M}$ *has a pp-defined, almost reflexive, antisymmetric, transitive, binary relation* $\preceq$*. Suppose further:*

1. *For some $\mathbf{B}_0 \leq \mathbf{M}^n$, there exists a homomorphism $h' : \mathbf{B}_0 \to \mathbf{M}$ and there are distinct elements $a', b' \in B_0$ with $a' \preceq b'$ in $\mathbf{M}^n$, such that $h'(a') \not\preceq h'(b')$ and $\mathbf{B}_0 = Sg_{\mathbf{M}^n}(\{a', b'\})$.*

2. *The associate $h : \mathbf{B} \to \mathbf{M}$ of h' is a $D(\mathbf{A})$-fragment, where $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}_r$, with $r \geq 0$ are all defined as in Constructions 3.3.1 and 3.3.2.*

Then the set V_h is dense in $\mathrm{Frag}_{\mathrm{D}(\mathbf{A})}$ and contains no projections or constant homomorphisms, and therefore $\mathbf{M}$ cannot be strongly dualized.

Proof. Recall that a set of fragments is dense if it is dense at each of its members. We first show that V_h is dense at h, and then that it is dense for any other fragment, $g \in V_h$.

First, recall $h : \mathbf{C}_0 \to \mathbf{M}$ is a $D(\mathbf{A})$-fragment. Fix $r > 0$ and recall that $C_0 \subseteq C_r$, assume that $Y \subseteq \hom(\mathbf{C}_r, \mathbf{M})$ is a non-empty collection of $D(\mathbf{A})$-fragments such that h lifts through Y and $|Y| \leq r$. Therefore, there is some homomorphism $p : \sqcap Y(\mathbf{C}_r) \to \mathbf{M}$ such that $p \circ (\sqcap Y){\restriction_{\mathbf{C}_r}} = h$.

Assume for a contradiction, that $Y \cap V_h = \emptyset$. Then each $y \in Y$ is responsible with respect to $c_0^r \preceq c_1^r \preceq \cdots \preceq c_{1+k_r}^r$. Therefore, by Lemma 4.1.2, the homomorphism $\sqcap Y$ partitions $\{c_1^r, \ldots, c_{k_r}^r\}$ into at most

$$|Y|(|M|-1)+1 \leq r(|M|-1)+1$$

convex sets. Since $k_r \geq r(|M|-1)+1$ we have that

$$\sqcap Y(c_i^r) = \sqcap Y(c_s^r) = \sqcap Y(c_j^r)$$

for some $i < j$ and all s with $i \leq s \leq j$. Thus, we can choose i' even and j' odd with $i \leq i' \leq j$ and $i \leq j' \leq j$ so that $\sqcap Y(c_{i'}^r) = \sqcap Y(c_{j'}^r)$, and $c_0^r \preceq c_{i'}^r$ in $\mathbf{C}_r$ and $c_{j'}^r \preceq c_{1+k_r}^r$ in $\mathbf{C}_r$. Then, because each

y respects $c_0^r \preceq c_{i'}^r$ and $c_{j'}^r \preceq c_{1+k_r}^r$ as $y \notin V_h$

$$\sqcap Y(c_0^r) \preceq \sqcap Y(c_{i'}^r) = \sqcap Y(c_{j'}^r) \preceq \sqcap Y(c_{1+k_r}^r) \text{ in } \sqcap Y(\mathbf{C}_r),$$

and by applying p we must have

$$(p \circ \sqcap Y)(c_0^r) \preceq (p \circ \sqcap Y)(c_{i'}^r) = (p \circ \sqcap Y)(c_{j'}^r) \preceq (p \circ \sqcap Y)(c_{1+k_r}^r) \text{ in } \mathbf{M}$$

and by transitivity this gives

$$h(a) = h(c_0^r) = (p \circ \sqcap Y)(c_0^r) \preceq (p \circ \sqcap Y)(c_{1+k_r}^r) = h(c_{1+k_r}^r) = h(b)$$

or more precisely that $h(a) \preceq h(b)$ in $\mathbf{M}$. This is a contradiction, therefore $Y \cap V_h \neq \emptyset$ and V_h is dense at h.

To prove V_h is dense at each other fragment, pick $g \in V_h$. We know $g : \mathbf{C}_\nu \to \mathbf{M}$ for some fixed $\nu > 0$ and that there are i and j with $0 \leq i < j \leq 1 + k_\nu$ such that $c_i^\nu \preceq c_j^\nu$ but $g(c_i^\nu) \not\preceq g(c_j^\nu)$. Note that $j - i$ must be odd, otherwise $c_i^\nu \preceq c_j^\nu$ is witnessed in $\mathbf{C}_\nu$ and so we would have $g(c_i^\nu) \preceq g(c_j^\nu)$ witnessed in $\mathbf{M}$.

Again, fix $r > 0$, and use Lemma 3.3.3 to find integers t and μ such that t is odd, $t \geq k_r + 3$, $\mathbf{C}_\nu \leq \mathbf{C}_\mu$ and for each $0 \leq i \leq 1 + k_\nu$, $c_i^\nu = c_{ti}^\mu$. By the definition of dense at g let $Y \subseteq \hom(\mathbf{C}_\mu, \mathbf{M})$ be a collection of $D(\mathbf{A})$-fragments, with $|Y| \leq r$ and g lifts through Y.

Assume for a contradiction, that $Y \cap V_h = \emptyset$. Then as before Y is a collection of $D(\mathbf{A})$-fragments each of which is responsible with respect to $c_0^\mu \preceq c_1^\mu \preceq \cdots \preceq c_{1+k_\mu}^\mu$. Thus, $\sqcap Y$ partitions $S = \{c_{ti+1}^\mu, c_{ti+2}^\mu, \ldots, c_{tj-1}^\mu\} \subseteq C_\mu$ into at most $|Y|(|M|-1)+1 \leq r(|M|-1)+1$ convex sets. So S has at least $(tj-1)-(ti+1) = t(j-i)-2 \geq t-2$ elements. Now, $r(|M|-1)+1 < rm_0+1 <$

$k_r < t-2$ so there are i', j' such that $\sqcap Y(c^{\mu}_{i'}) = \sqcap Y(c^{\mu}_{j'})$ and $ti+1 \leq i' < j' \leq tj-1$. Therefore, $\sqcap Y(c^{\mu}_{i'}) = \sqcap Y(c^{\mu}_{s}) = \sqcap Y(c^{\mu}_{j'})$ for all s with $i' \leq s \leq j'$. Since $i' < j'$ it follows we can choose i^* and j^* such that $i' \leq i^* \leq j'$ and $i' \leq j^* \leq j'$ and both $i^* - ti$ and $tj - j^*$ are even and positive. This gives $c^{\mu}_{ti} \preceq c^{\mu}_{i^*}$ in $\mathbf{C}_{\mu}$ and $c^{\mu}_{j^*} \preceq c^{\mu}_{tj}$ in $\mathbf{C}_{\mu}$. Thus, because each y respects $c^{\mu}_{ti} \preceq c^{\mu}_{i*}$ and $c^{\mu}_{j*} \preceq c^{\mu}_{tj}$ as $y \notin V_h$

$$\sqcap Y(c^{\mu}_{ti}) \preceq \sqcap Y(c^{\mu}_{i^*}) = \sqcap Y(c^{\mu}_{j^*}) \preceq \sqcap Y(c^{\mu}_{tj}) \text{ in } \sqcap Y(\mathbf{C}_{\mu})$$

If $p' : \sqcap Y(\mathbf{C}_{\mu}) \to \mathbf{M}$ is the map that lifts g through Y, then applying p' and transitivity to the above gives:

$$g(c^{\nu}_{i}) = g(c^{\mu}_{ti}) = (p' \circ \sqcap Y)(c^{\mu}_{ti}) \preceq (p' \circ \sqcap Y)(c^{\mu}_{tj}) = g(c^{\mu}_{tj}) = g(c^{\nu}_{j})$$

This gives $g(c^{\nu}_{i}) \preceq g(c^{\nu}_{j})$ in $\mathbf{M}$. This is again a contradiction, therefore $Y \cap V_h \neq \emptyset$ and V_h is dense at g.

Since V_h is dense at each of its members, it is dense in $\mathrm{Frag}_{\mathrm{D}(\mathbf{A})}$. V_h also contains no projections or constant homomorphisms, therefore, the algebra $\mathbf{M}$ is not strongly dualizable by Corollary 3.1.2. $\square$

Chapter 5

Unary Algebras and Irresponsibility

In the previous chapter we proved a result regarding irresponsible homomorphisms and any algebra, of any type. Since our main focus is with unary algebras, in this chapter we attempt to apply the previous result to this class of algebras. Throughout this chapter we assume that an algebra $\mathbf{M}$ has a pp-defined binary relation that is almost reflexive, antisymmetric, and transitive; in Section 5.3 we demonstrate that escalator algebras do, in fact, have such a relation.

5.1 Irresponsible Fragments

One useful fact about unary algebras is that the union of subalgebras is again a subalgebra. This can be seen by looking at the arity of the operations. If $\mathbf{A}$ is a unary algebra and $\mathbf{B}$ and $\mathbf{C}$ are subalgebras, then $A \cup B$ is the universe of a different subalgebra. Indeed, since if $b \in B$ then for any operation f in the type of $\mathbf{A}$ we have that $f(b) \in B$ which implies that $f(b) \in B \cup C$; and similarly, if $c \in C$ then $f(c) \in C$ and therefore $f(c) \in B \cup C$.

This is important since in Construction 3.3.2 we build an algebra $\mathbf{A}$ by taking the union of the algebras $\mathbf{C}_r$. Since each $\mathbf{C}_r$ is a subalgebra of $\mathbf{A}$ we can never "pick up" the witness for $c_i^r \preceq c_j^r$ when $j-i$ is odd in the algebraic closure required to form the algebra $\mathbf{A}$. Therefore, we have the following.

Corollary 5.1.1. *Let $\mathbf{M}$ be a finite unary algebra. Suppose that $\mathbf{M}$ has a pp-defined binary relation that is almost reflexive, antisymmetric, and transitive. Suppose further that there exists a subalgebra $\mathbf{B}$ of $\mathbf{M}^n$ such that there exists a homomorphism $h : \mathbf{B} \to \mathbf{M}$ that is irresponsible with respect to $a \preceq b$ for some $a \neq b$, with $a, b \in B$. Then the algebra $\mathbf{M}$ cannot be strongly dualized.*

This means that finding unary algebras that are not strongly dualizable, provided the algebra has a pp-defined relation with the required properties, only requires finding irresponsible homomorphisms. Using the main theorem from [1] one still needs to verify that any homomorphism with domain C_r that happens to be irresponsible is, in fact, irresponsible with respect to $c_i^r \preceq c_j^r$ for some $i < j$.

5.2 Algebras with Zero

Suppose that $\mathbf{M}$ is a unary algebra with zero and with a pp-defined binary relation, $\preceq$, that is almost reflexive, antisymmetric, and transitive. Suppose that $\{(0,0),(0,c),(c,c)\}$ is contained in $\preceq$ for some $c \neq 0$. We claim that the subalgebra generated by the elements $(0,0,c)$ and $(0,c,c)$ in M^3 has an irresponsible homomorphism.

Let $a = (0,0,c)$ and $b = (0,c,c)$, let $\mathbf{B} = Sg_{\mathbf{M}^3}(\{a,b\})$, and define $h : B \to M$ by

$$h(x) = \begin{cases} \pi_3(x) & x \in Sg_{\mathbf{M}^3}(\{a\}) \\ \pi_1(x) & x \in Sg_{\mathbf{M}^3}(\{b\}) \end{cases}$$

Claim 5.2.1. *The map h defined above is an irresponsible homomorphism.*

Note that $a \preceq b$ and by definition $h(a) = 1 \not\preceq 0 = h(b)$ so h is irresponsible. Since both π_3 and π_1 are projection maps they are homomorphisms, so we need only show that h is well-defined on $Sg_{\mathbf{M}^3}(\{a\}) \cap Sg_{\mathbf{M}^3}(\{b\})$.

If $x \in Sg_{\mathbf{M}^3}(\{a\}) \cap Sg_{\mathbf{M}^3}(\{b\})$ then $x = t(a)$ and $x = s(b)$ for some terms t and s. Then $(t(0), t(0), t(1)) = (s(0), s(1), s(1))$, and so $0 = t(0) = s(1)$ and we must have that $x = (0,0,0)$. Hence, $\pi_3(x) = \pi_1(x) = 0$ and h is well-defined. Thus, by Corollary 5.1.1 unary algebras with zero and a pp-defined antisymmetric, transitive, and almost reflexive binary relation where $0 \preceq c$ for some $c \neq 0$ are not strongly dualizable.

5.3 Escalator Algebras

For $\mu \geq 2$ define the algebra $\mathbf{M}_\mu = \langle \{0,1,\ldots,\mu\}; f, g \rangle$, where $g(x) = \max(0, x-1)$ and $f(x) = \min(\mu, x+1)$. Such an algebra is called an **escalator algebra** and these algebras have been studied extensively; Figure 5.1 shows the escalator algebra $\mathbf{M}_3$. In [8] it is shown that the three element escalator algebra is dualizable but not fully dualizable, in [2] it is shown that each escalator algebra is dualizable and that no escalator algebra is strongly dualizable. In fact, in [1] a new proof using height is used to prove the same result. We provide this example here because the proof using Corollary 5.1.1 is shorter than that provided in [1].

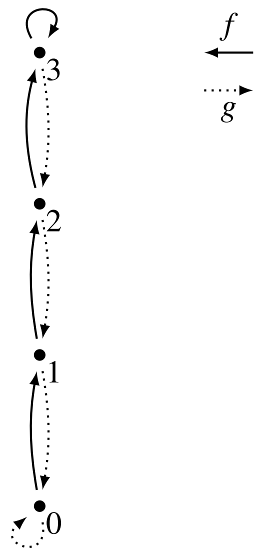

Figure 5.1: The escalator algebra $\mathbf{M}_3$

For $\mu \geq 2$, define the terms $t = g^{\mu-1} f^{\mu-2}$ and $s = g^{\mu-1} f^{\mu-1}$ on $\mathbf{M}_\mu$, where f^0 is the identity. Then the pp-formula

$$\Phi: \ \exists w[t(w) \approx x \;\&\; s(w) \approx y]$$

defines the almost reflexive, antisymmetric, and transitive relation $\{(0,0),(0,1),(1,1)\}$. Then define $a = (0,0,1)$, $b = (0,1,1)$, and $\mathbf{B} = Sg_{\mathbf{M}_\mu^3}(\{a,b\})$. There is an irresponsible homomorphism $h: \mathbf{B} \to \mathbf{M}_\mu$ determined by $h(a) = 1$ and $h(b) = 0$. So we have $a \preceq b$ in $\mathbf{M}_\mu^3$ but $h(a) \not\preceq h(b)$ in $\mathbf{M}_\mu$. Therefore, escalator algebras are not strongly dualizable.

5.4 One More Example

Let $\mathbf{M}_{5.4} = \langle \{0,1,2,3,4,5\}, f, g \rangle$ be the unary algebra defined as in the table below.

	g	f
0	3	3
1	4	4
2	0	1
3	0	0
4	1	1
5	2	2

Table 5.1: $\mathbf{M}_{5.4} = \langle\{0,1,2,3,4,5\}, f, g\rangle$

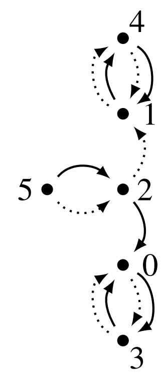

Figure 5.2: The associated diagram for $\mathbf{M}_{5.4}$

There is an obvious pp-formula,

$$\Phi : \exists w[x \approx g(w) \;\&\; y \approx f(w)]$$

which defines the relation $\preceq = \{(0,0),(0,1),(1,1),(2,2),(3,3),(4,4)\}$ that is almost reflexive, antisymmetric, and transitive. In particular we have that $0 \preceq 1$ and the subalgebra $\mathbf{B}_0 = Sg_{\mathbf{M}_{5.4}}(\{0,1\})$ has universe $\{0,1,3,4\}$.

Defining $h : B_0 \to M_{5.4}$ by $h(0) = 1, h(1) = 0, h(3) = 4, h(4) = 3$ is indeed a homomorphism. This claim is easy to verify as $f^{\mathbf{B}} = g^{\mathbf{B}}$ and $h(g(0)) = h(3) = 4 = g(1) = g(h(0)), h(g(1)) = h(4) = 3 = g(0) = g(h(1)), h(g(3)) = h(0) = 1 = g(4) = g(h(3))$, and $h(g(4)) = h(1) = 0 = g(3) = g(h(4))$. Further note that this homomorphism is irresponsible as $0 \preceq 1$ in $\mathbf{M}_{5.4}$ but $h(0) = 1 \not\preceq 0 =$

$h(1)$ in $\mathbf{M}_{5.4}$. Thus by Corollary 5.1.1 the algebra $\mathbf{M}_{5.4}$ is not strongly dualizable.

There are some properties to note here: First, it has not been investigated in this thesis whether $\mathbf{M}_{5.4}$ is in fact dualizable and this fact is actually irrelevant in determining that it is not strongly dualizable, the algebra $\mathbf{M}_{5.4}$ does not have a zero and so is different than those structures studied in Section 5.2, and the element 2 witnesses $0 \preceq 1$ in $\mathbf{M}_{5.4}$ yet for any term t we have $t(0) \neq 2 \neq t(1)$ an essential part of generating irresponsible homomorphisms.

Chapter 6

Conclusion

6.1 Summary

In summary, we have shown that the existence of an irresponsible homomorphism indicates that an algebra of any type is likely to not be strongly dualizable. In fact, if we can construct, in a particular way, a subalgebra of a product of the algebra which has a corresponding irresponsible fragment then the algebra is not strongly dualizable. More specifically, in Chapter 5 we discussed that the existence of an irresponsible homomorphism directly implies that a unary algebra is not strongly dualizable. In addition we looked at some cases where irresponsible homomorphisms are guaranteed to exist. We also provided a new method for constructing dense-in-$\mathrm{Frag}_{\mathrm{D}(\mathbf{A})}$ sets.

6.2 Future Work

In this thesis we were able to show that for unary algebras, irresponsibility guarantees that the algebra is not strongly dualizable. It is possible that future work can refine this process so that irresponsibility guarantees that an algebra is not strongly dualizable even if that algebra is not unary. This work would likely be done in steps, perhaps algebras with one binary operation first and moving on to more complicated structures. Furthermore, within the class of unary algebras more investigation can be done into what properties are necessary for irresponsibility. Future work may also investigate if irresponsibility with respect to other relations also indicates that an algebra is not strongly dualizable.

One problem that occurred in research was with meet homomorphisms. A meet homomorphism provides a natural way to define an ordering on the universe of the algebra which may or may not agree with the pp-defined order $\preceq$. In the case where the two orders agree (where they are defined) it seems likely that an irresponsible homomorphism must exist, however, we were not able to prove that either

a) The order defined by the meet homomorphism must agree with the pp-defined order $\preceq$, or

b) If the orders disagree whether irresponsibility is guaranteed or not.

The algebra in Section 5.4 does not have a meet homomorphism, so this provides a perfect example that meet homomorphisms are not the only place to look despite being interesting. This example suggests that we should look to find easily identifiable structure that guarantees that subalgebras do not pick up undesired witnesses for the pp-defined relation.

List of Symbols

$\mathbf{B} \leq \mathbf{A}$	$\mathbf{B}$ is a subalgebra of $\mathbf{A}$
$\mathbb{ISP}(\mathbf{M})$	The class of all isomorphic copies of subalgebras of products of $\mathbf{M}$
$\mathbb{IS}_c\mathbb{P}^+(\mathbb{M})$	The class of all isomorphic copies of closed substructures of positive products of $\mathbb{M}$
$\mathbb{SP}(\mathbf{M})$	The class of all subalgebras of products of $\mathbf{M}$
$\mathbf{B} \ll \mathbf{A}$	$\mathbf{B}$ is a finite subalgebra of $\mathbf{A}$
$\mathrm{Frag}_{\mathrm{D}(\mathbf{A})}$	The set of all $D(\mathbf{A})$-fragments
$Sg_{\mathbf{A}}(B)$	The subalgebra of $\mathbf{A}$ generated by the set B

Table 1: A list of important symbols used in this thesis

Bibliography

[1] E. Beveridge, D. Casperson, J. Hyndman, and T. Niven, *Irresponsibility indicates an inability to be strong*, Algebra Universalis **55** (2006), 457–477.

[2] Erin Natalie Beveridge, *Rank and duality of escalator algebras*, Master's thesis, University of Northern British Columbia, 2002.

[3] Stanley Burris and H. P. Sankappanavar, *A course in universal algebra*, Graduate Texts in Mathematics, vol. 78, Springer-Verlag, New York-Berlin, 1981. MR 648287 (83k:08001)

[4] D. Casperson, J. Hyndman, J. Mason, J. B. Nation, and B. Schaan, *Existence of finite bases for quasi-equations of unary algebras with 0*, Internat. J. Algebra Comput. **25** (2015), no. 6, 927–950. MR 3402848

[5] D. M. Clark and Brian A. Davey, *Natural dualities for the working algebraist*, vol. 57, Cambridge University Press, Cambridge, 1998.

[6] J. Hyndman, *Mono-unary algebras are strongly dualizable*, J. Austral. Math **72** (2002), 161–172.

[7] ———, *Positive primitive formulas preventing enough algebraic operations*, Algebra Universalis **52** (2004), 303–312.

[8] J. Hyndman and R. Willard, *An algebra that is dualizable but not fully dualizable*, Journal of Pure and Applied Algebra **151** (2000), 31–42.

[9] W. A. Lampe, G. F. McNulty, and R. Willard, *Full duality among graph algebras and flat graph algebras*, Algebra Universalis **45** (2001), 311–334.

[10] James R. Munkres, *Topology: a first course*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1975. MR 0464128

[11] J. G. Pitkethly and B. A. Davey, *Dualisability: Unary algebras and beyond*, Springer, 2005.

[12] Jane G. Pitkethly, *A full duality that cannot be upgraded to a strong duality*, Houston J. Math. **35** (2009), no. 3, 757–774. MR 2534279

[13] R. Willard, *New tools for proving dualizability*, Dualities, interpretability and ordered structures, 1999, pp. 69–74.