WHEN IS AN ALTER EGO A MINIMAL DUALIZING STRUCTURE?

by

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Abstract

We show that a particular five-element algebra is dualized by a particular alter-ego. We also show that, in some limited sense, the alter ego we chose is minimal. The algebra that we examine is a $\{0,1\}$-valued unary algebra with 0. It has meet and join as homomorphisms.

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Dedication

To my late Grandfather.

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Chapter 1

Introduction

A new branch of mathematics, called universal algebra, was first developed in Garrett Birkhoff's 1935 paper, "On the structure of Abstract Algebra" [15]. He stated a famous theorem, now call the Birkhoff's theorem. It says that the equational classes of algebras are closed under $\mathbb{H}, \mathbb{S}$, and $\mathbb{P}$; where $\mathbb{H}, \mathbb{S}$, and $\mathbb{P}$ stand for the closure operations of homomorphism, subalgebra, and product respectively. Duality starts with Birkhoff for finite distributive lattices. Later Duality theories were developed. These included Pontryagin's duality for Abelian groups and Stone's duality for Boolean algebras [10]. In 1975 H.A. Priestley provided a duality theory for distributive lattices [8]. An overview of duality theory from 1980 to 1992 was submitted by Brian A. Davey in his paper, "Duality Theory on Ten Dollars a Day" [9].

Now we discuss work related to dualizability, full dualizability, strong dualizability, unary algebras, unary algebras with zero, rank, and finitely based quasi-equational theories. In 2000 Jennifer Hyndman and Ross Willard displayed the first known example 3-element algebra which is dualizable but not fully dualizable by any alter-ego of bounded cardinality [2]. In 2002 Jennifer Hyndman showed that mono-unary algebras have rank at most two and are thus strongly dualizable [7]. In 2003 Clark, Davey, and Pitkethly classified of three-element unary algebras that are dualizable [13]. In 2004, J. Hyndman showed that a finite unary algebra with a certain positive primitive formula does not have enough algebraic operations, and consequently does not have a finite basis for its quasi-equation [5]. In 2005 J. Hyndman and J.G. Pitkethly showed that, within the class of three-element unary algebras, there is a tight connection between a finitely based quasi-equational theory, finite rank, enough algebraic operations and a speacial injectivity condition [6]. Under the supervision of Jennifer Hyndman, in 2006 E. Beveridge generalized the work of J. Hyndman and R. Willard [2] and defined escalator algebras that have infinite rank, and are dualizable but not strongly dualizable in her master's thesis [16]. In 2006 E. Beveridge, D. Caspersion, J. Hyndman, and T. Niven provided sufficient conditions on a finite algebra to prove that certain unary algebras are not strongly dualizable [18]. In 2015 D. Casperson, J. Hyndman, J. Mason, J.B. Nation, and B. Schaan looked at $\{0,1\}$-valued unary algebras with zero with respect to having a finite for the quasi-equations [3]. Under the supervision of Jennifer Hyndman, in 2014 B. Schaan provided results regarding the natural dualisability of certain $\{0,1\}$-valued unary algebras with zero in his master's thesis [19]. In 2014 D. Casperson, J. Hyndman, and B. Schaan defined tangled functions which need to be included for an extension of the algebra to be dualizable [4]. Under the supervision of Jennifer Hyndman, in 2016 J. Danyluk looked at $\{0,1\}$-valued finite unary algebras with zero where meet is defined on the algebra and join is partially defined in her master's thesis [20].

Our result fits in to the works by Beveridge [16] and Schaan [19]. In this thesis we look for when an alter ego is a minimal dualizing structure. In Chapter 2, we present preliminary materials such as notations, definitions, examples, theorems for algebras, lattices, quasivarieties, topology, and dualizability. In Chapter 3, we define a five-element algebra, an alter ego and conclude that our defined alter ego satisfies the interpolation condition relative to the algebra and thus dualizes the algebra. In Chapter 4, we show that the set of relations is a necessary part of a dualizing structure for a five-element algebra by finding a specific morphism that is a non-evaluation morphism. In Chapter 5, we summarize the main points of this document and give our future research idea.

Chapter 2

Preliminary and Background Materials

2.1 Algebras

In this section we discuss the concept and notions of algebras from universal algebra. All materials came from [1] which are used throughout this paper.

First we start with definitions and examples of algebras. If A is a non-empty set, called the **universe** or **underlying set** of $\mathbf{A}$ and $\mathscr{F}$ is a **language or type** of algebras then an ordered pair

$$\langle A, F \rangle$$

is written $\mathbf{A}$ and is called an **algebra** of type $\mathscr{F}$, where F is a family of finitary operations on A indexed by $\mathscr{F}$ such that there is an n**-ary** operation $f^{\mathbf{A}}$ on A if for every n**-ary function symbol** $f \in \mathscr{F}$. This non-negative integer n is called the **arity** of f and $f^{\mathbf{A}}$'s are called **fundamental operations** of $\mathbf{A}$. An operation whose arity is one is called **unary**. Similarly, an operation f on $\mathbf{A}$ is called a **binary** or **nullary** if its arity is two or zero respectively. If all of an algebra's operations are unary, then an algebra $\mathbf{A}$ is also called **unary**.

For example an algebra $\langle G, \cdot^G, -1^G, 1\rangle$ or $\langle G, \cdot, ^{-1}, 1\rangle$ with a binary, a unary, and a nullary operation respectively can define a **group** if the following three identities:

- $x_1 \cdot (x_2 \cdot x_3) = (x_1 \cdot x_2) \cdot x_3$,
- $x_1 \cdot 1 = x_1 = 1 \cdot x_1$, and
- $x_1 \cdot x_1^{-1} = 1 = x_1^{-1} \cdot x_1$

are true for all $x_1, x_2, x_3 \in G$ where the language of type $\mathscr{F}$ is $\{\cdot, ^{-1}, 1\}$. If $x_1 \cdot x_2 = x_2 \cdot x_1$ then the group G is called **Abelian** or **commutative**.

Similarly, a **semigroup** is an algebra $\langle G, \cdot\rangle$ satisfying

$$x_1 \cdot (x_2 \cdot x_3) = (x_1 \cdot x_2) \cdot x_3$$

for $x_1, x_2, x_3 \in G$.

For another example, we define a **ring** of type $\mathscr{F} = \{+, \cdot, -, 0\}$ to be an algebra $\langle R, +, \cdot, -, 0\rangle$ where $+$ and $\cdot$ are binary, $-$ is unary, and 0 is nullary which satisfies the following four properties:

- The algebra $\langle R, +, -, 0\rangle$ is a commutative group,
- $x_1 \cdot (x_2 \cdot x_3) = (x_1 \cdot x_2) \cdot x_3$ [the algebra $\langle R, \cdot\rangle$ is a semi-group],
- $x_1 \cdot (x_2 + x_3) = (x_1 \cdot x_2) + (x_1 \cdot x_3)$, and
- $(x_1 + x_2) \cdot x_3 = (x_1 \cdot x_3) + (x_2 \cdot x_3)$

for all $x_1, x_2, x_3 \in R$.

We often write $\mathbf{A} = \langle A, f_1, \ldots, f_k\rangle$ for $\langle A, F\rangle$ and say the algebra is of **finite type** if $\mathscr{F} = \langle f_1, \ldots, f_k\rangle$ is finite. If for all f in $\mathscr{F}$ the operation f is unary, then $\mathbf{A} = \langle A, F\rangle$ is called a **unary**

algebra. A unary algebra $\mathbf{A} = \langle A, F \rangle$ is called a $\{0,1\}$**-valued unary algebra** if the range of f is contained in $\{0,1\}$, for each $f \in F$ which is not identity map.

Example 2.1.1. *An algebra* $\mathbf{A} = \langle \{0,1,2,3\}, f_1, f_2 \rangle$ *shown in the Table 2.1 contains two unary operations* f_1 *and* f_2 *which take each element in* $\{0,1,2,3\}$ *to either* 0 *or* 1.

	f_1	f_2
0	0	0
1	1	1
2	0	1
3	1	0

Table 2.1: **A** is a $\{0,1\}$-valued unary algebra.

We define the **rows of A** of a unary algebra $\mathbf{A} = \langle A, F \rangle$ of finite type by the following k-ary relation

$$Rows(\mathbf{A}) = \{row(a) \mid a \in A\}$$

where $row(a) = \langle (f_1(a), \ldots, f_k(a)) \mid a \in A \rangle$ and $\{f_1, \ldots, f_k\}$ is a fixed enumeration of the non-constant, non-identity operations in F.

Now we suppose $X \subseteq A$ and $f : A \to B$ and $\alpha : X \to B$ are two functions such that for all $x \in X$

$$\alpha(x) = f(x)$$

then α is called the **restriction** of f to X, and is denoted $f\lceil_X$. If **A** and **B** are two algebras of the same type, $B \subseteq A$, and $f^{\mathbf{B}} = f^{\mathbf{A}}\lceil_B$ for all $f \in F$ then **B** is a **subalgebra** of **A**, denoted $\mathbf{B} \leq \mathbf{A}$. Any subset C of **A** that is closed under every operation in F is called a **subuniverse** of **A**. If **B** is a subalgebra of **A** then **B** is a subuniverse of **A**. For every $X \subseteq A$, the **subuniverse generated by**

X is

$$\mathrm{Sg}_{\mathbf{A}}(X) = \cap\{B : X \subseteq B \text{ and } B \text{ is a subuniverse of an algebra } \mathbf{A}\}.$$

Note that $\mathrm{Sg}_{\mathbf{A}}(X)$ is a subuniverse.

Let $\mathbf{A}$ and $\mathbf{B}$ be two algebras of the same type $\mathscr{F}$. A mapping $\alpha : A \to B$ is called a **homomorphism** from $\mathbf{A}$ to $\mathbf{B}$ if

$$\alpha(f^{\mathbf{A}}(a_1,\ldots,a_k)) = f^{\mathbf{B}}(\alpha(a_1),\ldots,\alpha(a_k))$$

for each k-ary f in $\mathscr{F}$ and each sequence $a_1,\ldots,a_k$ from $\mathbf{A}$. An **isomorphism** is a homomorphism which is one-to-one and onto.

If A is a non-empty set and n is a positive integer, then an n**-ary relation** on A is a subset of A^n, the set of all n-tuples of A. If $\alpha : \mathbf{A} \to \mathbf{B}$ is a homomorphism, then the **kernel** of α, written $\ker(\alpha)$, is defined by

$$\ker(\alpha) = \{\langle a,b\rangle \in A^2 : \alpha(a) = \alpha(b)\}.$$

The kernel of α is a binary relation on A.

Let $\mathbf{A}$ be an algebra of type $\mathscr{F}$. For every $1 \le i \le k$, we have a k-ary **projection operation** π_i on a non-empty set A defined by

$$\pi_i(a_1,\ldots,a_k) = a_i$$

for all $a_i \in A$. For a k-ary operation β, and k j-ary operations $\{\gamma_i\}_{i=1}^{k}$, there is a j-ary operation α defined by **composition of operations**, that is,

$$\alpha(a_1,\ldots,a_j) = \beta(\gamma_1(a_1,\ldots,a_j),\ldots,\gamma_k(a_1,\ldots,a_j)).$$

Moreover, all the operations constructed by the composition of operations using the basic operations of $\mathbf{A}$ and the projections on A are called the **term functions** or **term operations** of $\mathbf{A}$.

2.2 Lattices

In this section we introduce the idea of a lattice that came from [1]. We define lattice in two standard ways which are based on the same (algebraic) footing as groups or rings, and the notion of order.

The first way is to define a **lattice** as an algebra

$$\mathbf{L} = \langle L, \wedge, \vee \rangle$$

where L is a nonempty set with two binary operations $\wedge$ (**meet**) and $\vee$ (**join**) on L in which the following identities hold:

- Commutative: $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$
- Associative: $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ and $a \vee (b \vee c) = (a \vee b) \vee c$
- Idempotent: $a \wedge a = a$ and $a \vee a = a$
- Absorption: $a = a \wedge (a \vee b)$ and $a = a \vee (a \wedge b)$

for all $a, b, c \in L$.

Moreover if the above commutative, associative and idempotent identities are true for only one binary operation $\wedge$ on a set S, then the algebra

$$\mathbf{S} = \langle S, \wedge \rangle$$

is called a $\wedge$**-semilattice**.

The second way to define a lattice comes from the notion of a partial order. A binary relation $\leq$ on a set X is called a **partial order** on X if for each $x, y, z \in X$, the following three identities hold in X:

- Reflexivity: $x \leq x$,
- Antisymmetry: $x \leq y$ and $y \leq x$ imply $x = y$, and
- Transitivity: $x \leq y$ and $y \leq z$ imply $x \leq z$

We also say that $\leq$ on a set X is a **total order** on A if $x \leq y$ or $y \leq x$ for each $x, y \in X$. A **partially ordered set** is defined by a set X with a partial order on X. We write $x < y$ to mean $x \leq y$ but $x \neq y$ in X. Note that every lattice has a partial order given by defining $x \leq y$ to mean $x \wedge y = x$. When the relation is a total order we call the partially ordered set **totally ordered**.

Let S be a subset of a partially ordered set X. An element $x \in X$ is an **upper bound** for S if $s \leq x$ for every $s \in S$. An element $x \in X$ is the **supremum** of S, written $\sup S$, if x is an upper bound of S, and $s \leq y$ for every $s \in S$ implies $x \leq y$. A supremum is unique if it exists. Similarly, an element $x \in X$ is a **lower bound** for S if $x \leq s$ for every $s \in S$. An element $x \in X$ is called the **infimum** of S, written $\inf S$, if x is an lower bound of S and $y \leq x$ for all lower bounds y of S. An infimum is unique if it exists. A partially ordered set X is a **lattice** if and only if for every $x, y \in X$ both $\sup\{x, y\}$ and $\inf\{x, y\}$ exist in X.

2.3 Quasivarieties

First we define isomorphic, direct product and homomorphic images, and then finally look at quasivariety.

If a mapping α is an isomorphism from an algebra $\mathbf{A}_1$ to another algebra $\mathbf{A}_2$, then $\mathbf{A}_1$ is **isomorphic** to $\mathbf{A}_2$, written $\mathbf{A}_1 \cong \mathbf{A}_2$.

The **direct product** of two algebras $\mathbf{A}$ and $\mathbf{B}$ of the same type $\mathscr{F}$ is the one algebra whose universe is the set of cartesian product of **A** and **B** and the operations are defined by

$$f^{\mathbf{A}\times\mathbf{B}}(\langle a_1,b_1\rangle,\dots,\langle a_k,b_k\rangle) = \langle f^{\mathbf{A}}(a_1,\dots,a_k), f^{\mathbf{B}}(b_1,\dots,b_k)\rangle$$

for $a_i \in A, b_i \in B, 1 \leq i \leq k$ and for $f \in \mathscr{F}_k$ (the subset of k-ary function symbols in $\mathscr{F}$ is denoted by $\mathscr{F}_k$).

For i in some set I, if $(\mathbf{A}_i)_{i\in I}$ is a collection of algebras of type $\mathscr{F}$, then the **product algebra**, written $\mathbf{A} = \prod_{i\in I}\mathbf{A}_i$, is the algebra with universe $\prod_{i\in I} A_i$ and whose operations are defined by

$$f^{\mathbf{A}}(a_1,\dots,a_k)(i) = f^{\mathbf{A}_i}(a_1(i),\dots,a_k(i))$$

for $f \in \mathscr{F}_k$ and $\langle a_1,\dots,a_k\rangle \in \prod_{i\in I} A_i$.

Now we can define a **quasivariety** $\mathbb{ISP}(\mathbf{M})$ generated by an algebra $\mathbf{M}$ to be the class of algebras that are obtained by taking all isomorphic copies of subalgebras of products of $\mathbf{M}$, that is $\mathbb{ISP}(\mathbf{M})$, where $\mathbb{I}$ is the isomorphic copies, $\mathbb{S}$ is a subalgebras, and $\mathbb{P}$ is the products of an algebra $\mathbf{M}$. Similar to Birkhoff's theorem which says $\mathbb{HSP}(\mathbf{M})$ is equationally defined, we note that $\mathbb{ISP}(\mathbf{M})$ is quasi-equationally defined. This explains the interest in determining when the quasi-equational theory of quasivariety is finitely based.

2.4 Topologies

In this section we state the concept of topological space [11] and some algebraic operation, and then define topological quasi-variety generated by an alter ego, and some definitions of maps.

The collection of subsets of a set X is called a **topology** $\mathscr{T}$ of a space X if the following three properties hold:

- The empty set and X are in $\mathscr{T}$,
- If $\{T_i\}_{i\in I} \subset \mathscr{T}$ for any set I, then $\bigcup_{i\in I} T_i \in \mathscr{T}$,
- If $\{T_i\}_{i=1}^{n} \subset \mathscr{T}$ is a finite set of sets, then $\bigcap_{i=1}^{n} T_i \in \mathscr{T}$.

The ordered pair $(X, \mathscr{T})$ is called **topological space** where X is a set and $\mathscr{T}$ is a topology of X whose elements are called **open sets** of X. The **closed sets** are the complements to the open sets $S \in \mathscr{T}$. The **discrete topology** is the topology where all sets are open (and hence closed).

For some finite and non-zero n, an n-ary relation on an algebra $\mathbf{A}$ is called an **algebraic relation** on A if it is the universe of a subalgebra of $\mathbf{A}^n$. A homomorphism from $\mathbf{A}^n$ to $\mathbf{A}$ is called **algebraic (total) operation** on $\mathbf{A}$ for finite and non-negative n. An **algebraic partial operation** is also defined by the homomorphism from a subalgebra of $\mathbf{A}^n$ for some finite non-negative n to an algebra $\mathbf{A}$.

Let $\mathbf{M}$ be a finite algebra. If $\mathscr{G}$ is a set of algebraic total operations on M, $\mathscr{H}$ is a set of algebraic partial operations on M, $\mathscr{R}$ is a set of algebraic relations on M, and $\mathscr{T}$ is the discrete topology on M, then the structured topological space

$$\mathbb{M} = \langle M; \mathscr{G}, \mathscr{H}, \mathscr{R}, \mathscr{T} \rangle$$

is called an **alter ego** of $\mathbf{M}$. Both the algebra $\mathbf{M}$ and the alter ego $\mathbb{M}$ have the same underlying set M.

Now using the knowledge of alter ego, we can define a **topological quasi-variety**, $\mathbb{IS}_c\mathbb{P}^+(\mathbb{M})$, which consists of all isomorphic copies of topologically closed substructures of non-zero powers of the alter ego $\mathbb{M}$, where $\mathbb{I}$ is all isomorphic copies, $\mathbb{S}_c$ is all closed sub-structured topological spaces and $\mathbb{P}^+$ is all non-zero powers of the alter ego $\mathbb{M}$. See [10] for full definitions.

Moreover, for a function α from a topological space $\mathbb{U}$ to a topological space $\mathbb{V}$ if the set $\alpha^{-1}(S)$ is an open subset of X for all open subsets $S \in V$ then α from U to V is called a **continuous function**. For $\mathbb{U}, \mathbb{V} \in \mathbb{IS}_c\mathbb{P}^+(\mathbf{M})$ a **morphism** is a continuous map $\alpha : \mathbb{U} \to \mathbb{V}$ which preserves the total operations, partial operations, and relations on $\mathbf{M}$. This map is called an **embedding** if $\alpha : \mathbb{U} \to \alpha(\mathbb{U})$ is isomorphism and $\alpha(\mathbb{U})$ is a substructure of $\mathbb{V}$. The map α is called a **homeomorphism** when α is a bijection and both α and α^{-1} are continuous.

2.5 Dualizability

In this section we introduce the basic idea of natural duality theory, and we also look into what it means for an algebra to be dualisable.

Let $\mathbf{M} = \langle M, \mathscr{F} \rangle$ be a finite algebra. For an alter-ego $\mathbb{M} = \langle M; \mathscr{G}, \mathscr{H}, \mathscr{R}, \mathscr{T} \rangle$, and for any algebra $\mathbf{A} \in \mathbb{ISP}(\mathbf{M})$, define the **dual** of $\mathbf{A}$ (with respect to $\mathbb{M}$) to be

$$D(\mathbf{A}) = \langle \mathrm{Hom}(\mathbf{A}, \mathbf{M}); \mathscr{G}, \mathscr{H}, \mathscr{R}, \mathscr{T} \rangle$$

seen as a topologically-closed substructure of $\mathbb{M}^{\mathbf{A}}$, where $\mathrm{Hom}(\mathbf{A}, \mathbf{M})$ is the set of all homomorphisms from $\mathbf{A}$ to $\mathbf{M}$. Note $\mathrm{Hom}(\mathbf{A}, \mathbf{M})$ is a subset of $\mathbb{M}^{\mathbf{A}}$. This follows from $\mathscr{G}, \mathscr{H}$, and $\mathscr{R}$ being algebraic. It turns out that $D(\mathbf{A})$ is in the topological quasivariety $\mathbb{IS}_c\mathbb{P}^+(\mathbb{M})$. The proof of this non-trivial fact is given in Chapter 1 of [10]. See Figure 2.1 on the next page.

Likewise, for any topological structure $\mathbb{X} \in \mathbb{IS}_c\mathbb{P}^+(\mathbb{M})$, define the **dual of** $\mathbb{X}$ to be

$$E(\mathbb{X}) = \langle \mathrm{Hom}(\mathbb{X}, \mathbb{M}); \mathscr{F} \rangle$$

seen as a subalgebra of $\mathbf{M}^X$, where $\mathrm{Hom}(\mathbb{X}, \mathbb{M})$ is the set of all $\mathbb{M}$-morphisms from $\mathbb{X}$ to $\mathbb{M}$. We have $\mathrm{Hom}(\mathbb{X}, \mathbb{M})$ is a subset of $\mathbf{M}^X$. The proof that $E(\mathbb{X}) \in \mathbb{ISP}(\mathbf{M})$ is given in Chapter 1 of [10]. See Figure 2.2 on page 14.

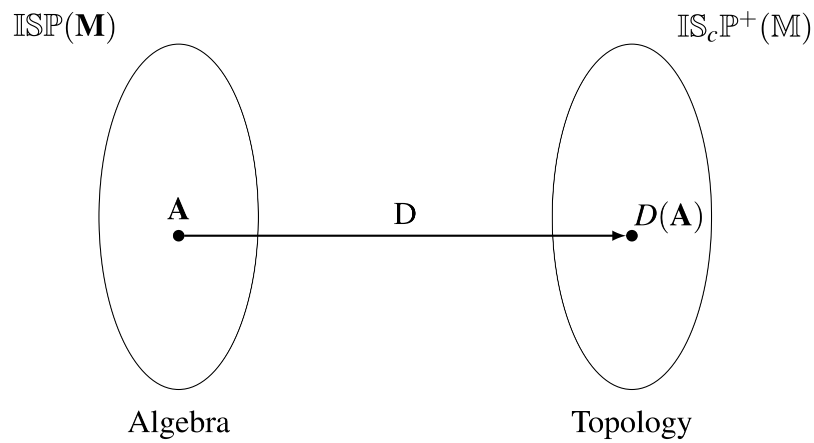

Figure 2.1: The Dual of an algebra is a topological structure.

If an algebra $\mathbf{A}$ is in $\mathbb{ISP}(\mathbf{M})$ then the **evaluation map** is a natural embedding $e_{\mathbf{A}} : \mathbf{A} \to E(D(\mathbf{A}))$, defined by

$$e_{\mathbf{A}}(a)(x) = x(a)$$

for all elements $a \in A$ and homomorphisms $x \in D(\mathbf{A})$. The map $e_{\mathbf{A}}(a) : \mathrm{Hom}(\mathbf{A}, \mathbf{M}) \to A$ is called an **evaluation**.

Similarly, for a structure $\mathbb{X} \in \mathbb{IS}_c\mathbb{P}^+(\mathbf{M})$ there is a natural embedding $\varepsilon_{\mathbb{X}}$, called an **evaluation map** from an algebra $\mathbb{X}$ to $D(E(\mathbb{X}))$, defined by

$$\varepsilon_{\mathbb{X}}(x)(a) = a(x)$$

for all homomorphisms $x \in \mathbb{X}$ and morphisms $a \in E(\mathbb{X})$.

Furthermore, a map $\beta : D(\mathbf{A}) \to M$ is **given by evaluation at** a **on B** where $B \subseteq D(\mathbf{A})$ if $\beta \restriction_B = e_{\mathbf{A}}(a) \restriction_B$ for $a \in A$.

If for each algebra $\mathbf{A}$ in $\mathbb{ISP}(\mathbf{M})$ we have $\mathbf{A}$ is isomorphic to a double dual of $\mathbf{A}$, $(ED(\mathbf{A}))$ through the evaluation map $e_{\mathbf{A}}$, then an alter ego $\mathbb{M}$ is said to **yield a duality on** the algebra $\mathbf{M}$. If

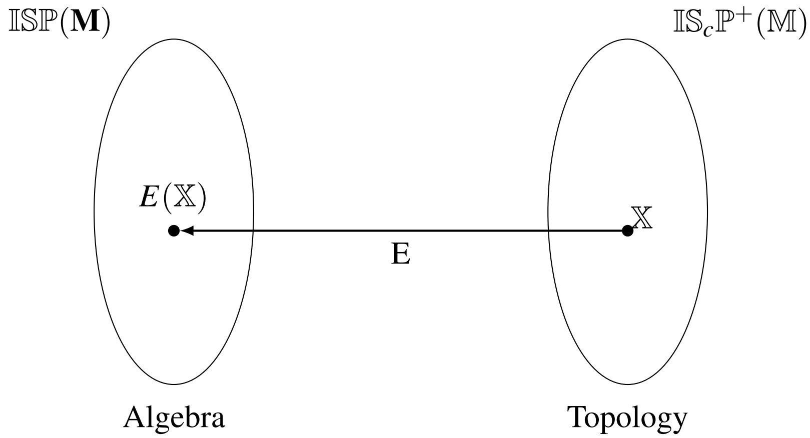

Figure 2.2: The Dual of a topological structure is an algebra.

there is an alter ego $\mathbb{M}$ such that for every $\mathbf{A} \in \mathbb{ISP}(\mathbf{M})$ we have that

$$\mathbf{A} \cong ED(\mathbf{A})$$

via $e_{\mathbf{A}}$, then an algebra $\mathbf{M}$ is said to be **dualizable**. See Figure 2.3.

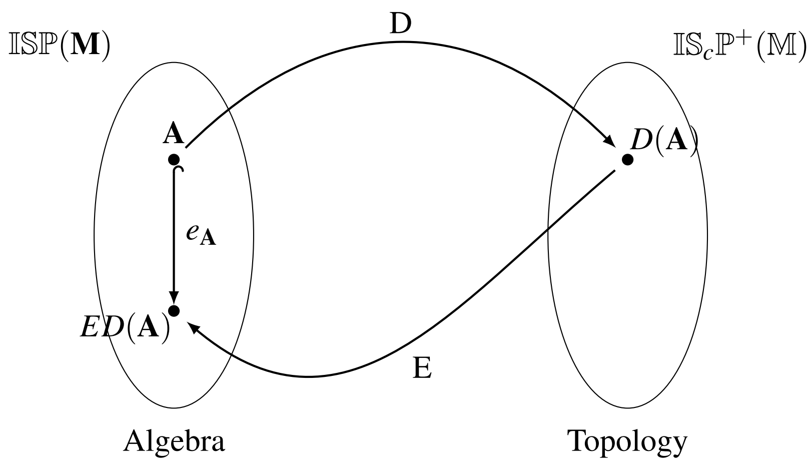

Figure 2.3: Dualizability of an algebra occurs when $e_{\mathbf{A}} : \mathbf{A} \to E(D(\mathbf{A}))$ is an isomorphism for all $\mathbf{A}$.

Moreover, we say that an algebra $\mathbf{M}$ is called **non-dualizable** when no alter ego dualizes $\mathbf{M}$.

Example 2.5.1. *(Although bounded distributive lattices are not used to this document, we give this as a example in the area of natural duality theory) H.A Priestley first found that the two elements bounded distributive lattice,*

$$\mathbf{M} = \langle \{0,1\}; \vee, \wedge, 0, 1 \rangle$$

is fully dualized by the alter ego

$$\mathbb{M} = \langle \{0,1\}; \leq, \mathscr{T} \rangle$$

where $\mathscr{T}$ is a discrete topology, and the partial operation $\leq$ is defined on $\{0,1\}$ with $0 \leq 1$. See [8, 9].

Example 2.5.2. *J. Hyndman and R. Willard showed that the algebra $\mathbf{M} = \langle \{0,1,2\}; f, g \rangle$ (where f, g are unary operations shown in the following Table 2.2)*

	f	g
0	0	1
1	0	2
2	1	2

Table 2.2: The escalator algebra

is dualized by the alter ego

$$\mathbb{M} = \langle M, \wedge, \vee, E, R, \mathscr{T} \rangle$$

where $\wedge$ and $\vee$ are the lattice meet and join operations on an order chain $\langle M, \leq \rangle$ with $0 < 1 < 2$, and $E \subseteq M^2$ and $R \subseteq M^4$, defined by

$E = \{(x,y) : x \leq y$ and $(x,y) \neq (0,2)\}$,

$R = \{(x,y,z,w) : x \leq y \leq z \leq w$ and $x = y$ or $z = w\}$,

and $\mathscr{T}$ is a discrete topology. However, $\mathbf{M}$ *is not fully dualizable. See [2].*

2.6 The Interpolation Condition and Duality Theorems

In this section we define the interpolation condition and state some duality theorems which are helpful for proving that an algebra is dualizable.

First, we provide the following lemma which is explicitly proved in [14] and was used to prove Theorem 2.6.3.

Lemma 2.6.1. *[14] Let* $\mathbf{M}$ *be a finite algebra and* $\mathbf{A} \in \mathbb{ISP}(\mathbf{M})$*, and let* $\alpha : D(\mathbf{A}) \to M$ *and* $n \in \mathbb{N}$*. Then* α *preserves the set of n-ary algebric relations on* $\mathbf{M}$*, denoted* R_n*, if and only if* α *agrees with an evaluation on each subset of* $D(\mathbf{A})$ *with at most n elements.*

We say that the alter ego $\mathbb{M}$ satisfies the **Interpolation Condition** relative to $\mathbf{M}$ if for each $n \in \mathbb{N}$ and each closed substructure $\mathbb{X}$ of $\mathbb{M}^n$, every morphism $\alpha : \mathbb{X} \to \mathbb{M}$ extends to a term function $t : M^n \to M$ of the algebra $\mathbf{M}$.

Now we need the definition of a total structure before we introduce the next theorem. An alter ego $\mathbb{M}$ is said to be a **total structure** if the only algebric operations in $\mathbb{M}$ are the total ones. If $\mathbb{M}$ is a total structure with finitely many algebric relations and $\mathbb{M}$ satisfies the interpolation condition relative to $\mathbf{M}$, then the Second Duality Theorem says that $\mathbf{M}$ is dualizable.

Theorem 2.6.1 (Second Duality Theorem). *([10]) Assume that an alter ego* $\mathbb{M}$ *is a total structure in which R is finitely many algebraic relations. If the interpolation condition holds, then* $\mathbb{M}$ *yields a duality on an algebra* $\mathbf{M}$*.*

The next theorem allows the use of partial operations in the type of $\mathbb{M}$ and was used to prove Theorem 2.6.3 [14].

Theorem 2.6.2 (Duality Compactness Theorem). *([10,12]) Let* $\mathbf{M}$ *be a finite algebra and let* $\mathbb{M}$

be an alter ego of $\mathbf{M}$ *with finite type. If* $\mathbb{M}$ *yields a duality on each finite algebra in* $\mathbb{ISP}(\mathbf{M})$*, then* $\mathbb{M}$ *dualizes* $\mathbf{M}$*.*

The following theorem is also helpful to prove the dualisability of a finite algebra.

Theorem 2.6.3. *[14] Let* $\mathbf{M}$ *be a finite algebra which has binary homomorphisms* $\wedge$ *and* $\vee$ *such that* $\langle \mathbf{M}; \wedge, \vee \rangle$ *is a lattice. Then the alter ego* $\mathbb{M} := \langle M, \wedge, \vee, R_{2|M|}, \mathscr{T} \rangle$ *yields a duality on* $\mathbb{ISP}(\mathbf{M})$*.*

We now have the necessary notation and background to formulate the results of this document.

Chapter 3

An Example of a Finite Algebra and Dualizing Structure

In this chapter we define a five-element algebra which is our central example in our thesis. Also we define an alter ego for our algebra. Then we show that the alter ego satisfies the interpolation condition relative to the algebra and thus dualizes the algebra. We look at this particular alter ego because it may turn out to be minimal.

3.1 The Five-Element Algebra

In this section we give an example of a finite algebra, and lattice diagram for the algebra. This five-element algebra is our central example in this paper.

Let **M** be the algebra $\langle M, p, q, r, \bar{0}\rangle$ where p, q, r and $\bar{0}$ are the unary operations defined on the underlying set $M = \{0, 1, 2, 3, 7\}$ as shown in Table 3.1 on page 20.

From the concept of the *row* of an algebra which is introduced in [3], we have the *row* of an algebra $\mathbf{M}$ as follows:

$$\begin{aligned}
row(0) &= (p(0), q(0), r(0)) \\
&= (0,0,0) \\
\text{or, } row(0) &= 000, \\
row(1) &= (p(1), q(1), r(1)) = 001, \\
row(2) &= (p(2), q(2), r(2)) = 010, \\
row(3) &= (p(3), q(3), r(3)) = 011, \\
\text{and } row(7) &= (p(7), q(7), r(7)) = 111.
\end{aligned}$$

Moreover, we have $Rows(\mathbf{M})$ as follows:

$$\begin{aligned}
Rows(\mathbf{M}) &= \{row(0), row(1), row(2), row(3), row(7)\} \\
&= \{000, 001, 010, 011, 111\}.
\end{aligned}$$

Note that neither $\bar{0}$ nor id are used in computing rows.

Now we use $Rows(\mathbf{M})$ to draw the lattice diagram induced by $\mathbf{M}$. (Note that we use $Rows(\mathbf{M})$ to define the lattice order on M, and that meet $\wedge$ and join $\vee$ are homomorphisms $\mathbf{M}^2 \to \mathbf{M}$.) This is done by drawing a Hasse diagram using the point-wise order induced by the elements in $Rows(\mathbf{M})$ by the order induced by $0 < 1$. This lattice diagram for this algebra shown in Figure 3.1 on the following page is drawn by looking at each row of $\mathbf{M}$ and comparing them coordinate-wise.

	p	q	r	$\bar{0}$
0	0	0	0	0
1	0	0	1	0
2	0	1	0	0
3	0	1	1	0
7	1	1	1	0

Table 3.1: The five-element algebra **M**

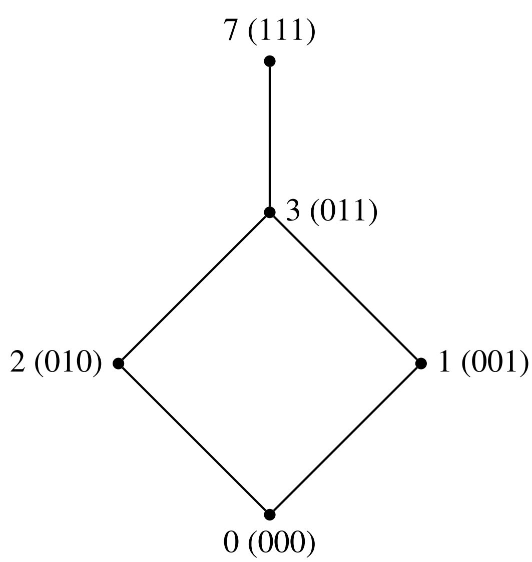

Figure 3.1: The Lattice Diagram for the five-element algebra **M**.

3.2 An Alter Ego

In this section we define an alter ego which is our central dualizing structure. First we introduce the relations Q_i for $1 \le i \le 13$. These are gathered together in Table 3.2. Here k is the arity of the relation being defined. The relation Q_i' is all k-tuples that satisfy the co-ordinate restrictions. The relation Q_i is the largest subuniverse of Q_i' that excludes β_i, given in the fifth column of the tables.

Range(α)	Lemma	k	Name	β_i	Co-ordinate restrictions
$\{0,7\}$	3.3.15	2	Q_1	$\langle 0,7\rangle$	$Y_1 \le Y_2$
$\{0,1\}$	3.3.14	4	Q_2	$\langle 0,1,1,0\rangle$	$Y_1 \le Y_2 \le Y_3$; $\quad Y_1 \le Y_4$
$\{0,2\}$	3.3.13	4	Q_3	$\langle 0,2,2,0\rangle$	$Y_1 \le Y_2 \le Y_3$; $\quad Y_1 \le Y_4$
$\{0,3\}$	3.3.12	4	Q_4	$\langle 0,3,3,0\rangle$	$Y_1 \le Y_2 \le Y_3$; $\quad Y_1 \le Y_4$
$\{0,1,7\}$	3.3.11	5	Q_5	$\langle 0,1,1,7,0\rangle$	$Y_1 \le Y_2 \le Y_3 \le Y_4$; $\quad Y_1 \le Y_5$
$\{0,2,7\}$	3.3.10	5	Q_6	$\langle 0,2,2,7,0\rangle$	$Y_1 \le Y_2 \le Y_3 \le Y_4$; $\quad Y_1 \le Y_5$
$\{0,3,7\}$	3.3.9	5	Q_7	$\langle 0,3,3,7,0\rangle$	$Y_1 \le Y_2 \le Y_3 \le Y_4$; $\quad Y_1 \le Y_5$
$\{0,1,3\}$	3.3.8	6	Q_8	$\langle 0,1,1,3,3,0\rangle$	$Y_1 \le Y_2 \le Y_3 \le Y_4 \le Y_5$; $\quad Y_1 \le Y_6$
$\{0,2,3\}$	3.3.7	6	Q_9	$\langle 0,2,2,3,3,0\rangle$	$Y_1 \le Y_2 \le Y_3 \le Y_4 \le Y_5$; $\quad Y_1 \le Y_6$
$\{0,1,3,7\}$	3.3.6	7	Q_{10}	$\langle 0,1,1,3,3,7,0\rangle$	$Y_1 \le Y_2 \le Y_3 \le Y_4 \le Y_5 \le Y_6$; $\quad Y_1 \le Y_7$
$\{0,2,3,7\}$	3.3.5	7	Q_{11}	$\langle 0,2,2,3,3,7,0\rangle$	$Y_1 \le Y_2 \le Y_3 \le Y_4 \le Y_5 \le Y_6$; $\quad Y_1 \le Y_7$
$\{0,1,2,3\}$	3.3.4	8	Q_{12}	$\langle 0,1,1,3,3,0,2,2\rangle$	$Y_1 \le Y_2 \le Y_3 \le Y_4 \le Y_5$; $Y_6 \le Y_7 \le Y_8 \le Y_4$
$\{0,1,2,3,7\}$	3.3.3	9	Q_{13}	$\langle 0,1,1,3,3,7,0,2,2\rangle$	$Y_1 \le Y_2 \le Y_3 \le Y_4 \le Y_5 \le Y_6$; $Y_7 \le Y_8 \le Y_9 \le Y_4$

Table 3.2: Relations Q_i in the alter-ego $\mathbb{M}$. Q_i omits β_i but satisfies the co-ordinate restrictions.

More precisely, we define sets Q'_i and Q_i by

$$Q'_i = \{ \quad \langle Y_1, \ldots, Y_k \rangle \in M^k : \text{ The co-ordinate restrictions hold on } \langle Y_1, \ldots, Y_k \rangle \quad \}$$

and $$Q_i = \{ \quad x \in Q'_i : \beta_i \notin \mathbf{Sg}_{M^k}(\{x\}) \quad \}$$

where the co-ordinate restrictions and β_i values are given in Table 3.2 on the preceding page. Note that Q_i and Q'_i are subuniverses of $\mathbf{M}^k$ and Q_i is a subalgebra of Q'_i.

Lemma 3.2.1. *For $i \neq 2$, we have $Q'_i = Q_i \cup \{\beta_i\}$.*

Proof. If $x \in Q'_i \setminus Q_i$ then by definition, $\beta_i = f(x)$ for $f \in \{p, q, r, \bar{0}, \mathrm{id}\}$ but $\beta_i \notin \{0,1\}^k$, so $f = \mathrm{id}$, so $x = \beta_i$. □

Let Q be the set of Q_i. Let $\mathbb{M} := \langle M, \wedge, \vee, \mathbf{0}, Q, \mathscr{T} \rangle$ be an alter ego for the algebra $\mathbf{M}$. In the alter ego $\mathbb{M}$, $\{\wedge, \vee, \mathbf{0}\}$ is a set of algebraic total operations on M, $Q = \{Q_1, \ldots, Q_{13}\}$ is a set of algebraic relations on M, and $\mathscr{T}$ is the discrete topology on M. (Here the empty set ($\emptyset$) is the set of algebraic partial operations on M.) Note that $\wedge$ and $\vee$ are homomorphisms, and $\mathbf{0}$ is the zero homomorphism. Also note that all Q_i's are subalgebras and are closed under p, q, r and $\bar{0}$.

3.3 Morphism Ranges

In this section we enumerate all possible cases for range(α), where $\alpha : \mathbb{X} \to \mathbb{M}$ is a morphism with respect to the alter ego $\mathbb{M} := \langle M, \wedge, \vee, \mathbf{0}, Q, \mathscr{T} \rangle$ and $\mathbb{X}$ is finite. All possible cases for range(α) are shown in Table 3.2 on the previous page. This list is exhaustive because range(α) is closed under meet $\wedge$, join $\vee$ and contains $\mathbf{0}$. Note that the alter ego does not contain the unary relation $\{0\}$ and for any $\alpha : D(\mathbf{A}) \to \mathbf{M}$ we have $\mathbf{0}$ (the zero homomorphism) in $D(\mathbf{A})$. We know that $\mathbf{0}$ is the unary operation $M \to M$ defined by the homomorphism $\mathbf{0}(x) = 0$, and $\alpha(\mathbf{0}(x)) = \mathbf{0}(\alpha(x)) = 0$. So $0 \in \text{range}(\alpha)$.

We wish to show that $\mathbb{M}$ satisfies the Interpolation Condition relative to $\mathbf{M}$. That is, we have that a morphism $\alpha : \mathbb{X} \to \mathbb{M}$ is the restriction to the substructure X of an n-ary term operation of $\mathbf{M}$ for each finite n, $\mathbb{X} \leq \mathbb{M}^n$, and $\alpha \in \text{Hom}(\mathbb{X}, \mathbb{M})$.

3.3.1 Notation

In the following lemmas we consistently use the following notation. We let $\mathbb{X}$ be a topologically-closed substructure of $\mathbb{M}^n$ where n is some fixed finite positive integer; $\alpha : \mathbb{X} \to \mathbb{M}$ is a morphism in the topological quasivariety.

For $j \in \text{range}(\alpha) \subseteq M$, we set $X_j = \{x \in X : \alpha(x) = j\}$, and set $a_j = \bigwedge(X_j)$ and $b_j = \bigvee(X_j)$. Note that a_j and b_j are well defined, as these are finite meets and joins respectively.

3.3.2 The General Argument

To show that the term interpolation condition holds, in each case we must show that $\alpha(x) = t(x)$ for $x \in \mathbb{X}$ where t is some term on M^n. Because the algebra is unary, so are all terms, and we must have $t(x) = f(\pi_i(x))$ (or $t = f \circ \pi_i$) for some co-ordinate i, where $1 \leq i \leq n$, and some $f \in \{p, q, r, \bar{0}, \text{id}\}$.

In each of the following lemmas, we shall find a co-ordinate i (where $1 \leq i \leq n$), and argue that there must be a function $f \in \{p, q, r, \bar{0}, \text{id}\}$ such that for all $j \in \text{range}(\alpha)$ we have

$$j = f(\pi_i(a_j)) = f(\pi_i(b_j)).$$

Lemma 3.3.1. *For all $j \in \mathrm{range}(\alpha)$, if $f(a_j(i)) = f(b_j(i)) = j$ for some co-ordinate i, where $1 \leq i \leq n$, and some $f \in \{p, q, r, \bar{0}, \mathrm{id}\}$ then we have $\alpha(x) = f(x(i))$ for every $x \in X$, and so $\alpha(x) = t(x)$ for $x \in \mathbb{X}$ where t is some term on M^n.*

Proof. We have for $x \in X_j$ that

$$a_j \leq x \leq b_j,$$

which means that each co-ordinate respects the order, so $\pi_i(a_j) \leq \pi_i(x) \leq \pi_i(b_j)$. Because meet and join are homomorphisms, f is order preserving, and thus

$$f(\pi_i(a_j)) \leq f(\pi_i(x)) \leq f(\pi_i(b_j));$$

whence

$$f(\pi_i(x)) = j = \alpha(x). \qquad \square$$

Because $f = \mathrm{id}$ is a possibility, it certainly suffices to show that

$$j = \pi_i(a_j) = \pi_i(b_j) \quad \text{for } j \in \mathrm{range}(\alpha).$$

Now we analyze all possible range (α) and find the term function of $\mathbf{M}$ for each range one by one.

Lemma 3.3.2. *If* $\mathrm{range}(\alpha) = \{0\}$, *then* $\alpha = \bar{0} \circ \pi_1 \restriction_X$.

Proof. Suppose $\mathrm{range}(\alpha) = \{0\}$. Since $\bar{0}$ is the constant 0 function on the algebra $\mathbf{M}$, we have $\alpha = \bar{0} \circ \pi_1 \restriction_X$. $\square$

Lemma 3.3.3. *Suppose $\alpha : \mathbb{X} \to \mathbb{M}$ is a morphism, where $\mathbb{X} \leq \mathbb{M}^n$, and $\mathbb{M} := \langle M, \wedge, \vee, \mathbf{0}, Q, \mathscr{T} \rangle$. If $\mathrm{range}(\alpha) = \{0, 1, 2, 3, 7\}$ then $\alpha = \pi_i \restriction_X$, for some i where $1 \leq i \leq n$.*

Proof. Here we use Q_{13}. Suppose $\mathrm{range}(\alpha) = \{0, 1, 2, 3, 7\}$. (See Figure 3.2 on the following page.) With X_j, a_j, and b_j defined as usual, let $d_2 = b_2 \wedge a_3$, $d_1 = a_3 \wedge b_1$, $d_0 = b_0 \wedge a_1$, $c_7 = a_7 \vee b_3$ and $e_0 = a_2 \wedge b_0$.

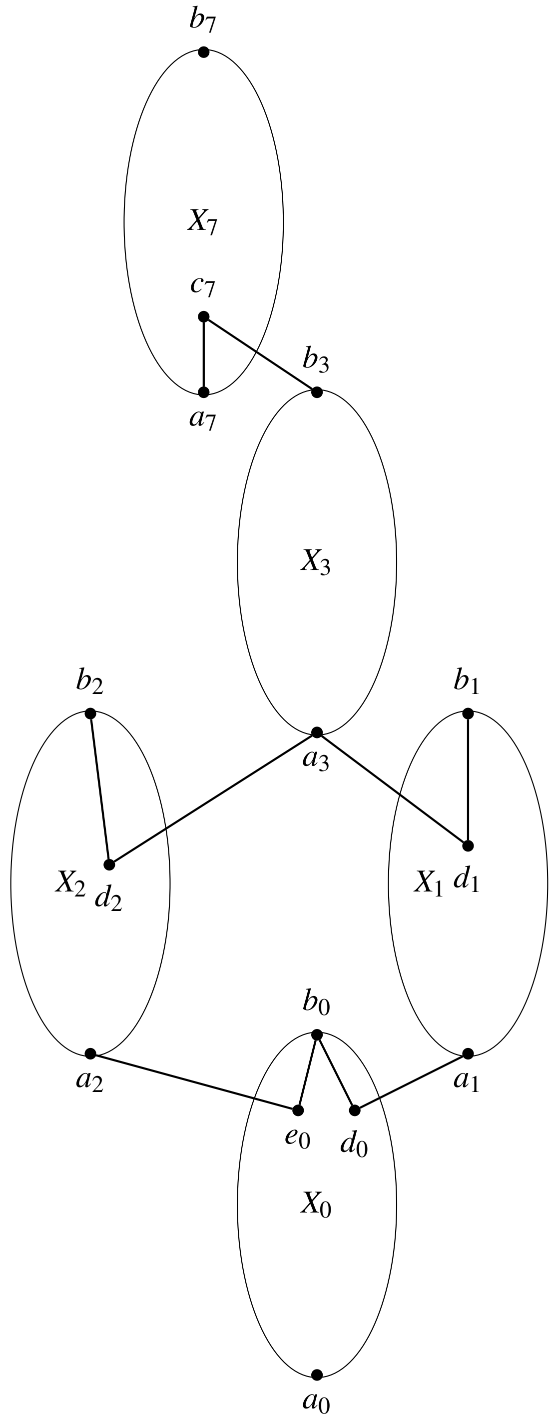

Figure 3.2: Diagram for range$(\alpha) = \{0, 1, 2, 3, 7\}$.

Here we see that $(d_0 \leq a_1 \leq d_1 \leq a_3 \leq b_3 \leq c_7)$ is one instance of the relation $(Y_1 \leq Y_2 \leq Y_3 \leq Y_4 \leq Y_5 \leq Y_6)$ and that $(e_0 \leq a_2 \leq d_2 \leq a_3)$ is one instance of $(Y_7 \leq Y_8 \leq Y_9 \leq Y_4)$.

Let $s = \langle d_0, a_1, d_1, a_3, b_3, c_7, e_0, a_2, d_2, \rangle$. Note that $s \in Q'_{13}$ by the above. Suppose also that $s \in Q_{13}$ then

$$\begin{aligned}\alpha(s) &= \langle \alpha(d_0), \alpha(a_1), \alpha(d_1), \alpha(a_3), \alpha(b_3), \alpha(c_7), \alpha(e_0), \alpha(a_2), \alpha(d_2) \rangle \\ &= \langle 0,1,1,3,3,7,0,2,2 \rangle.\end{aligned}$$

But $\langle 0,1,1,3,3,7,0,2,2 \rangle \notin Q_{13}$, so $\alpha(s) \notin Q_{13}$, and thus $s \notin Q_{13}$ because α respects Q_{13}. Therefore there exists a coordinate, say i, with $1 \leq i \leq n$ for which

$$s(i) = \langle d_0(i), a_1(i), d_1(i), a_3(i), b_3(i), c_7(i), e_0(i), a_2(i), d_2(i) \rangle \notin Q_{13}.$$

Because

$$d_0 \leq a_1 \leq d_1 \leq a_3 \leq b_3 \leq c_7, \text{ and } e_0 \leq a_2 \leq d_2 \leq a_3$$

we have that

$$d_0(i) \leq a_1(i) \leq d_1(i) \leq a_3(i) \leq b_3(i) \leq c_7(i), \text{ and } e_0(i) \leq a_2(i) \leq d_2(i) \leq a_3(i).$$

So $s(i) \in Q'_{13}$ and $s(i) \notin Q_{13}$, and there exists an $f \in \mathrm{Clo}_{\mathrm{u}}(\mathbf{M})$ with $f(s(i)) = \langle 0,1,1,3,3,7,0,2,2 \rangle$ but this forces $f = \mathrm{id}$, as the only term operation of $\mathbf{M}$ with 7 in the range is the identity operation. Thus we have $s(i) = \langle 0,1,1,3,3,7,0,2,2 \rangle$.

Now we show that α is the restriction of the term function π_i, that is, that α satisfies the interpolation condition for this case, and that this means $\alpha(x) = x(i)$ for all $x \in X$. By Lemma 3.3.1, this reduces to showing that

$$j = a_j(i) = b_j(i) \quad \text{ for } j \in \mathrm{range}(\alpha) = \{0,1,2,3,7\}.$$

Since $c_7(i) = b_3(i) \vee a_7(i)$, and $\vee$ is a homomorphism, we have $7 = 3 \vee a_7(i)$, so $a_7(i) = 7$. As $b_7 \geq a_7$ we have $b_7(i) \geq a_7(i) = 7$, so we also have $b_7(i) = 7$.

We know that $a_2(i) = 2$. Again since $d_2(i) = b_2(i) \wedge a_3(i)$, and $\wedge$ is a homomorphism, we have $2 = b_2(i) \wedge 3$, so $b_2(i) = 2$.

Similarly, we know that $a_1(i) = 1$, and since $d_1(i) = b_1(i) \wedge a_3(i)$ we have $1 = b_1(i) \wedge 3$, so $b_1(i) = 1$.

Likewise, since

$$\begin{aligned} 0 &= e_0(i) \vee d_0(i) \\ &= (a_2(i) \wedge b_0(i)) \vee (a_1(i) \wedge b_0(i)) \\ &= (2 \wedge b_0(i)) \vee (1 \wedge b_0(i)) \end{aligned}$$

we must have $b_0(i) = 0$. As $a_0 \leq b_0$ we have $a_0(i) \leq b_0(i) = 0$, so we also have $a_0(i) = 0$.

We already know that $a_3(i) = b_3(i) = 3$ from the co-ordinates of s.

Hence, by Lemma 3.3.1 we have $\alpha = \pi_i{\restriction}_X$, a term function of $\mathbf{M}$. □

We omit the detailed explanations in the following cases because all are very similar to the above case.

Lemma 3.3.4. *Suppose* $\alpha : \mathbb{X} \to \mathbb{M}$ *is a morphism, where* $\mathbb{X} \leq \mathbb{M}^n$*, and* $\mathbb{M} := \langle M, \wedge, \vee, \mathbf{0}, Q, \mathscr{T} \rangle$*. If* $\mathrm{range}(\alpha) = \{0,1,2,3\}$ *then* $\alpha = \pi_i{\restriction}_X$*, for some* i *where* $1 \leq i \leq n$*.*

Proof. Here we use Q_{12}. Suppose $\mathrm{range}(\alpha) = \{0,1,2,3\}$. (See Figure 3.3 on the next page.) With X_j, a_j and b_j defined as usual, let $d_2 = b_2 \wedge a_3$, $d_1 = a_3 \wedge b_1$, $d_0 = a_1 \wedge b_0$ and $e_0 = a_2 \wedge b_0$. We know that $\alpha(a_j) = j$, essentially by definition (and α respects finite meets), and similarly we know that $\alpha(b_j) = j$. We get $\alpha(e_0) = 0$ and $\alpha(d_j) = j$ from definitions and the fact that α respects meets.

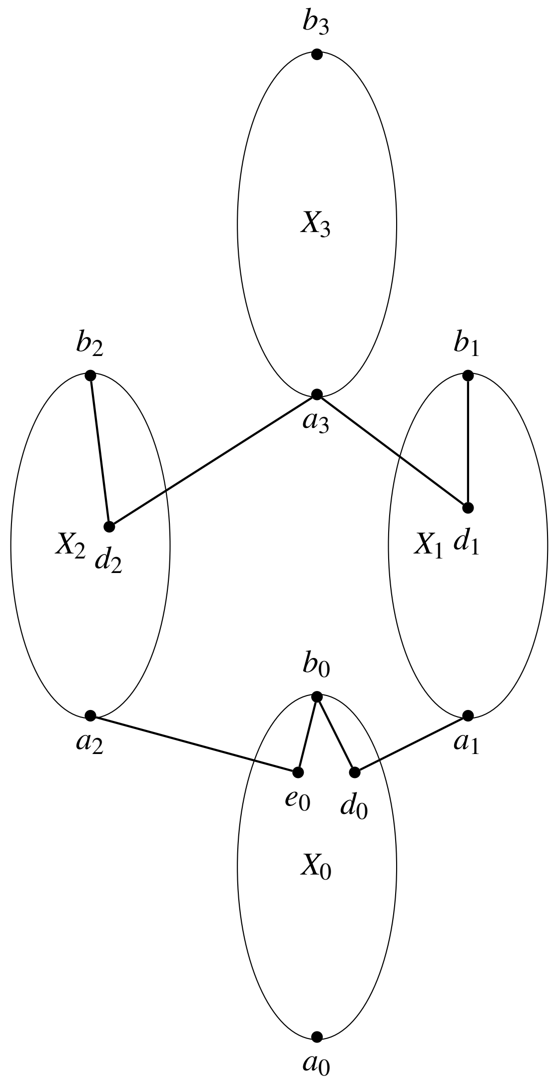

Figure 3.3: Diagram for range$(\alpha) = \{0,1,2,3\}$.

Here we see that $(d_0 \leq a_1 \leq d_1 \leq a_3 \leq b_3)$ is one instance of the relation $(Y_1 \leq Y_2 \leq Y_3 \leq Y_4 \leq Y_5)$ and that $(e_0 \leq a_2 \leq d_2 \leq a_3)$ is one instance of $(Y_6 \leq Y_7 \leq Y_8 \leq Y_4)$.

Let $s = \langle d_0, a_1, d_1, a_3, b_3, e_0, a_2, d_2 \rangle$. Note that $s \in Q'_{12}$ by the above. Then the tuple $\alpha(s) = \langle 0,1,1,3,3,0,2,2 \rangle$, and so $\alpha(s) \notin Q_{12}$. Thus $s \notin Q_{12}$ and there exists a coordinate i for which $s(i) \notin Q_{12}$. From this we have $s(i) = \langle 0,1,1,3,3,0,2,2 \rangle$. We now show that for all $x \in X$ we have $x(i) = \alpha(x)$. By Lemma 3.3.1, it suffices to show that for $j \in \text{range}(\alpha)$ we have $a_j(i) = b_j(i) = j$.

For $j = 0$ we know that

$$
\begin{aligned}
0 &= e_0(i) \vee d_0(i) \\
&= (a_2(i) \wedge b_0(i)) \vee (a_1(i) \wedge b_0(i)) \\
&= (2 \wedge b_0(i)) \vee (1 \wedge b_0(i))
\end{aligned}
$$

we must have $b_0(i) = 0$. As $a_0 \leq b_0$ we have $a_0(i) \leq b_0(i) = 0$, so $a_0(i) = 0$.

For $j = 1$ we argue as follows. From the value of s we have $a_1(i) = 1$, $d_1(i) = 1$, and $a_3(i) = 3$. Now $1 = d_1(i) = b_1(i) \wedge a_3(i) = b_1(i) \wedge 3$, so $b_1(i) = 1$.

For $j = 2$ we argue similarly. From the value of s we have $a_2(i) = 2$, $d_2(i) = 2$, and $a_3(i) = 3$. Now $2 = d_2(i) = b_2(i) \wedge a_3(i) = b_2(i) \wedge 3$, so $b_2(i) = 2$.

For $j = 3$ we have $a_3(i) = b_3(i) = 3$ from the co-ordinates of s.

Hence, by Lemma 3.3.1 we have $\alpha(x) = x(i)$ for every $x \in X$, and so $\alpha = \pi_i{\restriction}_X$, a term function of $\mathbf{M}$. □

Lemma 3.3.5. *Suppose $\alpha : \mathbb{X} \to \mathbb{M}$ is a morphism, where $\mathbb{X} \leq \mathbb{M}^n$, and $\mathbb{M} := \langle M, \wedge, \vee, \mathbf{0}, Q, \mathscr{T} \rangle$. If $\text{range}(\alpha) = \{0,2,3,7\}$ then $\alpha = \pi_i{\restriction}_X$, for some i where $1 \leq i \leq n$.*

Proof. Here we use Q_{11}. Suppose $\text{range}(\alpha) = \{0,2,3,7\}$ (See Figure 3.4 on the following page.)

With X_j, a_j, and b_j defined as usual, let $d_0 = b_0 \wedge a_2$, $d_2 = a_3 \wedge b_2$ and $c_7 = a_7 \vee b_3$.

Here we see that $(d_0 \leq a_2 \leq d_2 \leq a_3 \leq b_3 \leq c_7)$ is one instance of the relation $(Y_1 \leq Y_2 \leq Y_3 \leq Y_4 \leq Y_5 \leq Y_6)$ and that $(d_0 \leq b_0)$ is one instant of $(Y_1 \leq Y_7)$.

Let $s = \langle d_0, a_2, d_2, a_3, b_3, c_7, b_0 \rangle$. Note that $s \in Q'_{11}$ by the above. Then $\alpha(s) = \langle 0,2,2,3,3,7,0 \rangle$, and so $\alpha(s) \notin Q_{11}$. Thus $s \notin Q_{11}$ and there is some coordinate i for which $s(i) \notin Q_{11}$. From this we have $s(i) = \langle 0,2,2,3,3,7,0 \rangle$. Here we use that fact that the unique solution of

$$2 = d_2(i) = b_2(i) \wedge a_3(i) = b_2(i) \wedge 3 \quad \text{is} \quad b_2(i) = 2,$$

and the unique solution of

$$7 = c_7(i) = b_3(i) \vee a_7(i) = 3 \vee a_7(i) \quad \text{is} \quad a_7(i) = 7,$$

and the unique solution of

$$7 = a_7(i) \leq c_7(i) \leq b_7(i) \quad \text{is} \quad b_7(i) = 7.$$

Hence, by Lemma 3.3.1, we have $\alpha(x) = x(i)$ for every $x \in X$, and so $\alpha = \pi_i \restriction_X$, a term function of $\mathbf{M}$. □

Lemma 3.3.6. *Suppose* $\alpha : \mathbb{X} \to \mathbb{M}$ *is a morphism, where* $\mathbb{X} \leq \mathbb{M}^n$*, and* $\mathbb{M} := \langle M, \wedge, \vee, \mathbf{0}, Q, \mathscr{T} \rangle$*. If* $\mathrm{range}(\alpha) = \{0,1,3,7\}$ *then* $\alpha = \pi_i \restriction_X$*, for some* i *where* $1 \leq i \leq n$*.*

Proof. Here we use Q_{10}. Suppose $\mathrm{range}(\alpha) = \{0,1,3,7\}$. (See Figure 3.5 on the next page.) With X_j, a_j, and b_j defined as usual, let $d_0 = b_0 \wedge a_1$, $d_1 = b_1 \wedge a_3$ and $c_7 = b_3 \vee a_7$.

Here we see that $(d_0 \leq a_1 \leq d_1 \leq a_3 \leq b_3 \leq c_7)$ is one instance of the relation $(Y_1 \leq Y_2 \leq Y_3 \leq Y_4 \leq Y_5 \leq Y_6)$ and that $(d_0 \leq b_0)$ is one instance of $(Y_1 \leq Y_7)$.

Let $s = \langle d_0, a_1, d_1, a_3, b_3, c_7, b_0 \rangle$. Note that $s \in Q'_{10}$ by the above. Then $\alpha(s) = \langle 0,1,1,3,3,7,0 \rangle$, and so $\alpha(s) \notin Q_{10}$. Thus $s \notin Q_{10}$ and hence there exists a coordinate i for which $s(i) \notin Q_{10}$. From

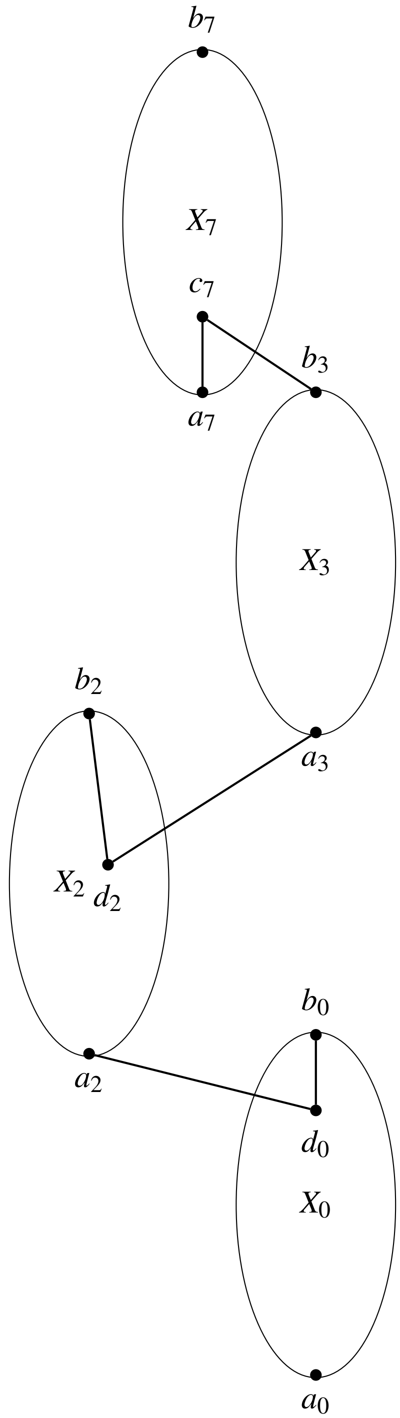

Figure 3.4: Diagram for range$(\alpha) = \{0,2,3,7\}$.

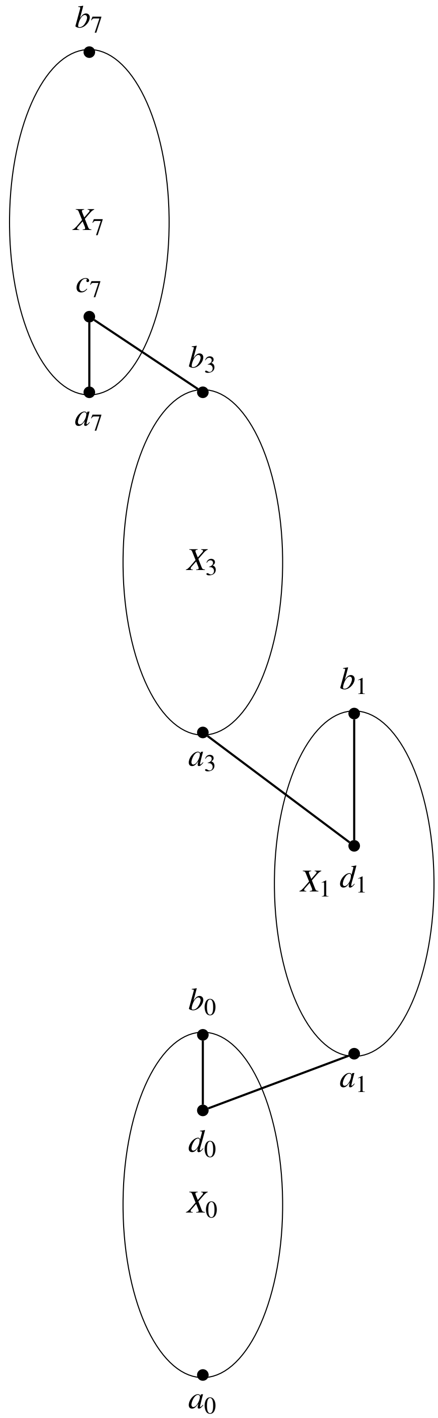

Figure 3.5: Diagram for range$(\alpha) = \{0, 1, 3, 7\}$.

this we have $s(i) = \langle 0,1,1,3,3,7,0\rangle$. Here we use that fact that the unique solution of

$$7 = c_7(i) = b_3(i) \vee a_7(i) = 3 \vee a_7(i) \quad \text{is} \quad a_7(i) = 7,$$

and the unique solution of

$$1 = d_1(i) = b_1(i) \wedge a_3(i) = b_1(i) \wedge 3 \quad \text{is} \quad b_1(i) = 1.$$

Hence, by Lemma 3.3.1, we have $\alpha(x) = x(i)$ for every $x \in X$, and so $\alpha = \pi_i{\restriction}_X$, a term function of $\mathbf{M}$. $\square$

Lemma 3.3.7. *Suppose $\alpha : \mathbb{X} \to \mathbb{M}$ is a morphism, where $\mathbb{X} \leq \mathbb{M}^n$, and $\mathbb{M} := \langle M, \wedge, \vee, \mathbf{0}, Q, \mathscr{T}\rangle$. If* $\mathrm{range}(\alpha) = \{0,2,3\}$ *then $\alpha = \pi_i{\restriction}_X$, for some i where $1 \leq i \leq n$.*

Proof. Here we use Q_9. Suppose $\mathrm{range}(\alpha) = \{0,2,3\}$. (See Figure 3.6.) With X_j, a_j, and b_j

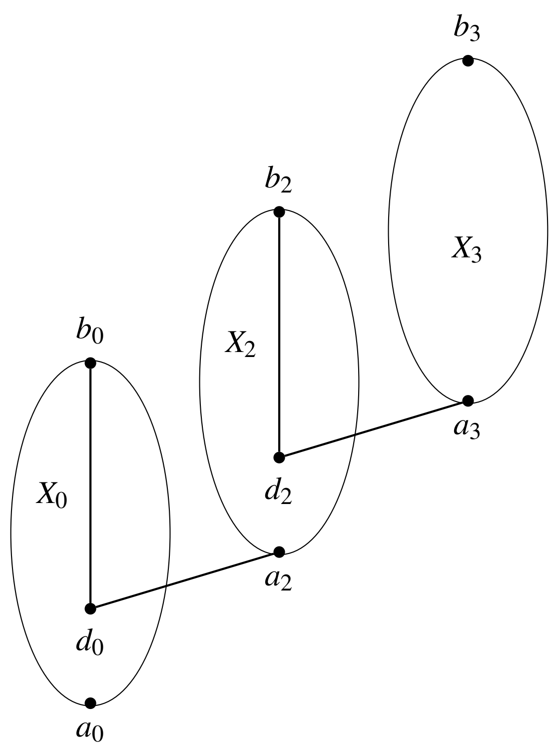

Figure 3.6: Diagram for $\mathrm{range}(\alpha) = \{0,2,3\}$.

defined as usual, let $d_0 = b_0 \wedge a_2$ and $d_2 = b_2 \wedge a_3$.

Here we see that $(d_0 \leq a_2 \leq d_2 \leq a_3 \leq b_3)$ is one instance of the relation $(Y_1 \leq Y_2 \leq Y_3 \leq Y_4 \leq Y_5)$ and that $(d_0 \leq b_0)$ is one instance of $(Y_1 \leq Y_6)$.

Let $s = \langle d_0, a_2, d_2, a_3, b_3, b_0 \rangle$. Note that $s \in Q_9'$ by the above. Then $\alpha(s) = \langle 0,2,2,3,3,0 \rangle$, and so $\alpha(s) \notin Q_9$. Thus $s \notin Q_9$ and hence there is some coordinate i for which $s(i) \notin Q_9$. From this we have $s(i) = \langle 0,2,2,3,3,0 \rangle$. We now show that for all $x \in X$ we have $x(i) = \alpha(x)$. By Lemma 3.3.1 it suffices to show that for $j \in \text{range}(\alpha)$ we have $a_j(i) = b_j(i) = j$.

For $j = 0$ we have $b_0(i) = 0$. As $a_0 \leq b_0$ we have $a_0(i) \leq b_0(i) = 0$.

For $j = 2$ we argue as follows. From the value of s we have $a_2(i) = 2$, $d_2(i) = 2$, and $a_3(i) = 3$. Now $2 = d_2(i) = b_2(i) \wedge a_3(i) = b_2(i) \wedge 3$, so $b_2(i) = 2$.

For $j = 3$ we have $a_3(i) = b_3(i) = 3$ from the co-ordinates of s.

Hence, by Lemma 3.3.1, we have $\alpha(x) = x(i)$ for every $x \in X$, and so $\alpha = \pi_i{\restriction}_X$, a term function of $\mathbf{M}$. □

Lemma 3.3.8. *Suppose* $\alpha : \mathbb{X} \to \mathbb{M}$ *is a morphism, where* $\mathbb{X} \leq \mathbb{M}^n$, *and* $\mathbb{M} := \langle M, \wedge, \vee, \mathbf{0}, Q, \mathscr{T} \rangle$. *If* $\text{range}(\alpha) = \{0,1,3\}$ *then* $\alpha = \pi_i{\restriction}_X$, *for some i where* $1 \leq i \leq n$.

Proof. Here we use Q_8. Suppose $\text{range}(\alpha) = \{0,1,3\}$. (See Figure 3.7 on the next page.) With X_j, a_j, and b_j defined as usual, let $d_0 = b_0 \wedge a_1$ and $d_1 = b_1 \wedge a_3$.

Here we see that $(d_0 \leq a_1 \leq d_1 \leq a_3 \leq b_3)$ is one instance of the relation $(Y_1 \leq Y_2 \leq Y_3 \leq Y_4 \leq Y_5)$ and that $(d_0 \leq b_0)$ is one instance of $(Y_1 \leq Y_6)$.

Let $s = \langle d_0, a_1, d_1, a_3, b_3, b_0 \rangle$. Note that $s \in Q_8'$ by the above. Then $\alpha(s) = \langle 0,1,1,3,3,0 \rangle$, and

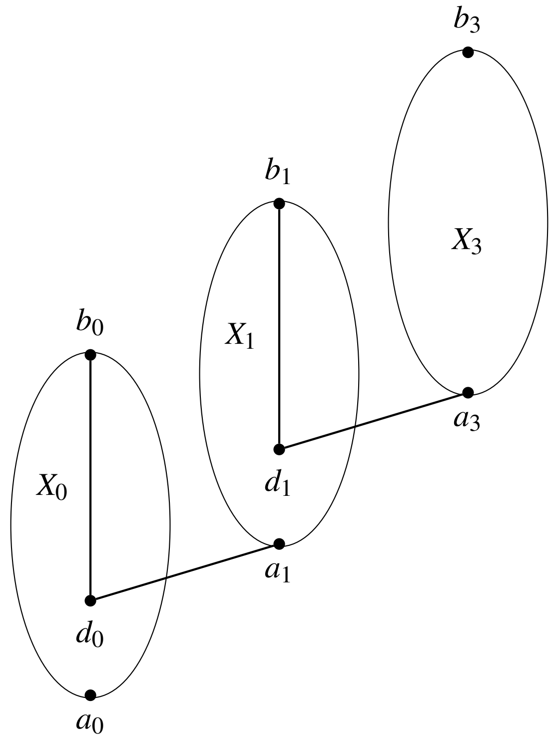

Figure 3.7: Diagram for $\text{range}(\alpha) = \{0, 1, 3\}$.

so $\alpha(s) \notin Q_8$. Thus $s \notin Q_8$ and hence there is some coordinate i for which $s(i) \notin Q_8$. From this we have $s(i) = \langle 0, 1, 1, 3, 3, 0 \rangle$. Here we use that fact that the unique solution of

$$1 = d_1(i) = b_1(i) \wedge a_3(i) = b_1(i) \wedge 3 \quad \text{is} \quad b_1(i) = 1.$$

Hence, by Lemma 3.3.1, we have $\alpha(x) = x(i)$ for every $x \in X$, and so $\alpha = \pi_i\restriction_X$, a term function of $\mathbf{M}$. □

Lemma 3.3.9. *Suppose* $\alpha : \mathbb{X} \to \mathbb{M}$ *is a morphism, where* $\mathbb{X} \leq \mathbb{M}^n$*, and* $\mathbb{M} := \langle M, \wedge, \vee, \mathbf{0}, Q, \mathscr{T} \rangle$. *If* $\text{range}(\alpha) = \{0, 3, 7\}$ *then* $\alpha = \pi_i\restriction_X$*, for some* i *where* $1 \leq i \leq n$.

Proof. Here we use Q_7. We have $\text{range}(\alpha) = \{0, 3, 7\}$. (See Figure 3.8 on the following page.) With X_j, a_j, and b_j defined as usual, let $d_0 = b_0 \wedge a_3$ and $d_3 = b_3 \wedge a_7$.

Here we see that $(d_0 \leq a_3 \leq d_3 \leq a_7)$ is one instance of the relation $(Y_1 \leq Y_2 \leq Y_3 \leq Y_4)$ and

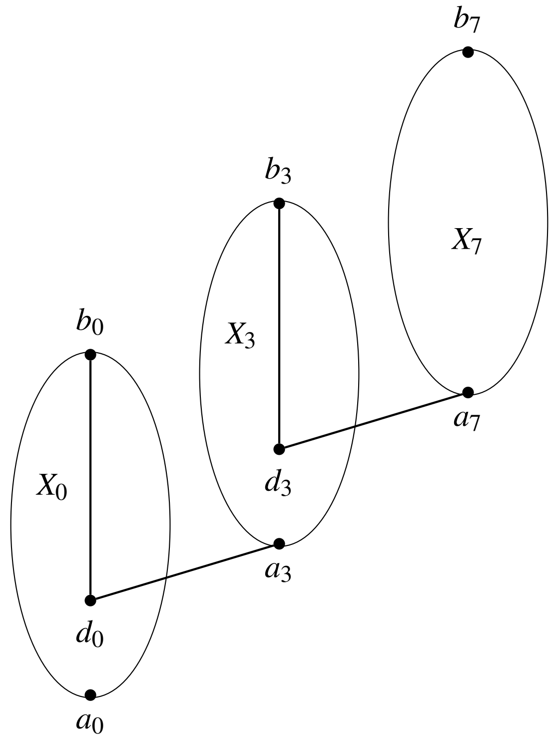

Figure 3.8: Diagram for range$(\alpha) = \{0,3,7\}$.

$(d_0 \leq b_0)$ is one instance of $(Y_1 \leq Y_5)$.

Let $s = \langle d_0, a_3, d_3, a_7, b_0 \rangle$. Note that $s \in Q_7'$ by the above. Then $\alpha(s) = \langle 0,3,3,7,0 \rangle$, and so $\alpha(s) \notin Q_7$. Thus $s \notin Q_7$ and we get that there is a coordinate i for which $s(i) \notin Q_7$. From this we have $s(i) = \langle 0,3,3,7,0 \rangle$. Here we use that fact that the unique solution of

$$3 = d_3(i) = b_3(i) \wedge a_7(i) = b_3(i) \wedge 7 \quad \text{is} \quad b_3(i) = 3.$$

Hence, by Lemma 3.3.1, we have $\alpha(x) = x(i)$ for every $x \in X$, and so $\alpha = \pi_i\restriction_X$, a term function of $\mathbf{M}$. $\square$

Lemma 3.3.10. *Suppose $\alpha : \mathbb{X} \to \mathbb{M}$ is a morphism, where $\mathbb{X} \leq \mathbb{M}^n$, and $\mathbb{M} := \langle M, \wedge, \vee, \mathbf{0}, Q, \mathscr{T} \rangle$. If* range$(\alpha) = \{0,2,7\}$ *then $\alpha = \pi_i\restriction_X$, for some i where $1 \leq i \leq n$.*

Proof. Here we use Q_6. Suppose range$(\alpha) = \{0,2,7\}$. (See Figure 3.9 on the next page.) With

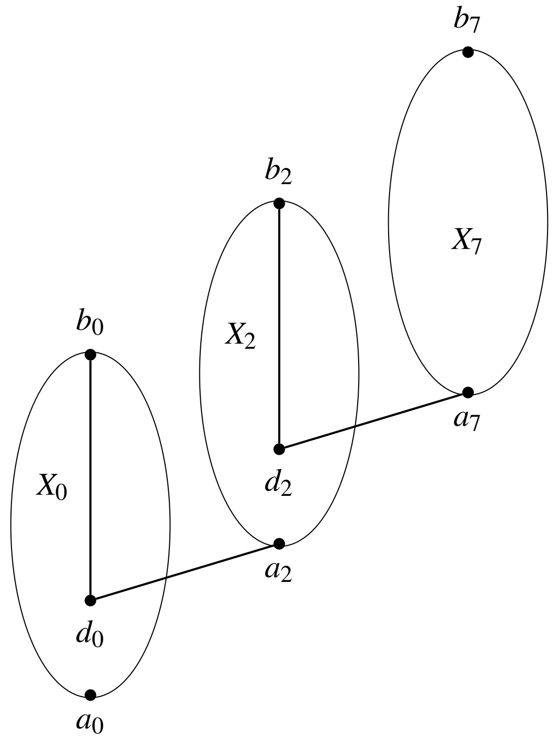

Figure 3.9: Diagram for range$(\alpha) = \{0,2,7\}$.

X_j, a_j, and b_j defined as usual, let $d_0 = b_0 \wedge a_2$ and $d_2 = b_2 \wedge a_7$.

Here we see that $(d_0 \leq a_2 \leq d_2 \leq a_7)$ is one instance of the relation $(Y_1 \leq Y_2 \leq Y_3 \leq Y_4)$ and that $(d_0 \leq b_0)$ is one instance of $(Y_1 \leq Y_5)$.

Let $s = \langle d_0, a_2, d_2, a_7, b_0 \rangle$. Note that $s \in Q_6'$ by the above. Then $\alpha(s) = \langle 0,2,2,7,0 \rangle$, and so $\alpha(s) \notin Q_6$. Thus $s \notin Q_6$ and we get that there exists i for which $s(i) \notin Q_6$. From this we have $s(i) = \langle 0,2,2,7,0 \rangle$. Here we use that fact that the unique solution of

$$2 = d_2(i) = b_2(i) \wedge a_7(i) = b_2(i) \wedge 7 \quad \text{is} \quad b_2(i) = 2.$$

Hence, by Lemma 3.3.1, we have $\alpha(x) = x(i)$ for each $x \in X$, and so $\alpha = \pi_i\restriction_X$, a term function of **M**. $\square$

Lemma 3.3.11. *Suppose $\alpha : \mathbb{X} \to \mathbb{M}$ is a morphism, where $\mathbb{X} \leq \mathbb{M}^n$, and $\mathbb{M} := \langle M, \wedge, \vee, \mathbf{0}, Q, \mathscr{T} \rangle$.*

If $\operatorname{range}(\alpha) = \{0, 1, 7\}$ *then* $\alpha = \pi_i\restriction_X$*, for some* i *where* $1 \leq i \leq n$*.*

Proof. Here we use Q_5. Suppose $\operatorname{range}(\alpha) = \{0, 1, 7\}$. (See Figure 3.10.) With X_j, a_j, and b_j

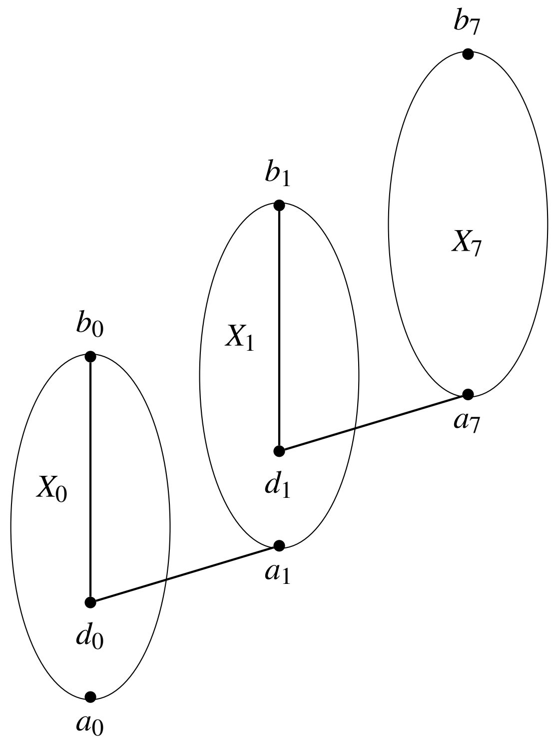

Figure 3.10: Diagram for $\operatorname{range}(\alpha) = \{0, 1, 7\}$.

defined as usual, let $d_0 = b_0 \wedge a_1$, and $d_1 = b_1 \wedge a_7$.

Here we see that $(d_0 \leq a_1 \leq d_1 \leq a_7)$ is one instance of the relation $(Y_1 \leq Y_2 \leq Y_3 \leq Y_4)$ and that $(d_0 \leq b_0)$ is one instance of $(Y_1 \leq Y_5)$.

Let $s = \langle d_0, a_1, d_1, a_7, b_0\rangle$. Note that $s \in Q_5'$ by the above. Then $\alpha(s) = \langle 0, 1, 1, 7, 0\rangle$, and so $\alpha(s) \notin Q_5$. Thus $s \notin Q_5$ and hence there is some coordinate i for which $s(i) \notin Q_5$. From this we have $s(i) = \langle 0, 1, 1, 7, 0\rangle$. Here we use that fact that the unique solution of

$$1 = d_1(i) = b_1(i) \wedge a_7(i) = b_1(i) \wedge 7 \quad \text{is} \quad b_1(i) = 1.$$

Hence, by Lemma 3.3.1, we have $\alpha(x) = x(i)$ for every $x \in X$, and so $\alpha = \pi_i{\restriction}_X$, a term function of $\mathbf{M}$. $\square$

Lemma 3.3.12. *Suppose $\alpha : \mathbb{X} \to \mathbb{M}$ is a morphism, where $\mathbb{X} \leq \mathbb{M}^n$, and $\mathbb{M} := \langle M, \wedge, \vee, \mathbf{0}, Q, \mathscr{T} \rangle$. If* $\operatorname{range}(\alpha) = \{0,3\}$ *then $\alpha = \pi_i{\restriction}_X$, for some i where $1 \leq i \leq n$.*

Proof. Here we use Q_4. Suppose $\operatorname{range}(\alpha) = \{0,3\}$. (See Figure 3.11.) With X_j, a_j, and b_j defined

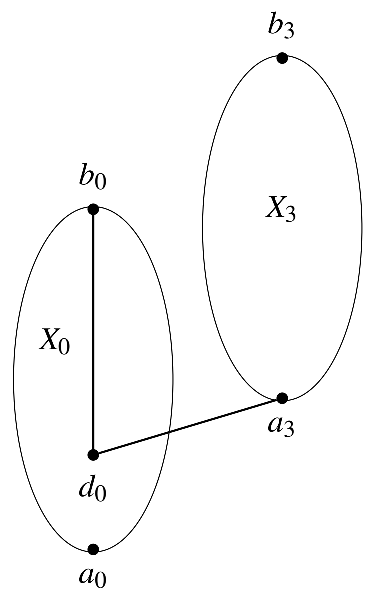

Figure 3.11: Diagram for $\operatorname{range}(\alpha) = \{0,3\}$.

as usual, let $d_0 = b_0 \wedge a_3$.

Here we see that $(d_0 \leq a_3 \leq b_3)$ is one instance of the relation $(Y_1 \leq Y_2 \leq Y_3)$ and that $(d_0 \leq b_0)$ is one instance of $(Y_1 \leq Y_4)$.

Let $s = \langle d_0, a_3, b_3, b_0 \rangle$. Note that $s \in Q_4'$ by the above. Then $\alpha(s) = \langle 0,3,3,0 \rangle$, and so $\alpha(s) \notin Q_4$. Thus $s \notin Q_4$ and hence there is some coordinate i for which $s(i) \notin Q_4$. From this we have $s(i) = \langle 0,3,3,0 \rangle$.

Hence, by Lemma 3.3.1, we have $\alpha(x) = x(i)$ for every $x \in X$, and so $\alpha = \pi_i{\restriction}_X$, a term function of $\mathbf{M}$. $\square$

Lemma 3.3.13. *Suppose $\alpha : \mathbb{X} \to \mathbb{M}$ is a morphism, where $\mathbb{X} \leq \mathbb{M}^n$, and $\mathbb{M} := \langle M, \wedge, \vee, \mathbf{0}, Q, \mathscr{T} \rangle$. If $\mathrm{range}(\alpha) = \{0,2\}$ then $\alpha = \pi_i{\restriction}_X$, for some i where $1 \leq i \leq n$.*

Proof. Here we use Q_3. Suppose $\mathrm{range}(\alpha) = \{0,2\}$. (See Figure 3.12.) With X_j, a_j, and b_j defined

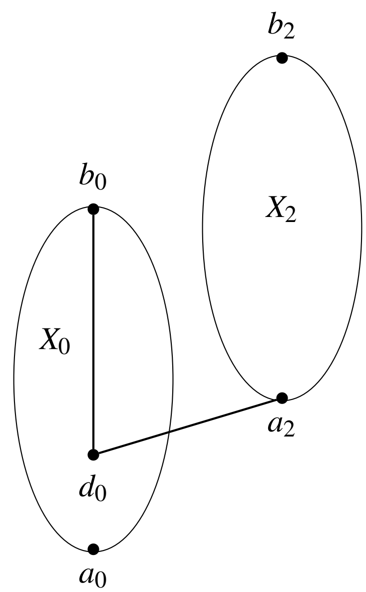

Figure 3.12: Diagram for $\mathrm{range}(\alpha) = \{0,2\}$.

as usual, let $d_0 = b_0 \wedge a_2$.

Here we see that $(d_0 \leq a_2 \leq b_2)$ is one instance of the relation $(Y_1 \leq Y_2 \leq Y_3)$, and that $(d_0 \leq b_0)$ is one instance of $(Y_1 \leq Y_4)$.

Let $s = \langle d_0, a_2, b_2, b_0 \rangle$. Note that $s \in Q_3'$ by the above. Then $\alpha(s) = \langle 0,2,2,0 \rangle$, and so $\alpha(s) \notin Q_3$. Thus $s \notin Q_3$ and hence there is some coordinate i for which $s(i) \notin Q_3$. From this we have $s(i) = \langle 0,2,2,0 \rangle$.

Hence, by the Lemma 3.3.1, we have $\alpha(x) = x(i)$ for every $x \in X$, and so $\alpha = \pi_i{\restriction}_X$, a term function of $\mathbf{M}$. $\square$

Lemma 3.3.2 and the next lemma are the cases where α might not be the restriction of a projection.

Lemma 3.3.14. *Suppose* $\alpha : \mathbb{X} \to \mathbb{M}$ *is a morphism, where* $\mathbb{X} \leq \mathbb{M}^n$*, and* $\mathbb{M} := \langle M, \wedge, \vee, \mathbf{0}, Q, \mathscr{T} \rangle$*. If* $\operatorname{range}(\alpha) = \{0, 1\}$ *then* $\alpha = f(\pi_i){\restriction}_X$*, for some* i *where* $1 \leq i \leq n$*.*

Proof. Here we use Q_2. Suppose $\operatorname{range}(\alpha) = \{0, 1\}$. (See Figure 3.13.) With X_j, a_j, and b_j defined

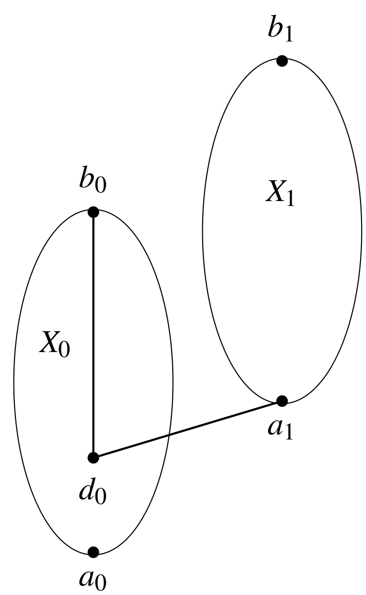

Figure 3.13: Diagram for $\operatorname{range}(\alpha) = \{0, 1\}$.

as usual, let $d_0 = b_0 \wedge a_1$.

Let U be the subset of M^4 defined by $\{\langle Y_1, Y_2, Y_3, Y_4 \rangle \in M^4 : Y_1 \leq Y_2 \leq Y_3 \;\&\; Y_1 \leq Y_4\}$. Let S be the subset of U defined by $S = \{u \in U : \langle 0, 1, 1, 0 \rangle \in \mathbf{Sg}_{M^4}(\{u\})\}$. We note that if $t \in U \setminus S$ then $\langle 0, 1, 1, 0 \rangle \notin \mathbf{Sg}_{M^4}(\{t\})$.

Note that $Q_2 = U \setminus S$, and $Q_2' = U \cup S$. Let $s = \langle d_0, a_1, b_1, b_0 \rangle$. Note that $s \in Q_2'$ by the above. Then $\alpha(s) = \langle 0,1,1,0 \rangle \notin Q_2$, and therefore $s \notin Q_2$ and hence there is some coordinate i for which $s(i) \notin Q_2$. However $d_0 \leq a_1 \leq b_1$ and $d_0 \leq b_0$, so $s(i) \in U$. Thus $s(i) \in S$, and there exists an $f \in \mathrm{Clo_u}(\mathbf{M})$, such that $f(s(i)) = \langle 0,1,1,0 \rangle$. We now show that for all $x \in X$ we have $f(x(i)) = \alpha(x)$. By Lemma 3.3.1 it suffices to show that for $j \in \mathrm{range}(\alpha)$ we have $f(a_j(i)) = f(b_j(i)) = j$.

For $j = 0$ we have $f(b_0(i)) = 0$. As $a_0 \leq b_0$, and each possible f is order preserving, we have $f(a_0(i)) \leq f(b_0(i)) = 0$, so $f(a_0(i)) = 0$.

For $j = 1$ we have $f(a_1(i)) = f(b_1(i)) = 1$ from the co-ordinates of s.

Hence, by Lemma 3.3.1, we have $\alpha(x) = f(x(i))$ for every $x \in X$, and so $\alpha = f(\pi_i)\restriction_X$, the restriction of a term function of $\mathbf{M}$. $\square$

Lemma 3.3.15. *Suppose $\alpha : \mathbb{X} \to \mathbb{M}$ is a morphism, where $\mathbb{X} \leq \mathbb{M}^n$, and $\mathbb{M} := \langle M, \wedge, \vee, \mathbf{0}, Q, \mathscr{T} \rangle$. If $\mathrm{range}(\alpha) = \{0,7\}$ then $\alpha = \pi_i\restriction_X$, for some i where $1 \leq i \leq n$.*

Proof. Here we use Q_1. Suppose $\mathrm{range}(\alpha) = \{0,7\}$. (See Figure 3.14 on the following page.) With X_j, a_j, and b_j defined as usual, let $d_0 = b_0 \wedge a_7$.

Here we see that $(d_0 \leq a_7)$ is an instance of the relation $(Y_1 \leq Y_2)$.

Let $s = \langle d_0, a_7 \rangle$. Note that $s \in Q_1'$ by the above. Then $\alpha(s) = \langle 0,7 \rangle$, and so $\alpha(s) \notin Q_1$. Thus $s \notin Q_1$ and there is some coordinate i for which $s(i) \notin Q_1$. From this we have $s(i) = \langle 0,7 \rangle$. Here we use that fact that the unique solution of

$$0 = d_0(i) = b_0(i) \wedge a_7(i) = b_0(i) \wedge 7 \quad \text{is} \quad b_0(i) = 0,$$

and the unique solution of

$$7 = a_7(i) \leq b_7(i) \quad \text{is} \quad b_7(i) = 7.$$

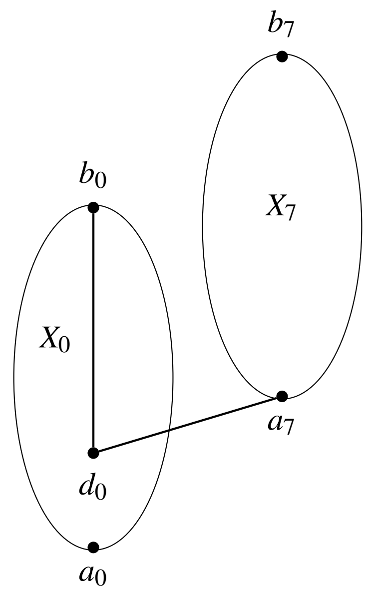

Figure 3.14: Diagram for range$(\alpha) = \{0,7\}$.

Hence, the Lemma 3.3.1, we have $\alpha(x) = x(i)$ for every $x \in X$, and so $\alpha = \pi_i\restriction_X$, a term function of $\mathbf{M}$. $\square$

Now we prove the following Theorem by using Interpolation Condition.

Theorem 3.3.1. *The algebra* $\mathbf{M}$ *is dualized by the alter ego* $\langle \mathbb{M}, \wedge, \vee, \mathbf{0}, Q, \mathscr{T} \rangle$*, where the set* $Q = \{Q_1, \ldots, Q_{13}\}$ *as given in Table 3.2 on page 21 .*

Proof. For range$(\alpha) = \{0\}$, we have shown that $\alpha = \bar{0} \circ \pi_1\restriction_X$ (Lemma 3.3.2).

Moreover, for range$(\alpha) = \{0,1\}$, we have proved that

$$\alpha = f(\pi_i)\restriction_X$$

is a term function of the algebra $\mathbb{M}$ (Lemma 3.3.14).

Furthermore, we have shown (Lemmas 3.3.3, 3.3.4, 3.3.5, 3.3.6, 3.3.7, 3.3.8, 3.3.9, 3.3.10, 3.3.11, 3.3.12, 3.3.13, and 3.3.15) that for any range(α) (except range$(\alpha) = \{0\}$ and range$(\alpha) = \{0, 1\}$), for every $x \in X$, and $1 \leq i \leq n$ we have

$$\alpha = \pi_i \restriction_X,$$

a term function of $\mathbf{M}$. This concludes that the alter ego $\langle \mathbb{M}, \wedge, \vee, \mathbf{0}, Q, \mathscr{T} \rangle$ satisfies the interpolation condition relative to the algebra $\mathbf{M}$, and hence $\mathbb{M}$ dualizes $\mathbf{M}$; that means $\mathbf{M}$ is dualizable. $\square$

The previous Theorem 3.3.1 gives the following corollary.

Corollary 3.3.1. *The alter ego* $\langle M, \wedge, \vee, \mathbf{0}, R_9, \mathscr{T} \rangle$ *yields a duality on* $\mathbb{ISP}(\mathbf{M})$.

Proof. This follows from Theorem 3.3.1 because $Q \subseteq R_9$. $\square$

By Theorem 2.6.3, the alter ego $\langle M, \wedge, \vee, \mathbf{0}, R_{10}, \mathscr{T} \rangle$ yields a duality on $\mathbb{ISP}(\mathbf{M})$. From Theorem 3.3.1, the algebra $\mathbf{M}$ is dualized by the alter ego $\langle \mathbb{M}, \wedge, \vee, \mathbf{0}, Q, \mathscr{T} \rangle$. From Corollary 3.3.1, the alter ego $\langle M, \wedge, \vee, \mathbf{0}, R_9, \mathscr{T} \rangle$ yields a duality on $\mathbb{ISP}(\mathbf{M})$. In the alter ego $\mathbb{M}$, a set of algebraic relations Q is smaller than both R_{10} and R_9. We now are going to show whether or not the alter ego $\mathbb{M}$ is a minimal dualizing structure.

Chapter 4

Minimality

4.1 Non-evaluation morphisms

We now want to show that the set Q defined on page 22 is minimal. We must show that if Q' is a proper subset of Q containing Q_{12} and Q_{13} then the algebra $\mathbf{M}$ is not dualized by the alter ego $\langle M, \wedge, \vee, \mathbf{0}, Q', \mathscr{T} \rangle$. Note that if for some $\mathbf{L} \in \mathbb{ISP}(\mathbf{M})$, we have that $\mathbf{L}$ is not isomorphic to $ED(\mathbf{L})$ because of a non-evaluation morphism, then $\langle M, \wedge, \vee, \mathbf{0}, Q', \mathscr{T} \rangle$ does not yield a duality on $\mathbf{M}$.

We first want to prove that the dual of $\mathbf{L}$ with respect to $\langle M, \{\wedge, \vee, \mathbf{0}\}, \mathscr{T} \rangle$ has non-evaluation morphisms.

Lemma 4.1.1. *[17] Let L be a k-ary relation on M. Let β be an element of $M^k \setminus L$, where $L' \supseteq L \cup \{\beta\}$ and $L \leq L' \leq \mathbf{M}^k$ are all algebras.*

If there is an embedding ι of $\mathrm{Hom}(\mathbf{L}, \mathbf{M})$ *into* $\mathrm{Hom}(\mathbf{L}', \mathbf{M})$ *such that*

- $\iota(h_1 \wedge h_2) = \iota(h_1) \wedge \iota(h_2)$,

- *$\iota(h_1 \vee h_2) = \iota(h_1) \vee \iota(h_2)$,*

- *$\iota(\mathbf{0}) = \mathbf{0}$, and*

- *$[\iota(\pi_j)](\beta) = \pi_j(\beta)$;*

then the dual of $\mathbf{L}$ with respect to $\langle M, \{\wedge, \vee, \mathbf{0}\}, \mathscr{T}\rangle$ has a non-evaluation morphism.

Proof. Let ι be the embedding mentioned above. Define $\alpha : \mathrm{Hom}(\mathbf{L}, \mathbf{M}) \to \mathbf{M}$ by

$$\alpha(h) = [\iota h](\beta).$$

Note that for $a \in \mathbf{L}$ we have

$$\langle e_a(\pi_k)\rangle_k = \langle \pi_k(a)\rangle_k = \langle a_k\rangle_k = a \neq \beta;$$

whereas

$$\langle \alpha(\pi_k)\rangle_k = \beta.$$

Thus α is not an evaluation morphism. On the other hand, for $h \in D(\mathbf{L})$, $\alpha(h)$ is well defined. Because there is an embedding ι of $\mathrm{Hom}(\mathbf{L}, \mathbf{M})$ into $\mathrm{Hom}(\mathbf{L}', \mathbf{M})$ and $\alpha : \mathrm{Hom}(\mathbf{L}, \mathbf{M}) \to \mathbf{M}$ we have

$$\begin{aligned} \alpha(h_1 \wedge h_2) &= [\iota(h_1 \wedge h_2)](\beta) \\ &= [\iota(h_1) \wedge \iota(h_2)](\beta) \\ &= [\iota h_1](\beta) \wedge [\iota h_2](\beta) \\ &= \alpha(h_1) \wedge \alpha(h_2), \end{aligned}$$

and similarly, $\alpha(h_1 \vee h_2) = \alpha(h_1) \vee \alpha(h_2)$, and $\alpha(\mathbf{0}) = [\iota(\mathbf{0})](\beta) = \mathbf{0}(\beta) = \mathbf{0}$, hence α respects $\wedge$, $\vee$ and $\mathbf{0}$. Thus $D(\mathbf{L})$ contains a non-evaluation morphism with respect to $\langle M, \{\wedge, \vee, \mathbf{0}\}, \mathscr{T}\rangle$. $\square$

Any alter ego $\langle M, \wedge, \vee, \mathbf{0}, Q', \mathscr{T} \rangle$ that dualizes $\mathbf{M}$ must contain relations that the morphisms provided by Lemma 4.1.1 do not respect.

We want to prove that $\mathrm{Hom}(\mathbf{Q}_i, \mathbf{M})$ embeds into $\mathrm{Hom}(\mathbf{Q}'_i, \mathbf{M})$. We prove this in two separate lemmas. Firstly, we prove that $\mathrm{Hom}(\mathbf{Q}_i, \mathbf{M})$ embeds into $\mathrm{Hom}(\mathbf{Q}'_i, \mathbf{M})$ for $1 \le i \le 13$ $(i \neq 2)$ and secondly, we prove that $\mathrm{Hom}(\mathbf{Q}_2, \mathbf{M})$ embeds into $\mathrm{Hom}(\mathbf{Q}'_2, \mathbf{M})$. The general argument in Lemma 4.1.2 does not work for Q_2 as $|Q'_2 \setminus Q_2| > 1$, so we need a second lemma.

Lemma 4.1.2. *Let Q_i, Q'_i and β_i be defined as in Chapter 3. For $1 \le i \le 13$ $(i \neq 2)$, there is an embedding ι of $\mathrm{Hom}(\mathbf{Q}_i, \mathbf{M})$ into $\mathrm{Hom}(\mathbf{Q}'_i, \mathbf{M})$ satisfying the properties of Lemma 4.1.1.*

Proof. In general, note that we have $p(x) \le q(x)$ and $p(x) \le r(x)$. Consulting Table 4.1 on the following page and Table 4.2 on page 49, note that in each case we have b_i, c_i, d_i, u_i, and v_i are all elements of Q_i and that

$$b_i = p(\beta_i) = p(u_i) = p(v_i),$$

$$c_i = q(\beta_i) = q(v_i), \text{ and}$$

$$d_i = r(\beta_i) = r(u_i).$$

Let $h \in \mathrm{Hom}(\mathbf{Q}_i, \mathbf{M})$. We know that $\langle h(b_i), h(c_i), h(d_i) \rangle \in \{0,1\}^3$. We can establish that $h(b_i) \le h(c_i)$ and $h(b_i) \le h(d_i)$ by noting that

$$h(b_i) = h(p(v_i)) = p(h(v_i)) \le q(h(v_i)) = h(q(v_i)) = h(c_i),$$

$$h(b_i) = h(p(u_i)) = p(h(u_i)) \le r(h(u_i)) = h(r(u_i)) = h(d_i).$$

Thus, there exists an $m_i \in M$ such that $\langle h(b_i), h(c_i), h(d_i) \rangle = \langle p(m_i), q(m_i), r(m_i) \rangle$. By Lemma 3.2.1, extend h to $h' : Q'_i \to \mathbf{M}$ by setting $h'(\beta_i) = m_i$. We have

$$h'(p(\beta_i)) = h'(b_i) = h(b_i) = p(m_i) = p(h'(\beta_i)),$$

i	β_i	u_i	v_i
1	$\langle 0,7\rangle$	$\langle 2,7\rangle$	$\langle 1,7\rangle$
3	$\langle 0,2,2,0\rangle$	$\langle 0,0,2,2\rangle$	$\langle 0,2,3,0\rangle$
4	$\langle 0,3,3,0\rangle$	$\langle 0,1,3,0\rangle$	$\langle 0,2,3,1\rangle$
5	$\langle 0,1,1,7,0\rangle$	$\langle 0,1,3,7,0\rangle$	$\langle 0,0,1,7,0\rangle$
6	$\langle 0,2,2,7,0\rangle$	$\langle 0,0,0,7,0\rangle$	$\langle 0,2,3,7,0\rangle$
7	$\langle 0,3,3,7,0\rangle$	$\langle 0,1,3,7,2\rangle$	$\langle 0,2,2,7,1\rangle$
8	$\langle 0,1,1,3,3,0\rangle$	$\langle 0,3,3,3,3,0\rangle$	$\langle 0,0,0,3,3,1\rangle$
9	$\langle 0,2,2,3,3,0\rangle$	$\langle 0,0,0,3,3,2\rangle$	$\langle 0,2,2,2,3,0\rangle$
10	$\langle 0,1,1,3,3,7,0\rangle$	$\langle 0,3,3,3,3,7,0\rangle$	$\langle 0,1,1,2,3,7,0\rangle$
11	$\langle 0,2,2,3,3,7,0\rangle$	$\langle 0,0,0,3,3,7,0\rangle$	$\langle 0,3,3,3,3,7,1\rangle$
12	$\langle 0,1,1,3,3,0,2,2\rangle$	$\langle 0,1,1,1,3,0,0,0\rangle$	$\langle 0,0,0,2,3,0,2,2\rangle$
13	$\langle 0,1,1,3,3,7,0,2,2\rangle$	$\langle 0,3,3,3,3,7,0,2,2\rangle$	$\langle 0,0,0,3,3,7,0,2,2\rangle$

Table 4.1: Elements β_i, u_i, v_i for proof of the Lemma 4.1.2

i	b_i	c_i	d_i
1	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$
3	$\langle 0,0,0,0\rangle$	$\langle 0,1,1,0\rangle$	$\langle 0,0,0,0\rangle$
4	$\langle 0,0,0,0\rangle$	$\langle 0,1,1,0\rangle$	$\langle 0,1,1,0\rangle$
5	$\langle 0,0,0,1,0\rangle$	$\langle 0,0,0,1,0\rangle$	$\langle 0,1,1,1,0\rangle$
6	$\langle 0,0,0,1,0\rangle$	$\langle 0,1,1,1,0\rangle$	$\langle 0,0,0,1,0\rangle$
7	$\langle 0,0,0,1,0\rangle$	$\langle 0,1,1,1,0\rangle$	$\langle 0,1,1,1,0\rangle$
8	$\langle 0,0,0,0,0,0\rangle$	$\langle 0,0,0,1,1,0\rangle$	$\langle 0,1,1,1,1,0\rangle$
9	$\langle 0,0,0,0,0,0\rangle$	$\langle 0,1,1,1,1,0\rangle$	$\langle 0,0,0,1,1,0\rangle$
10	$\langle 0,0,0,0,0,1,0\rangle$	$\langle 0,0,0,1,1,1,0\rangle$	$\langle 0,1,1,1,1,1,0\rangle$
11	$\langle 0,0,0,0,0,1,0\rangle$	$\langle 0,1,1,1,1,1,0\rangle$	$\langle 0,0,0,1,1,1,0\rangle$
12	$\langle 0,0,0,0,0,0,0,0,0\rangle$	$\langle 0,0,0,1,1,0,1,1\rangle$	$\langle 0,1,1,1,1,0,0,0\rangle$
13	$\langle 0,0,0,0,0,1,0,0,0\rangle$	$\langle 0,0,0,1,1,1,0,1,1\rangle$	$\langle 0,1,1,1,1,1,0,0,0\rangle$

Table 4.2: Elements b_i, c_i, d_i for proof of the Lemma 4.1.2

$$h'(q(\beta_i)) = h'(c_i) = h(c_i) = q(m_i) = q(h'(\beta_i)),$$

$$h'(r(\beta_i)) = h'(d_i) = h(d_i) = r(m_i) = r(h'(\beta_i)).$$

Thus h' is a homomorphism, and every homomorphisms in $\mathrm{Hom}(\mathbf{Q}_i, \mathbf{M})$ extends to $\mathrm{Hom}(\mathbf{Q}'_i, \mathbf{M})$.

The properties of the Lemma 4.1.1 hold because $\wedge$, $\vee$ and $\mathbf{0}$ are homomorphism; all Q_i's are subalgebras and are closed under p, q, r and $\bar{0}$; Q_i is a k-ary relation on M; β be an element of $M^k \setminus Q_i$, where $Q'_i \supseteq Q_i \cup \{\beta\}$ and $Q_i \leq Q'_i \leq \mathbf{M}^k$ are all algebras; and there is an embedding ι of $\mathrm{Hom}(\mathbf{Q}_i, \mathbf{M})$ into $\mathrm{Hom}(\mathbf{Q}'_i, \mathbf{M})$.

Therefore, for $1 \leq i \leq 13$ $(i \neq 2)$, $\mathrm{Hom}(\mathbf{Q}_i, \mathbf{M})$ embeds into $\mathrm{Hom}(\mathbf{Q}'_i, \mathbf{M})$ in a way that satisfies the Lemma 4.1.1. $\square$

As Q'_2 is strictly larger $Q_2 \cup \{\beta_2\}$, a slightly different argument is needed for Q_2.

Lemma 4.1.3. *Let Q_2, Q'_2 and β_2 be defined as in Chapter 3. There is an embedding ι of $\mathrm{Hom}(\mathbf{Q}_2, \mathbf{M})$ into $\mathrm{Hom}(\mathbf{Q}'_2, \mathbf{M})$ satisfying the properties of Lemma 4.1.1.*

Proof. First we determine that $\mathrm{Hom}(\mathbf{Q}_2, \mathbf{M}) = \{\bar{0}, \pi_1, \pi_2, \pi_3 \wedge \pi_4, \pi_3, \pi_3 \vee \pi_4, \pi_4\}$, illustrated in Table 4.3 on the following page.

We can do this by direct calculation (or by using Lemma 4.2.1 and Lemma 4.2.2). It is clear that each of these extends to $D(\mathbf{Q}'_2)$. Thus every homomorphism in $\mathrm{Hom}(\mathbf{Q}_2, \mathbf{M})$ embeds into $\mathrm{Hom}(\mathbf{Q}'_2, \mathbf{M})$ in a way that satisfies the Lemma 4.1.1. $\square$

Now we define the maps $\alpha_i : D(\mathbf{Q}_i) \to M$. Recall that Q_i and Q'_i are subuniverses of $\mathbf{M}^k$ as defined in Chapter 3. Q_i values are given in the fourth column of the Table 3.2 on page 21 and β_i is given in the fifth column of the Table 3.2 on page 21. Recall that for h in $\mathrm{Hom}(\mathbf{Q}_i, \mathbf{M})$ there

	$\bar{0}$	π_1	π_2	π_3	π_4	$\pi_3 \wedge \pi_4$	$\pi_3 \vee \pi_4$
0000	0	0	0	0	0	0	0
0001	0	0	0	0	1	0	1
0010	0	0	0	1	0	0	1
0011	0	0	0	1	1	1	1
0111	0	0	1	1	1	1	1
1111	0	1	1	1	1	1	1

Table 4.3: Homomorphisms of Q_2 on $\{0,1\}$-elements.

is a unique h' in $\mathrm{Hom}(\mathbf{Q}_i', \mathbf{M})$ such that $h = h' \restriction_{Q_i}$. From Lemma 4.1.1 we see the non-evaluation morphism α_i with respect to $\langle M, \wedge, \vee, \mathbf{0}, \mathscr{T} \rangle$ for $\mathbf{Q}_i \leq \mathbf{Q}_i'$ is given by:

$$\alpha_i(h) = e_{\beta_i}(h') \quad \text{for} \quad 1 \leq i \leq 13.$$

Note that:

- α_i is "evaluate at β_i", an element in $Q_i' \setminus Q_i$;
- α_i does not respect Q_i;
- α_i respects $\wedge$, $\vee$, $\mathbf{0}$.

Lemma 4.1.4. *For* $1 \leq i \leq 13$*, we have* $\operatorname{range} \alpha_i \supseteq \{\pi_j(\beta_i) \mid 1 \leq j \leq n\}$.

Proof. We have $\alpha_i(\pi_j) = \pi_j(\beta_i)$, and $\beta_i \in \operatorname{range}(\alpha_i)^k$, where β_i is k-ary and we are done. □

4.2 When does the Map α_i Respect Q_j for $j \neq i$?

In this section, we verify whether or not the map $\alpha_i : D(\mathbf{Q}_i) \to M$ respects all of the relations $Q_j \in Q \setminus \{Q_i\}$. First recall that the morphism $\alpha_i : D(\mathbf{Q}_i) \to M$ respects the relation Q_j of arity k, means if $(h_1, \ldots, h_k) \in Q_i^k$ is such that $(h_1, \ldots, h_k) \in Q_j^{D(\mathbf{Q}_i)}$ then $(\alpha_i(h_1), \ldots, \alpha_i(h_k)) \in Q_j^{\mathbf{M}}$. Thus to check that α_i preserves a k-ary relation R, we need to know $\mathrm{Hom}(\mathbf{Q}_i, \mathbf{M}) = D(Q_i)$. Second recall that a subset S of a partial ordered set is an up-set if $x \in S$ and $y \geq x$ implies $y \in S$. We define

$$S_{01}(X) := X \cap \{0,1\}^s \quad \text{for} \quad X \subseteq M^s.$$

Lemma 4.2.1. *Let $\mathbf{A} \leq \mathbf{M}^s$ be an algebra with the property that for all s, and t in $S_{01}(A)$ where $s < t$ there is an element $u \in A$ such that $s = p(u)$ and $t = q(u)$. Then for $h \in D(\mathbf{A})$ we have $S_{01}(h^{-1}(1))$ is an up-set of $S_{01}(A)$.*

Proof. Let $s \in S_{01}(h^{-1}(1))$ and $t \in S_{01}(A)$ be elements such that $s < t$. Pick u as in the hypothesis. We have $1 = h(s) = h(p(u)) = p(h(u)) \leq q(h(u)) = h(q(u)) = h(t)$, so $t \in S_{01}(h^{-1}(1))$, and $S_{01}(h^{-1}(1))$ is an up-set. □

Corollary 4.2.1. *For $\mathbf{A} \in Q \cup \{\mathbf{Q}_i' \mid 1 \leq i \leq 13\}$, and $h \in D(\mathbf{A})$ we have $S_{01}(h^{-1}(1))$ is an up-set of $S_{01}(A)$.*

Proof. This follows from Lemma 4.2.1 because if $\pi_j(s) \leq \pi_k(s)$ and $\pi_j(t) \leq \pi_k(t)$ then $\pi_j(u) \leq \pi_k(u)$ for $s \in S_{01}(h^{-1}(1))$, $t \in S_{01}(A)$ and $u \in A$. □

Lemma 4.2.2. *If $h_1, h_2 \in \mathrm{Hom}(\mathbf{Q}_i, \mathbf{M})$ and $S_{01}(h_1^{-1}(1)) = S_{01}(h_2^{-1}(1))$ then $h_1 = h_2$.*

Proof. For $a \in Q_i$, we have $\langle p(a), q(a), r(a) \rangle \in \{0,1\}^s$. Since $h_1, h_2 \in \mathrm{Hom}(\mathbf{Q}_i, \mathbf{M})$ and $S_{01}(h_1^{-1}(1)) = S_{01}(h_2^{-1}(1))$ we have $h_2(p(a)) = h_1(p(a))$, $h_2(q(a)) = h_1(q(a))$, and $h_2(r(a)) =$

$h_1(r(a))$ or, we have $p(h_2(a)) = p(h_1(a))$, $q(h_2(a)) = q(h_1(a))$, and $r(h_2(a)) = r(h_1(a))$. From this we have $h_2(a) = h_1(a)$ (because rows of the Table 3.1 on page 20 are unique); that means $h_1 = h_2$. $\square$

The previous lemmas allow us to compute $\mathrm{Hom}(\mathbf{Q}_i, \mathbf{M})$ by up-sets of $S_{01}(Q_i) := Q_i \cap \{0,1\}^s$.

Before launching into the details of the proof, let us examine the logical structure of what we wish to show. Recall that α_i is a map from $D(\mathbf{Q}_i)$ to M. To say that α_i respects Q_j, where Q_j is k-ary, means that for all $\hat{h} = \langle h_1, \ldots, h_k \rangle \in D(\mathbf{Q}_i)^k$ we have that $\hat{h}$ in Q_j implies $\alpha(\hat{h}) := \langle \alpha(h_1), \ldots, \alpha(h_k) \rangle$ in Q_j, or contrapositively if $\alpha(\hat{h})$ is not in Q_j then $\hat{h}$ is not in Q_j. There are two ways to have $\alpha(\hat{h})$ not in Q_j :

1. $\alpha(\hat{h}) \in M^k \setminus Q'_j$,
2. $\alpha(\hat{h}) \in Q'_j \setminus Q_j$.

For $j \neq 2$, $\alpha(\hat{h}) \in Q'_j \setminus Q_j$ means $\alpha(\hat{h}) = \beta_j$. We first deal with $\alpha(\hat{h}) \in M^k \setminus Q'_j$. For the rest of this chapter we look at $\alpha(\hat{h}) \in Q'_j \setminus Q_j$.

Lemma 4.2.3. *For $i \neq j$, and Q_j a k-ary relation, we have that for all $\langle h_1, \ldots, h_k \rangle \in D(\mathbf{Q}_i)^k$*

$$\textit{if } \langle \alpha_i(h_1), \ldots, \alpha_i(h_k) \rangle \in M^k \setminus Q'_j \textit{ then } \langle h_1, \ldots, h_k \rangle \textit{ is not in } Q_j.$$

Proof. We prove the contrapositive. Suppose that $\langle h_1, \ldots, h_k \rangle$ in Q_j. Clearly $\langle h_1, \ldots, h_k \rangle$ in Q'_j. Then, because ι is order-preserving, $\langle (\iota h_1), \ldots, (\iota h_k) \rangle$ in Q'_j, and, in particular,

$$\langle (\iota h_1)(\beta_i), \ldots, (\iota h_k)(\beta_i) \rangle = \langle \alpha_i(h_1), \ldots, \alpha_i(h_k) \rangle \in Q'_j. \quad \square$$

The following lemma is helpful to prove that the map α_i respects Q_j for $j \neq i$ and $j \neq 2$.

Lemma 4.2.4. *For $i \neq j$, $j \neq 2$, and Q_j a k-ary relation, if* $\operatorname{range}\alpha_j \nsubseteq \operatorname{range}\alpha_i$ *we have that for all* $\langle h_1,\ldots,h_k\rangle \in D(\mathbf{Q}_i)^k$

$$\textit{if } \langle \alpha_i(h_1),\ldots,\alpha_i(h_k)\rangle = \beta_j \textit{ then } \langle h_1,\ldots,h_k\rangle \textit{ is not in } Q_j.$$

That is, α_i respects Q_j.

Proof. This follows because for all $\langle h_1,\ldots,h_k\rangle \in D(\mathbf{Q}_i)^k$ we have $\langle \alpha_i(h_1),\ldots,\alpha_i(h_k)\rangle \in (\operatorname{range}\alpha_i)^k$, but $\beta_j \notin (\operatorname{range}\alpha_i)^k$. □

Now we start to verify whether or not the map α_i respects all of the relations Q_j for $j \neq i$.

Lemma 4.2.5. *The map $\alpha_1 : D(\mathbf{Q}_1) \to M$ respects Q_j for $j \neq 1$.*

Proof. Note that Q_1 is defined in Table 3.2 on page 21 as $\{\langle Y_1,Y_2\rangle \in M^2 : Y_1 \leq Y_2\} \setminus \{\langle 0,7\rangle\}$. We now compute $\operatorname{Hom}(\mathbf{Q}_1,\mathbf{M})$. To do this, we make an up-sets table. To get all up-sets we first draw the lattice diagram of $S_{01}(Q_1)$. (See Figure 4.1.) From the lattice diagram we find the up-sets of

$$\begin{tikzcd} q_1^{(2)} := 11 \arrow[d, dash] \\ q_1^{(1)} := 01 \arrow[d, dash] \\ q_1^{(0)} := 00 \end{tikzcd}$$

Figure 4.1: The Lattice Diagram for $S_{01}(Q_1)$.

$S_{01}(h^{-1}(1))$, the corresponding $h \in \operatorname{Hom}(\mathbf{Q}_1,\mathbf{M})$, and $\alpha_1(h)$ which are given by Table 4.4 on the following page.

$S_{01}(h^{-1}(1))$	$h \in \mathrm{Hom}(\mathbf{Q}_1, \mathbf{M})$	$\alpha_1(h)$
$\emptyset$	$h_1^{(0)} = \bar{0}$	0
$\{q_1^{(2)}\}$	$h_1^{(1)} = \pi_1$	0
$\{q_1^{(2)}, q_1^{(1)}\}$	$h_1^{(2)} = \pi_2$	7

Table 4.4: Up-sets table of $S_{01}(Q_1)$

We can calculate that $\mathrm{Hom}(Q_1, \mathbf{M}) = \{h_1^{(0)}, h_1^{(1)}, h_1^{(2)}\}$, where $h_1^{(0)} = \bar{0}$, $h_1^{(1)} = \pi_1$, and $h_1^{(2)} = \pi_2$, given in Table 4.5.

	$h_1^{(0)} = \bar{0}$	$h_1^{(1)} = \pi_1$	$h_1^{(2)} = \pi_2$
00	0	0	0
01	0	0	1
11	0	1	1

Table 4.5: Homomorphism table of $S_{01}(Q_1)$

We have $\alpha_1 : \mathrm{Hom}(\mathbf{Q}_1, \mathbf{M}) \to M$ is defined by

$$\alpha_1(h) = (\imath h)(\beta_1), \text{ where } \beta_1 = \langle 0, 7 \rangle.$$

Suppose that α_1 does not respect R, a k-ary relation. Then there exists

$$\{h_1, h_2, \dots, h_k\} \subseteq \mathrm{Hom}(\mathbf{Q}_1, \mathbf{M})$$

such that $\langle h_1, h_2, \dots, h_k \rangle \in R$, but $\langle \alpha_1(h_1), \alpha_1(h_2), \dots, \alpha_1(h_k) \rangle \notin R$. From the up-set table, for $h \in \mathrm{Hom}(\mathbf{Q}_1, \mathbf{M})$ we have $\alpha_1(h) \in \{0, 0, 7\}$.

Firstly, we prove that α_1 respects Q_j for $j \neq 1, 2$. We have $\mathrm{range}\,\alpha_1 = \{0, 7\}$, which by Lemma 4.1.4 does not contain $\mathrm{range}\,\alpha_j$ for $j \geq 3$, so by Lemma 4.2.4, the map α_1 respects Q_j for $j \geq 3$.

Secondly, we prove that α_1 respects Q_2. Recall that Q_2 is defined in Table 3.2 on page 21. Suppose $\langle h_1, h_2, h_3, h_4 \rangle \in \mathrm{Hom}(\mathbf{Q}_1, \mathbf{M})^4$ where $h_1 \leq h_2 \leq h_3$ and $h_1 \leq h_4$ such that $\langle \alpha_1(h_1), \ldots, \alpha_1(h_4) \rangle \in Q_2' \setminus Q_2$, here $(h_1, h_4) \in \{h_1^{(0)}, h_1^{(1)}\}$, $h_2 = h_1^{(2)}$, and $h_3 = h_1^{(2)}$. As range $\alpha_1 = \{0, 7\}$ this means $\langle \alpha_1(h_1), \ldots, \alpha_1(h_4) \rangle = \langle 0, 7, 7, 0 \rangle$. Thus $\langle h_1, h_2, h_3, h_4 \rangle (0, 3) = \langle 0, 3, 3, 0 \rangle \notin Q_2$.
So $\langle h_1, h_2, h_3, h_4 \rangle \notin Q_2$. Thus α_1 respects Q_2.

Hence the map α_1 respects Q_j for $j \neq 1$. □

The above Lemma is looked at in detail. The following Lemmas are simplified.

Lemma 4.2.6. *The map $\alpha_2 : D(\mathbf{Q}_2) \to M$ respects Q_j for $j \neq 2$.*

Proof. Note that Q_2 is defined in Table 3.2 on page 21. By drawing the lattice diagram of $S_{01}(Q_2)$ (See Figure 4.2) we determine the up-sets of $S_{01}(h^{-1}(1))$, the corresponding $h \in \mathrm{Hom}(\mathbf{Q}_2, \mathbf{M})$, and $\alpha_2(h)$. See Table 4.6.

$S_{01}(h^{-1}(1))$	$h \in \mathrm{Hom}(\mathbf{Q}_2, \mathbf{M})$	$\alpha_2(h)$
$\emptyset$	$h_2^{(0)} = \bar{0}$	0
$\{q_2^{(5)}\}$	$h_2^{(1)} = \pi_1$	0
$\{q_2^{(5)}, q_2^{(4)}\}$	$h_2^{(2)} = \pi_2$	1
$\{q_2^{(5)}, q_2^{(4)}, q_2^{(3)}\}$	$h_2^{(3)} = \pi_3 \wedge \pi_4$	0
$\{q_2^{(5)}, q_2^{(4)}, q_2^{(3)}, q_2^{(2)}\}$	$h_2^{(4)} = \pi_3$	1
$\{q_2^{(5)}, q_2^{(4)}, q_2^{(3)}, q_2^{(1)}\}$	$h_2^{(5)} = \pi_4$	0
$\{q_2^{(5)}, q_2^{(4)}, q_2^{(3)}, q_2^{(2)}, q_2^{(1)}\}$	$h_2^{(6)} = \pi_3 \vee \pi_4$	1

Table 4.6: Up-sets table of $S_{01}(Q_2)$

We can calculate that $\mathrm{Hom}(\mathbf{Q}_2, \mathbf{M}) = \{h_2^{(0)}, h_2^{(1)}, \ldots, h_2^{(6)}\}$, where $h_2^{(0)} = \bar{0}$, $h_2^{(1)} = \pi_1$, $h_2^{(2)} = \pi_2$,

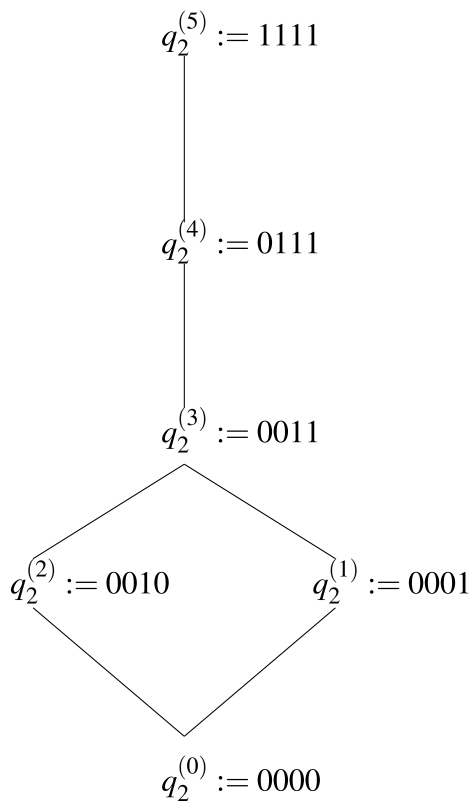

Figure 4.2: The Lattice Diagram for $S_{01}(Q_2)$.

$h_2^{(3)} = \pi_3 \wedge \pi_4$, $h_2^{(4)} = \pi_3$, $h_2^{(5)} = \pi_4$, and $h_2^{(6)} = \pi_3 \vee \pi_4$, defined in Table 4.7.

	$h_2^{(0)}$	$h_2^{(1)}$	$h_2^{(2)}$	$h_2^{(3)}$	$h_2^{(4)}$	$h_2^{(5)}$	$h_2^{(6)}$
0000	0	0	0	0	0	0	0
0001	0	0	0	0	0	1	1
0010	0	0	0	0	1	0	1
0011	0	0	0	1	1	1	1
0111	0	0	1	1	1	1	1
1111	0	1	1	1	1	1	1

Table 4.7: Homomorphism table of $S_{01}(Q_2)$

We have $\alpha_2 : \mathrm{Hom}(\mathbf{Q}_2, \mathbf{M}) \to M$ is defined by

$$\alpha_2(h) = (\imath h)(\beta_2), \text{ where } \beta_2 = \langle 0, 1, 1, 0 \rangle.$$

From the up-set table, for $h \in \mathrm{Hom}(\mathbf{Q}_2, \mathbf{M})$ we have $\alpha_2(h) \in \{0, 0, 1, 0, 1, 0, 1\}$, and from Table 3.2 we get $\beta_j \notin \{0, 1\}^k$ for $j \neq 2$. By Lemma 4.2.4 the map α_2 respects Q_j for $j \neq 2$. $\square$

Lemma 4.2.7. *The map $\alpha_4 : D(\mathbf{Q}_4) \to M$ respects Q_j for $j \neq 4$.*

Proof. Note that $Q_4 = \{\langle Y_1, Y_2, Y_3, Y_4 \rangle \in M^4 : Y_1 \leq Y_2 \leq Y_3 \text{ and } Y_1 \leq Y_4\} \setminus \{\langle 0, 3, 3, 0 \rangle\}$, defined in Table 3.2. Now we draw the lattice diagram of $S_{01}(Q_4)$. (See Figure 4.3 on the following page.) From the lattice diagram we find the up-sets of $S_{01}(h^{-1}(1))$, the corresponding $h \in \mathrm{Hom}(\mathbf{Q}_4, \mathbf{M})$, and $\alpha_4(h)$. See Table 4.8 on page 60.

We can determine that $\mathrm{Hom}(\mathbf{Q}_4, \mathbf{M}) = \{h_4^{(0)}, h_4^{(1)}, \ldots, h_4^{(9)}\}$, where $h_4^{(0)} = \bar{0}$, $h_4^{(1)} = \pi_1$, $h_4^{(2)} = \pi_2 \wedge \pi_4$, $h_4^{(3)} = \pi_2$, $h_4^{(4)} = \pi_3 \wedge \pi_4$, $h_4^{(5)} = \pi_2 \vee (\pi_3 \wedge \pi_4)$, $h_4^{(6)} = \pi_4$, $h_4^{(7)} = \pi_3$, $h_4^{(8)} = \pi_2 \vee \pi_4$, and $h_4^{(9)} = \pi_3 \vee \pi_4$, defined in Table 4.9 on page 60.

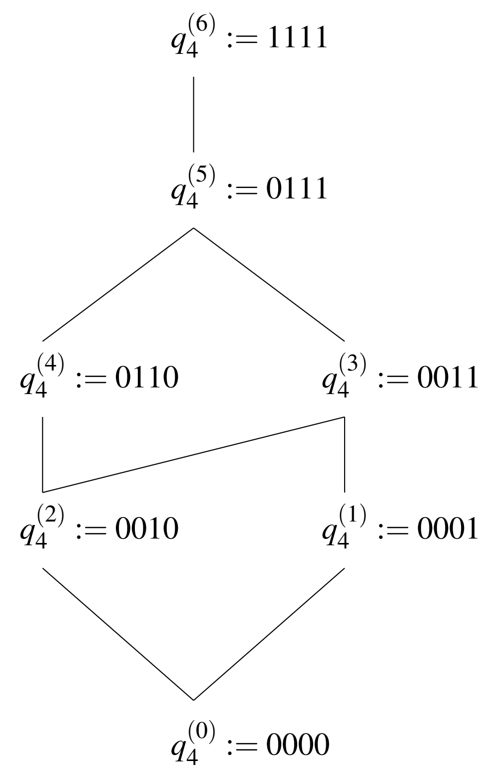

Figure 4.3: The Lattice Diagram for $S_{01}(Q_4)$.

$S_{01}(h^{-1}(1))$	$h \in \mathrm{Hom}(\mathbf{Q}_4, \mathbf{M})$	$\alpha_4(h)$
$\emptyset$	$h_4^{(0)} = \bar{0}$	0
$\{q_4^{(6)}\}$	$h_4^{(1)} = \pi_1$	0
$\{q_4^{(6)}, q_4^{(5)}\}$	$h_4^{(2)} = \pi_2 \wedge \pi_4$	0
$\{q_4^{(6)}, q_4^{(5)}, q_4^{(4)}\}$	$h_4^{(3)} = \pi_2$	3
$\{q_4^{(6)}, q_4^{(5)}, q_4^{(3)}\}$	$h_4^{(4)} = \pi_3 \wedge \pi_4$	0
$\{q_4^{(6)}, q_4^{(5)}, q_4^{(4)}, q_4^{(3)}\}$	$h_4^{(5)} = \pi_2 \vee (\pi_3 \wedge \pi_4)$	3
$\{q_4^{(6)}, q_4^{(5)}, q_4^{(4)}, q_4^{(1)}\}$	$h_4^{(6)} = \pi_4$	0
$\{q_4^{(6)}, q_4^{(5)}, q_4^{(4)}, q_4^{(3)}, q_4^{(2)}\}$	$h_4^{(7)} = \pi_3$	3
$\{q_4^{(6)}, q_4^{(5)}, q_4^{(4)}, q_4^{(3)}, q_4^{(1)}\}$	$h_4^{(8)} = \pi_2 \vee \pi_4$	3
$\{q_4^{(6)}, q_4^{(5)}, q_4^{(4)}, q_4^{(3)}, q_4^{(2)}, q_4^{(1)}\}$	$h_4^{(9)} = \pi_3 \vee \pi_4$	3

Table 4.8: Up-sets table of $S_{01}(Q_4)$

	$h_4^{(0)}$	$h_4^{(1)}$	$h_4^{(2)}$	$h_4^{(3)}$	$h_4^{(4)}$	$h_4^{(5)}$	$h_4^{(6)}$	$h_4^{(7)}$	$h_4^{(8)}$	$h_4^{(9)}$
0000	0	0	0	0	0	0	0	0	0	0
0001	0	0	0	0	0	0	1	0	1	1
0010	0	0	0	0	0	0	0	1	0	1
0011	0	0	0	0	1	1	1	1	1	1
0110	0	0	0	1	0	1	0	1	1	1
0111	0	0	1	1	1	1	1	1	1	1
1111	0	1	1	1	1	1	1	1	1	1

Table 4.9: Homomorphism table of $S_{01}(Q_4)$

We have $\alpha_4 : \mathrm{Hom}(\mathbf{Q}_4, \mathbf{M}) \to M$ is defined by

$$\alpha_4(h) = (\imath h)(\beta_4), \text{ where } \beta_4 = \langle 0,3,3,0 \rangle.$$

From the up-set table, we have $\alpha_4(h) \in \{0,0,0,3,0,3,0,3,3,3\}$.

Firstly, we prove that α_4 respects Q_j for $j \neq 2,4$. We have range $\alpha_4 = \{0,3\}$, which by Lemma 4.1.4 does not contain range α_j for $j \geq 3$, so by Lemma 4.2.4, the map α_4 respects Q_j for $j \geq 3$.

Secondly, we prove that α_4 respects Q_2. Recall that Q_2 is defined in Table 3.2 on page 21. Suppose $\langle h_1, h_2, h_3, h_4 \rangle \in \mathrm{Hom}(\mathbf{Q}_4, \mathbf{M})^4$ where $h_1 \leq h_2 \leq h_3$ and $h_1 \leq h_4$ such that $\langle \alpha_4(h_1), \ldots, \alpha_4(h_4) \rangle \in Q_2' \setminus Q_2$, here $(h_1, h_4) \in \{h_4^{(0)}, h_4^{(1)}, h_4^{(2)}, h_4^{(4)}, h_4^{(6)}\}$, and $(h_2, h_3) \in \{h_4^{(3)}, h_4^{(5)}, h_4^{(7)}, h_4^{(8)}, h_4^{(9)}\}$. As range $\alpha_3 = \{0,3\}$ this means $\langle \alpha_4(h_1), \ldots, \alpha_4(h_4) \rangle = \langle 0,3,3,0 \rangle$. Thus $\langle h_1, h_2, h_3, h_4 \rangle (0,7,7,0) = \langle 0,7,7,0 \rangle \notin Q_2$. So $\langle h_1, h_2, h_3, h_4 \rangle \notin Q_2$. Thus α_4 respects Q_2.

Hence the map $\alpha_4 : D(\mathbf{Q}_4) \to M$ respects Q_j for $j \neq 4$. $\square$

Lemma 4.2.8. *If* $7 \in \mathrm{range}\, \alpha_i$ *and* $7 \notin \mathrm{range}\, \alpha_j$ *and* $j \neq 2$*, then* α_i *respects* Q_j*.*

Proof. Let a be an element of Q_i gotten by replacing a 7 in β_i with a 3. Suppose that $\eta = \bigwedge_{k \in K} \pi_k$. If $\eta(\beta_i) \leq 3$ then $\eta(\beta_i) = \eta(a)$ as $3 \wedge x = x$ for $x \leq 3$. Now we know from the computation of $D(Q_i)$ that $h \in D(Q_i)$ can be written as the join of meets of projections. If $h(\beta_i) \leq 3$ then each of the joinands is less than or equal to 3, and by the preceding argument $h(\beta_i) = h(a)$.

As $7 \notin \mathrm{range}(\alpha_j)$, if there is a k-tuple of homomorphisms in $D(Q_i)$ such that $\langle h_1, \ldots, h_k \rangle (\beta_i) = \beta_j$ then $\langle h_1, \ldots, h_k \rangle (a) = \beta_j$, which proves that α_i respects Q_j. $\square$

Lemma 4.2.9. *For* $i \neq 2$*, if there is a four tuple* $\hat{h} \in D(Q_i)^4$ *such that* $\alpha(\hat{h}) \in Q_2' \setminus Q_2$ *then there is an element* $a \in \mathbf{Sg}(\{\beta_i\}) \cap Q_i$ *such that* $\hat{h}(a) = \langle 0,1,1,0 \rangle$*.*

Proof. Suppose that $\alpha(\hat{h}) \in Q_2' \setminus Q_2$. That means there is a term operation f such that $f(\alpha(\hat{h})) = \langle 0,1,1,0\rangle$. Set $a = r(f(\beta_i))$. Clearly $a \in \mathbf{Sg}(\{\beta_i\})$. As $a \neq \beta_i$ we have $a \in Q_i$ (Lemma 3.2.1). Now $\hat{h}(a) = \hat{h}(r(f(a)) = r(f(\alpha_i(\hat{h}))) = r(\langle 0,1,1,0\rangle) = \langle 0,1,1,0\rangle$. □

Corollary 4.2.2. *For $i \neq 2$, the morphism α_i respects the relation Q_2.*

Now we can prove the following theorem. It implies that removing any Q_i with $i \leq 11$ results in an alter ego that is not dualizing.

Theorem 4.2.1. *The map $\alpha_i : D(\mathbf{Q}_i) \to M$ respects Q_j for $j \neq i$, with the following exceptions.*

1. *α_{12} does not respect Q_8 or Q_9; and*

2. *α_{13} does not respect Q_{10} or Q_{11}.*

Proof. From Lemma 4.2.5, the map α_1 respects Q_j, for $j \neq 1$. From Lemma 4.2.6, the map α_2 respects Q_j, for $j \neq 2$. From Lemma 4.2.7, the map α_4 respects Q_j, for $j \neq 4$.

The remainder of the calculations are gathered in Tables 4.10 on page 64, 4.11 on page 65, and 4.12 on page 66. In these tables, in row α_i and column Q_j a string of digits (for instance 00070 in α_7 row and Q_1 column) represents an element a_i of Q_i and means the following. Suppose that $Q_j \leq \mathbf{M}^s$, and that $\hat{h}$ is an s-tuple of homomorphisms from $D(\alpha_i)$ such that $\alpha_i(\hat{h}) = \beta_j$ where β_j is the element excluded from Q_j. Then $\hat{h}(a_i) = \beta_j$ also.

Such a claim is verified by computing $D(Q_i)$, which is done by using Lemmas 4.2.1 and 4.2.2. The lattice diagrams for the up-sets of $S_{01}(Q_i)$ are shown in Figure 4.4 on page 67, Figure 4.5 on page 68, Figure 4.6 on page 69, Figure 4.7 on page 70, Figure 4.8 on page 71, Figure 4.9 on page 72.

In the four cases where α_i does not respect Q_j $(i \neq j)$, the calculations were verified by computer analysis by David Casperson. $\square$

Now we prove the following theorem:

Theorem 4.2.2. *The alter ego* $\langle \mathbb{M}, \wedge, \vee, \mathbf{0}, Q, \mathscr{T} \rangle$ *is a dualizing structure. The set* Q *is minimal with respect to the the alter egos that extend* $\langle M, \{\wedge, \vee, \mathbf{0}\}, \{Q_{12}, Q_{13}\}, \mathscr{T} \rangle$.

Proof. In Chapter 3 we showed that the algebra $\mathbf{M}$ is dualized by the alter-ego $\langle \mathbb{M}, \wedge, \vee, \mathbf{0}, Q, \mathscr{T} \rangle$. In this Chapter we proved that each relation Q_j in Q except Q_{12} and Q_{13} is a necessary part of a dualizing structure for the five-element algebra by finding a specific morphism which is an non-evaluation morphism, when Q_j is dropped. Thus the set Q is minimal with respect to the the alter egos that extend $\langle M, \{\wedge, \vee, \mathbf{0}\}, \{Q_{12}, Q_{13}\}, \mathscr{T} \rangle$. $\square$

From Theorem 4.2.2, the set Q is minimal with respect to the the alter egos that extend $\langle M, \{\wedge, \vee, \mathbf{0}\}, \{Q_{12}, Q_{13}\}, \mathscr{T} \rangle$.

	Q_1	Q_2	Q_3	Q_4	Q_5
α_1	No, Lem 4.1.1	Lem 4.2.9	Lem 4.2.8	Lem 4.2.8	Lem 4.2.5
α_2	Lem 4.2.6	No, Lem 4.1.1	Lem 4.2.6	Lem 4.2.6	Lem 4.2.6
α_3	Lem 4.2.4	Lem 4.2.9	No, Lem 4.1.1	Lemm 4.2.4	Lem 4.2.4
α_4	Lem 4.2.7	Lem 4.2.9	Lem 4.2.7	No, Lem 4.1.1	Lem 4.2.7
α_5	00070	Lem 4.2.9	Lem 4.2.8	Lem 4.2.8	No, Lem 4.1.1
α_6	00070	Lem 4.2.9	Lem 4.2.8	Lem 4.2.8	Lem 4.2.4
α_7	00070	Lem 4.2.9	Lem 4.2.8	Lem 4.2.8	Lem 4.2.4
α_8	Lem 4.2.4	Lem 4.2.9	Lem 4.2.4	000330	Lem 4.2.4
α_9	Lem 4.2.4	Lem 4.2.9	022220	000330	Lem 4.2.4
α_{10}	0000070	Lem 4.2.9	Lem 4.2.8	Lem 4.2.8	0111170
α_{11}	0000070	Lem 4.2.9	Lem 4.2.8	Lem 4.2.8	Lem 4.2.4
α_{12}	Lem 4.2.4	Lem 4.2.9	00022022	00033033	Lem 4.2.4
α_{13}	000007000	Lem 4.2.9	Lem 4.2.8	Lem 4.2.8	011117000

See the proof of Theorem 4.2.1 for the meaning of the entries.

Table 4.10: The map α_i respects Q_j for $j \neq i$ (Part I).

	Q_6	Q_7	Q_8	Q_9
α_1	Lem 4.2.5	Lem 4.2.5	Lem 4.2.8	Lem 4.2.8
α_2	Lem 4.2.6	Lem 4.2.6	Lem 4.2.6	Lem 4.2.6
α_3	Lem 4.2.4	Lem 4.2.4	Lem 4.2.4	Lem 4.2.4
α_4	Lem 4.2.7	Lem 4.2.7	Lem 4.2.7	Lem 4.2.7
α_5	Lem 4.2.4	Lem 4.2.4	Lem 4.2.8	Lem 4.2.8
α_6	No, (Lem 4.1.1)	Lem 4.2.4	Lem 4.2.8	Lem 4.2.8
α_7	Lem 4.2.4	No, (Lem 4.1.1)	Lem 4.2.8	Lem 4.2.8
α_8	Lem 4.2.4	Lem 4.2.4	No, (Lem 4.1.1)	Lem 4.2.4
α_9	Lem 4.2.4	Lem 4.2.4	Lem 4.2.4	No, (Lem 4.1.1)
α_{10}	Lem 4.2.4	0003370	Lem 4.2.8	Lem 4.2.8
α_{11}	0222270	0003370	Lem 4.2.8	Lem 4.2.8
α_{12}	Lem 4.2.4	Lem 4.2.4	**No!** (comp)	**No!** (comp)
α_{13}	000227022	000337033	Lem 4.2.8	Lem 4.2.8

See the proof of Theorem 4.2.1 for the meaning of the entries.

Table 4.11: Where α_i respects Q_j (Part II).

	Q_{10}	Q_{11}	Q_{12}	Q_{13}
α_1	Lem 4.2.5	Lem 4.2.5	Lem 4.2.8	Lem 4.2.5
α_2	Lem 4.2.6	Lem 4.2.6	Lem 4.2.6	Lem 4.2.6
α_3	Lem 4.2.4	Lem 4.2.4	Lem 4.2.4	Lem 4.2.4
α_4	Lem 4.2.7	Lem 4.2.7	Lem 4.2.7	Lem 4.2.7
α_5	Lem 4.2.4	Lem 4.2.4	Lem 4.2.8	Lem 4.2.4
α_6	Lem 4.2.4	Lem 4.2.4	Lem 4.2.8	Lem 4.2.4
α_7	Lem 4.2.4	Lem 4.2.4	Lem 4.2.8	Lem 4.2.4
α_8	Lem 4.2.4	Lem 4.2.4	Lem 4.2.4	Lem 4.2.4
α_9	Lem 4.2.4	Lem 4.2.4	Lem 4.2.4	Lem 4.2.4
α_{10}	No, (Lem 4.1.1)	Lem 4.2.4	Lem 4.2.8	Lem 4.2.4
α_{11}	Lem 4.2.4	No, (Lem 4.1.1)	Lem 4.2.8	Lem 4.2.4
α_{12}	Lem 4.2.4	Lem 4.2.4	No, (Lem 4.1.1)	Lem 4.2.4
α_{13}	**No!** (comp)	**No!** (comp)	Lem 4.2.8	No, (Lem 4.1.1)

See the proof of Theorem 4.2.1 for the meaning of the entries.

Table 4.12: Where α_i respects Q_j (Part III).

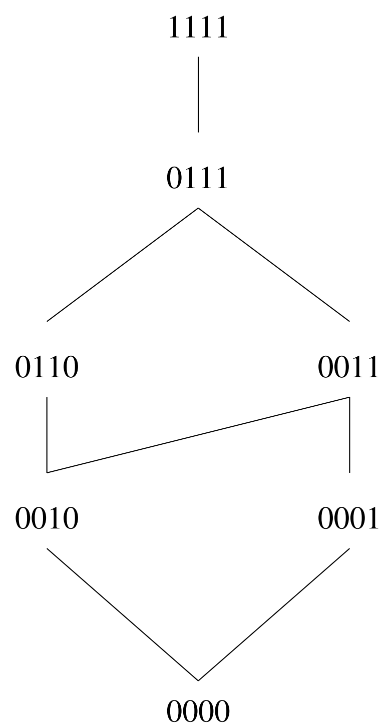

Figure 4.4: The Lattice Diagram for $S_{01}(Q_3)$, and $S_{01}(Q_4)$.

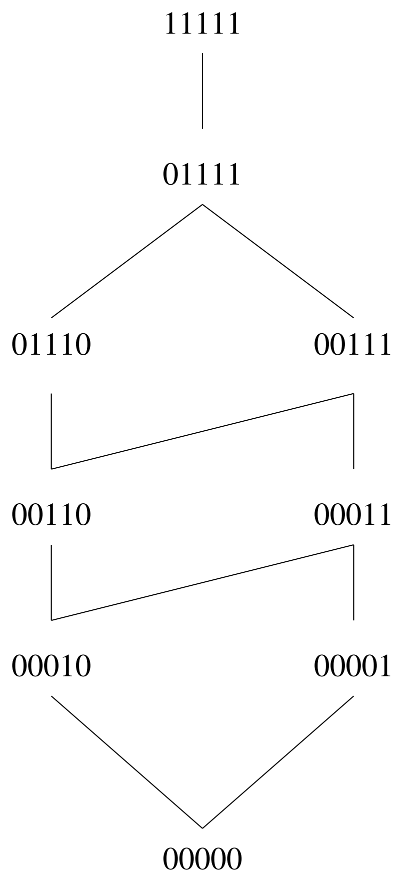

Figure 4.5: The Lattice Diagram for $S_{01}(Q_5)$, $S_{01}(Q_6)$, and $S_{01}(Q_7)$.

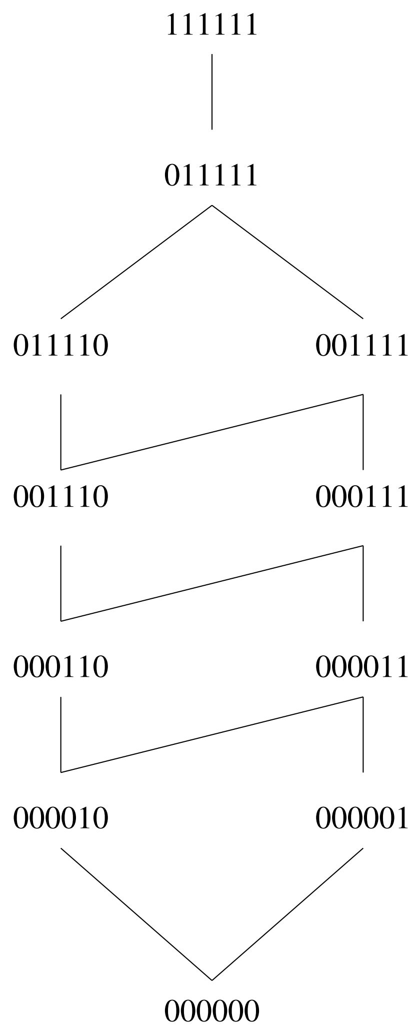

Figure 4.6: The Lattice Diagram for $S_{01}(Q_8)$, and $S_{01}(Q_9)$.

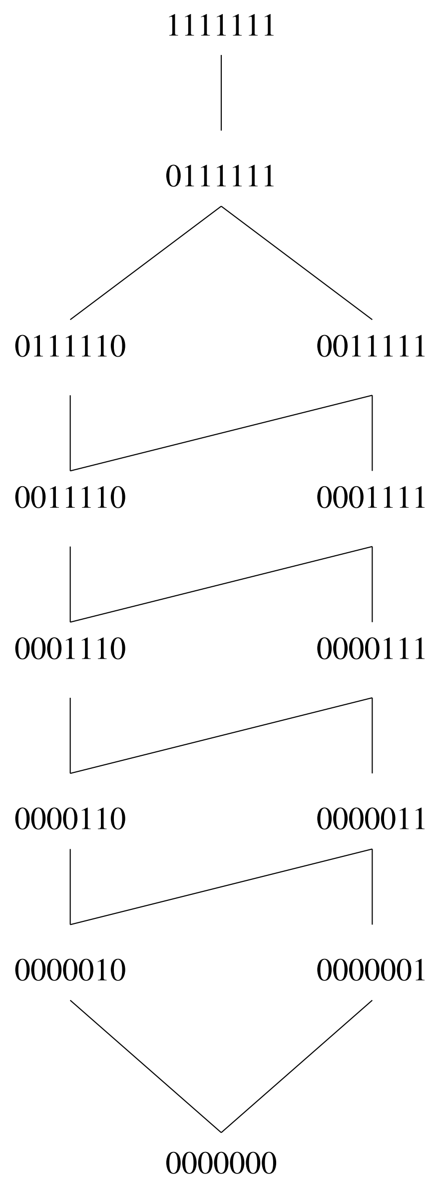

Figure 4.7: The Lattice Diagram for $S_{01}(Q_{10})$, and $S_{01}(Q_{11})$.

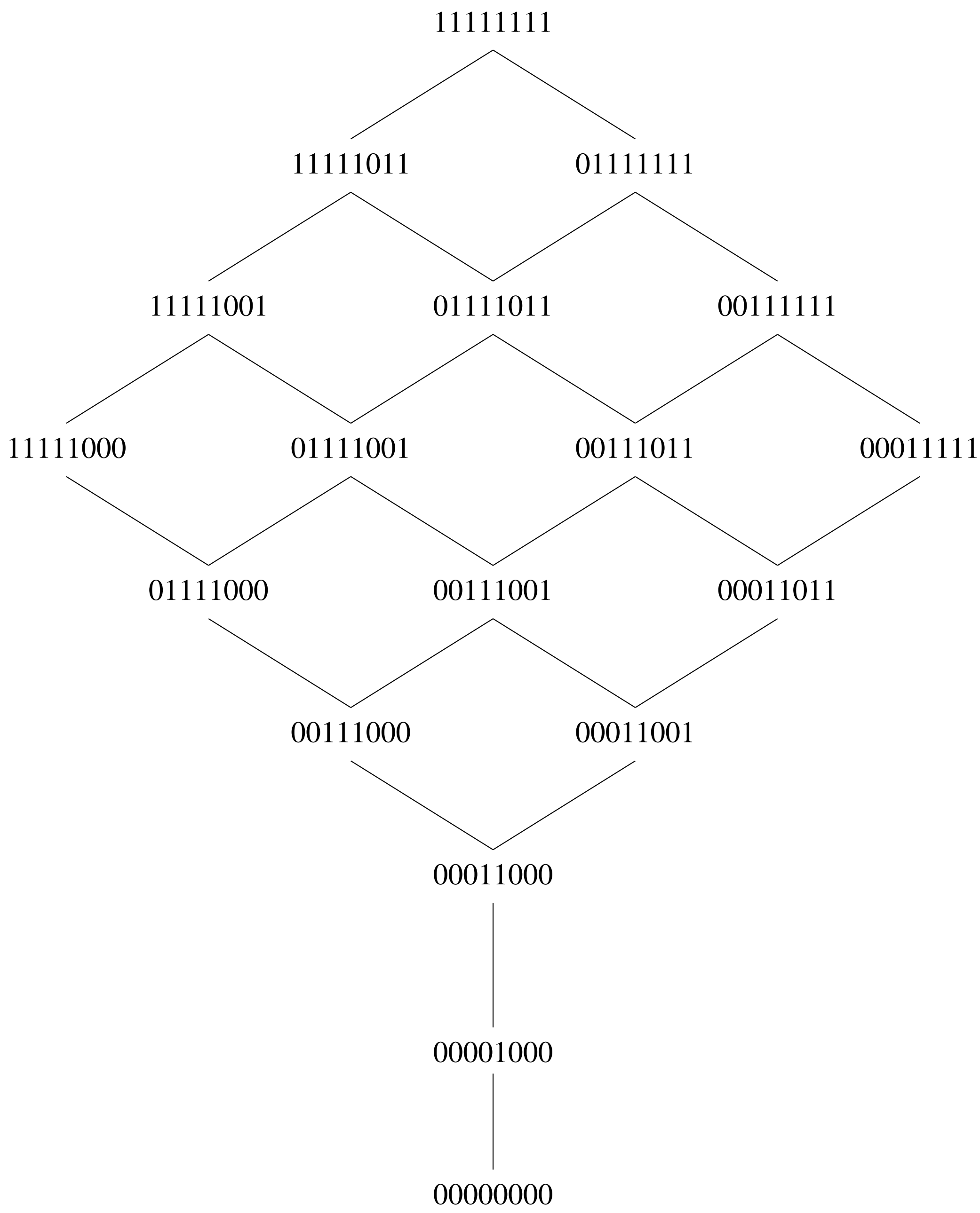

Figure 4.8: The Lattice Diagram for $S_{01}(Q_{12})$.

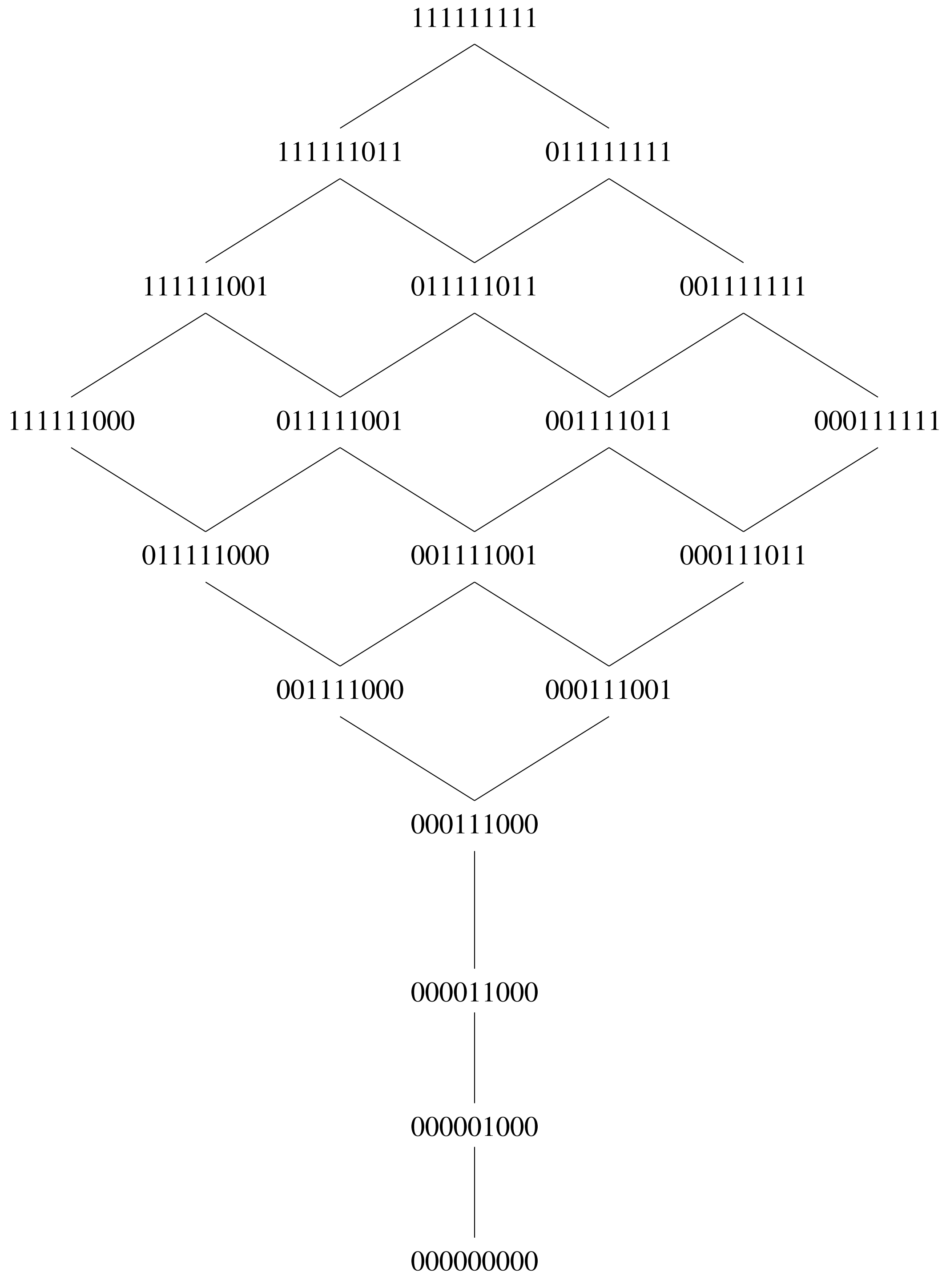

Figure 4.9: The Lattice Diagram for $S_{01}(Q_{13})$.

Chapter 5

Conclusion

5.1 Summary

In this thesis we looked for when an alter ego is a minimal dualizing structure.

To summarize, we covered the required preliminary materials such as notations, definitions, examples, and theorems for algebras, lattices, quasivarieties, topology, and dualizability in Chapter 2. In Chapter 3, we defined a five-element algebra, an alter ego and concluded that our defined alter ego satisfies the interpolation condition relative to the algebra and thus dualizes the algebra. That means, we proved the alter-ego $\mathbb{M} := \langle M, \wedge, \vee, \mathbf{0}, Q, \mathscr{T} \rangle$ dualizes the algebra $\mathbf{M}$. In Chapter 4, we showed that the set of relations is a necessary part of a dualizing structure including $\wedge$, $\vee$, and $\mathbf{0}$ by finding a specific morphism which is an non-evaluation morphism. We proved that the set Q is minimal with respect to the the alter egos that extend $\langle M, \{\wedge, \vee, \mathbf{0}\}, \{Q_{12}, Q_{13}\}, \mathscr{T} \rangle$ (see Theorem 4.2.2).

5.2 Future Research

In this thesis we showed that our defined alter-ego dualizes the five-element algebra and a particular relation of a dualizing structure is a necessary part for the alter ego. In this paper we did not show that $\mathbb{M}$ was minimal in the way that we originally claimed it was, we have some obvious material to include in future research.

For further research, with the current work of this paper, the following questions about unary algebras and their dualizability are:

In this paper we used tables and 13 cases. Can we simplify the cases of Chapter 3 and Chapter 4 of this thesis to determine the dualizability results?

In this thesis we proved that the set Q is minimal with respect to the the alter egos that extend $\langle M, \{\wedge, \vee, \mathbf{0}\}, \{Q_{12}, Q_{13}\}, \mathscr{T}\rangle$. Can we show that the set Q is a minimal dualizing structure?

Moreover, can we describe a minimal alter ego for any finite unary algebra with an underlying lattice structure?

However, we did not look at fully dualizable or strongly dualizable. Can we find that our defined alter-ego fully dualizes or strongly dualizes on the five-element algebra?

Furthermore, for $\{0,1\}$-valued unary algebras with more than five elements, can we find nice conditions for dualizability, full dualizability or not full dualizability, strong dualizability or not strong dualizability?

Following the work of this paper, is it possible to find whether or not any $\{0,1\}$-valued unary algebras are dualizable or fully dualizable? Can we research $\{0,1\}$-valued unary algebras that do not have a constant zero-valued function?

Finally, with the technique of this thesis, can we find that there exists an algebra which is fully dualizable but is not strongly dualizable?

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