STANDARD UNARY ALGEBRAS

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Abstract

This research studies three-element unary algebras to determine which of them are standard.

A quasi-variety generated by an algebra $\mathbb{M}$ is standard if it consists exactly of those algebraic structures being the same type as $\mathbb{M}$ which carry a compatible Boolean topology and are models of the quasi-equational theory of $\mathbb{M}$.

This work shows that two previously unclassified structures, $\mathbb{M}$, are standard.

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Chapter 1

Introduction and Background

1.1 Introduction

In 2003, Clark, Davey, Haviar, Pitkethly and Talukder [3] introduced the idea of *standard topological quasi-varieties.* This idea was motivated by the theory of natural dualities [2] which provides methods for understanding algebraic quasi-varieties, $\mathcal{A}$, whenever they can be represented by a category of structured Boolean spaces. The algebraic quasi-variety $\mathcal{A} := \mathbb{ISP}\,\mathbf{M}$, consists of all isomorphic copies of subalgebras of direct powers of $\mathbf{M}$. The category $\mathcal{A}$ is the smallest class of algebras closed under isomorphisms, subalgebras and products containing the finite algebra $\mathbf{M}$.

A natural duality on $\mathcal{A}$ yields a category called a *topological quasi-variety* of the form $\mathcal{X} := \mathbb{IS}_c\,\mathbb{P}^{+}\mathbb{M}$ where $\mathbb{M}$ is a finite structure such that $\mathbb{M} = \langle M; G, H, R, \tau\rangle$ with operations G, partial operations H, relations R and the discrete topology τ. This category $\mathcal{X}$ consists of all isomorphic copies of topologically closed substructures of non-zero direct powers of $\mathbb{M}$. The topological structure $\mathbb{M}$ has the same underlying set as $\mathbf{M}$ and has the discrete topology. One of the benefits of duality theory is that is that the objects in $\mathcal{X}$ can often be much simpler than their duals in $\mathcal{A}$.

The program of study initiated by Clark, Davey, Haviar, Pitkethly and Talukder is to determine which topological quasi-varieties are standard and which are not. By defining the quasi-equational theory of $\mathcal{X}$, that is, the set of all quasi-equations satisfied by $\mathbb{M}$, a nice axiomatic description of the members of $\mathcal{X}$ is obtained. The quasi-equational theory is denoted $\mathrm{Th}_{qe}(\mathbb{M})$, and $\mathrm{Mod}_\tau\,\mathrm{Th}_{qe}(\mathbb{M})$ is the class of all topological models of the quasi-equational theory of $\mathbb{M}$.

The category $\mathcal{X}$ is a *standard* topological quasi-variety provided it consists exactly of those algebraic structures being the same type as $\mathbf{M}$ which carry a compatible Boolean topology and are models of the quasi-equational theory of $\mathbb{M}$.

The purpose of this research is to determine which topological three-element unary algebras are standard. I was able to show that $\mathbb{M} = \langle\{0,1,2\};\,\mathcal{F};\,\tau\rangle$ where $\mathcal{F} = \{002, 110\}$, and $\mathbb{M}_2 = \langle\{0,1,2\};\mathcal{F};\,\tau\rangle$ where $\mathcal{F} = \{001, 002\}$, are standard. See Theorem 3.1 in Chapter 3 and Theorem 4.1 in Chapter 4 for these results.

1.2 Definitions

The following definitions are taken from [1]

An *algebra* is a set A, together with a collection of operations on A. An *n-ary operation* (or *function*) on A is any function f from A^n to A where n is the *arity* (or *rank*) of f. A *finitary* operation is an n-ary operation, for some n. The image of $\langle a_1 \dots a_n\rangle$ under an n-ary operation f is denoted by $f(a_1 \dots a_n)$. An operation f on A is called a *nullary* operation (or *constant*) if its arity is zero. An operation f on A is *unary*, *binary*, or *ternary* if its arity is $1, 2$, or 3, respectively. An algebra $\mathbf{A}$ is *unary* if all of its operations are unary, and it is *mono-unary*, or a *unar*, if it has just one unary operation.

A *language* (or *type*) of algebras is a set $\mathcal{F}$ of function symbols such that a nonnegative integer n is assigned to each member f of $\mathcal{F}$.

If $\mathcal{F}$ is a language of algebras then an algebra $\mathbf{A}$ of type $\mathcal{F}$ is an ordered pair $\langle A, F \rangle$ where A is a nonempty set and F is a family of finitary operations on A indexed by the language $\mathcal{F}$, such that corresponding to each n-ary function symbol f in $\mathcal{F}$ there is an n-ary operation $f^{\mathbf{A}}$ on A. The set A is called the *universe* or *underlying set* of $\mathbf{A} = \langle A; F \rangle$ and the $f^{\mathbf{A}}$'s are called the *fundamental operations* of $\mathbf{A}$.

For example, a monoid is an algebra $\langle M, \cdot, 1 \rangle$ with a binary and a nullary operation satisfying $x \cdot (y \cdot z) \approx (x \cdot y) \cdot z$, and $x \cdot 1 \approx 1 \cdot x \approx x$ for each $x, y, z \in M$.

Let $\mathbf{A}$ and $\mathbf{B}$ be two algebras of the same type. Then $\mathbf{B}$ is a *subalgebra* of $\mathbf{A}$ if $B \subseteq A$ and every fundamental operation of $\mathbf{B}$ is the restriction of the corresponding operation of $\mathbf{A}$, that is, for each function symbol $f, f^{\mathbf{B}}$ is $f^{\mathbf{A}}$ restricted to B; we write simply $\mathbf{B} \leq \mathbf{A}$. Where the context is clear, we drop the superscript in f^A and write f. A *subuniverse* of $\mathbf{A}$ is a subset B of A which is closed under the fundamental operations of $\mathbf{A}$, that is, if f is a fundamental n-ary operation of $\mathbf{A}$ and $a_1 \dots a_n \in B$, we would require $f(a_1 \dots a_n) \in B$. Thus if $\mathbf{B}$ is a subalgebra of $\mathbf{A}$; then B is a subuniverse of $\mathbf{A}$.

An algebra $\mathbf{A}$ is *locally finite* if, given any set $C \subseteq A$, the subalgebra generated by C is finite. A class $\mathbf{K}$ of algebras is locally finite if every member of $\mathbf{K}$ is locally finite. An example of a locally finite algebra is any algebra with a finite underlying set and finitely many operations.

Let $\mathbf{A}$ and $\mathbf{B}$ be two algebras of the same type $\mathcal{F}$. A mapping $\alpha \colon A \to B$ is a *homomorphism* from $\mathbf{A} \to \mathbf{B}$ if it preserves the operations in $\mathcal{F}$, that is for every n-ary $f \in \mathcal{F}$ and for $a_1, \dots, a_n \in A$ we have

$$\alpha f^{\mathbf{A}}(a_1, \dots, a_n) = f^{\mathbf{B}}(\alpha a_1, \dots, \alpha a_n)$$

Such a mapping $\alpha \colon A \to B$ is an *isomorphism*, that is $\mathbf{A}$ is *isomorphic* to $\mathbf{B}$, if α is one-to-one and onto.

A *variety* is a non-empty collection of algebras of a fixed type that is closed under homomorphic images, subalgebras and products. A *quasi-variety* is a non-empty collection of algebras of a fixed type that is closed under isomorphic images, subalgebras and products.

A *quasi-atomic formula* is an expression which is either an atomic formula, a neg-atomic formula, $\neg\alpha$, or an implication, $\beta_1 \wedge \ldots \wedge \beta_m \implies \alpha$, where $m \geq 1$ and $\beta_1, \ldots, \beta_m, \alpha$ are atomic formulae. A *quasi-equation* is a universally quantified formula of the form $\phi_1 \bigwedge \phi_2 \bigwedge \cdots \bigwedge \phi_m \implies \alpha$, where the ϕ_i are atomic equations. For an algebraic language, the ϕ_i are simply equations.

The *quasi-atomic theory* of $\mathbb{M}$ is denoted by $Th_{qa}(\mathbb{M})$. For unary algebras, where there are no relations, then $Th_{qa}(\mathbb{M})$ is called the *quasi-equational theory* of $\mathbb{M}$ and is denoted $\mathrm{Th}_{qe}(\mathbb{M})$. The quasi-equational theory of a class of a topological structures $\mathbb{M}$ is the set of quasi-equations that hold and are satisfied by $\mathbb{M}$, and $\mathbb{M}$ is a *model* of a set of quasi-equations if it satisfies every quasi-equation in the set.

The quasi-variety $\mathcal{A}$ generated by an algebra $\mathbf{M}$ is simultaneously the class of all algebras of the same type as $\mathbf{M}$ that are models of the quasi-equational theory of $\mathbf{M}$ and is the smallest class of algebras containing $\mathbf{M}$ that is closed under isomorphic copies, subalgebras and products. This quasi-variety is denoted $\mathbb{ISP}(\mathbf{M})$. Note that if $\mathbf{M}$ is finite with a finite language, then $\mathbb{ISP}(\mathbf{M})$ is locally finite.

1.3 Background

An area of on-going research in the theory of natural dualities is the problem of exactly which finite algebras generate a dualizable quasi-variety. Clark and Davey [2] have shown that two-element algebras are dualizable. The dualizability problem for three-element algebras, however, was more complicated but was later solved by Clark, Davey and Pitkethly [4].

Their classification system divides three-element unary algebras into zero, one, two and three-kernel algebras. This classification is important for the dualizability of three-element unary algebras and is useful in determining which are candidates for standardness.

The definition of kernel from [4], is as follows:

A *kernel* of $\mathbf{M}$ is an equivalence relation on M of the form $\ker(u)$, for some unary term function u of $\mathbf{M}$ that is neither a constant map nor a permutation. Where the triple abc denotes the map u from $\mathbf{M}$ to $\mathbf{M}$ with $u(0) = a, u(1) = b$ and $u(2) = c$, the kernel of 110 is $\{\{0,1\},\{2\}\}$. An algebra $\mathbf{M} = \langle M; F\rangle$ is *n-kernel* if there are n distinct kernels for the non-constant, non-permutation operations in F. For example, a 0-kernel algebra has operations that are permutations or constants, $\langle\{0,1,2\}; 210, 111\rangle$, and a 1-kernel algebra has operations that are permutations or constants, or operations with one fixed non-trivial kernel, e.g., $\langle\{0,1,2\}; 111, 000, 110, 002\rangle$.

Clark, Davey and Pitkethly [4] classified the dualizable three-element unary algebras as follows:

Theorem 1.1. *Let* $\mathbf{M}$ *be a three-element unary algebra, on the set* $\{0,1,2\}$

1. *If* $\mathbf{M}$ *is a zero-kernel or a one-kernel algebra,* $\mathbf{M}$ *is dualizable;*
2. *If* $\mathbf{M}$ *is a two-kernel algebra with kernels* $01|2$ *and* $02|1$*, then* $\mathbf{M}$ *is dualizable if and only if:*
   (a) $pp1$ *and* pqp *are term functions of* $\mathbf{M}$ *with* $p, q \in M$ *and* $p \neq q$*, but* 010 *or* 002 *is not a term function of* $\mathbf{M}$*;*
   (b) 010, 001 *and* 110 *are term functions of* $\mathbf{M}$*, but* 222 *isn't;*

(c) $002, 020$ *and* 202 *are term functions of* $\mathbf{M}$ *but* 111 *isn't.*

3. *If* $\mathbf{M}$ *is a three-kernel algebra, then* $\mathbf{M}$ *is not dualizable.*

It is this type of theorem that I am looking for in classifying which algebras are standard.

Given an algebraic quasi-variety $\mathcal{A} := \mathbb{ISP}\,\mathbf{M}$, the *Algebraic Separation Theorem* shows that the members of $\mathcal{A}$ are characterized by the existence of separating homomorphisms into $\mathbf{M}$.

Theorem 1.2 (Algebraic Separation Theorem). [2] *An algebra* $\mathbf{A}$ *is in* $\mathcal{A} := \mathbb{ISP}\,\mathbf{M}$ *if and only if, for each* $a, b \in A$ *where* $a \neq b$*, there is a homomorphism* $u_{ab} : \mathbf{A} \to \mathbf{M}$ *such that* $u_{ab}(a) \neq u_{ab}(b)$.

The $\mathbb{ISP}$ *Theorem* shows that every algebraic structure in $\mathcal{A}$ is a model of the quasi-equational theory of $\mathbf{M}$.

Theorem 1.3 ($\mathbb{ISP}$-Theorem). [2] *Let* $\mathbf{M}$ *be a finite algebra. For any algebra* $\mathbf{B}$ *of the same type as* $\mathbf{M}$*, the following are equivalent:*

(i) $\mathbf{B} \in \mathcal{A} := \mathbb{ISP}\,\mathbf{M}$*;*

(ii) $\mathbf{B}$ *is a model of the quasi-equational theory of* $\mathbf{M}$*;*

(iii) $\mathbf{B}$ *is obtainable from* $\mathbf{M}$ *by repeated applications of* $\mathbb{I}, \mathbb{S}$ *and* $\mathbb{P}$.

In particular, $\mathcal{A} := \mathbb{ISP}\,\mathbf{M}$ *is exactly the quasi-variety generated by* $\mathbf{M}$.

The spaces I am working with are structured Boolean spaces, defined as follows, beginning with the definition of a *Boolean space* [2].

A *topological space* $\mathbb{X}$ is a set X, together with a collection, τ, of open subsets of X, such that τ includes $\emptyset$ and X and is closed under finite intersections and arbitrary unions. The collection τ is called the *topology* on $\mathbb{X}$. A subset of X is *closed* if its complement is open, and it is *clopen* if it is both closed and open. A collection C of open subsets is a *cover* of X if the union of the elements of C is equal to X. A topological space is *compact* if every open cover contains a finite subcover. A topological space is *Hausdorff* if for all $a, b \in X$, there exist open sets $U, V \in X$ with $a \in U, b \in V$, such that $U \cap V = \emptyset$. If U and V can always be chosen to be clopen, then X is said to be *totally disconnected.* A *Boolean Space* is a topological space that is compact, Hausdorff and totally disconnected.

A *structured topological space* is defined in [2] to be a structure of the type $\langle G, H, R \rangle$, such that

$$\mathbb{X} = \langle X; G^{\mathbb{X}}, H^{\mathbb{X}}, R^{\mathbb{X}}, \tau^{\mathbb{X}} \rangle$$

where

(i) $g^{\mathbb{X}}$ consists of an n-ary total operation $g^{\mathbb{X}} : X^n \rightarrow X$ for each n-ary total operation symbol $g \in G$,

(ii) $h^{\mathbb{X}}$ consists of an n-ary partial operation $h^{\mathbb{X}} : \mathrm{dom}(h^{\mathbb{X}}) \rightarrow X$ for each n-ary partial operation symbol $h \in H$, where $\mathrm{dom}(h^{\mathbb{X}}) \subseteq X^n$,

(iii) $r^{\mathbb{X}}$ consists of an n-ary relation $r^{\mathbb{X}} \subseteq X^n$ on X for each n-ary relation symbol $r \in R$,

(iv) $\langle X, \tau^{\mathbb{X}} \rangle$ is a topological space.

The superscripts are omitted when the context is clear.

A *Boolean structure* [3], is a topological structure $\mathbb{X} = \langle X; G^{\mathbb{X}}, H^{\mathbb{X}}, R^{\mathbb{X}}, \tau^{\mathbb{X}} \rangle$, such that

(i) $\langle X; \tau^{\mathbb{X}} \rangle$ is a Boolean space,

(ii) if $h \in G \cup H$ is n-ary, then the domain $\mathrm{dom}(h^{\mathbb{X}})$ is a closed subset of X^n and $h^{\mathbb{X}} : \mathrm{dom}(h^{\mathbb{X}}) \to X$ is continuous, and

(iii) if $r \in R$ is n-ary, then $r^{\mathbb{X}}$ is a closed subset of X^n.

Given a structured topological space, $\mathbb{M} = \langle M; G, H, R, \tau \rangle$, where M is finite and the topology τ is discrete, we generate the class $\mathcal{X} := \mathbb{IS}_c\mathbb{P}^+\mathbb{M}$. Because $\mathbb{M}$ is finite and the topology is discrete, $\mathbb{M}$ is Boolean, so the topology on every member of $\mathcal{X}$ is Boolean and therefore $\mathcal{X} := \mathbb{IS}_c\mathbb{P}^+\mathbb{M}$, is a category of *structured Boolean spaces.*[2]

The following definitions explain a morphism between structured topological spaces and a substructure of a topological space.

Given a structured topological space $\mathbb{X} = \langle X; G^{\mathbb{X}}, H^{\mathbb{X}}, R^{\mathbb{X}}, \tau^{\mathbb{X}} \rangle$, and another structured topological space $\mathbb{Y} = \langle Y; G^{\mathbb{Y}}, H^{\mathbb{Y}}, R^{\mathbb{Y}}, \tau^{\mathbb{Y}} \rangle$, a continuous map $\varphi \colon \mathbb{X} \to \mathbb{Y}$ is a *morphism* if

(i) for each n-ary $g \in G$ and each $(x_1, x_2, \dots, x_n) \in \mathbb{X}$, we have

$$\varphi(g^{\mathbb{X}}(x_1, x_2, \dots, x_n)) = g^{\mathbb{Y}}(\varphi(x_1), \varphi(x_2), \dots, \varphi(x_n)),$$

(ii) for each n-ary $h \in H$ and each $(x_1, x_2, \dots, x_n) \in \mathrm{dom}(h^{\mathbb{X}})$ we have

$$(\varphi(x_1), \varphi(x_2), \dots, \varphi(x_n)) \in \mathrm{dom}(h^{\mathbb{Y}}) \text{ and}$$
$$\varphi(h^{\mathbb{X}}(x_1, x_2, \dots, x_n)) = h^{\mathbb{Y}}(\varphi(x_1), \varphi(x_2), \dots, \varphi(x_n)),$$

(iii) for each n-ary $r \in R$ and each $(x_1, x_2, \dots, x_n) \in r^{\mathbb{X}}$ we have $(\varphi(x_1), \varphi(x_2), \dots, \varphi(x_n)) \in r^{\mathbb{Y}}$, and

(iv) φ is continuous.

The structure $\mathbb{Y} = \langle Y; G^{\mathbb{Y}}, H^{\mathbb{Y}}, R^{\mathbb{Y}}, \tau^{\mathbb{Y}} \rangle$ is called a *substructure* of the structured topological space $\mathbb{X} = \langle X; G^{\mathbb{X}}, H^{\mathbb{X}}, R^{\mathbb{X}}, \tau^{\mathbb{X}} \rangle$, written $\mathbb{Y} \leq \mathbb{X}$, provided that $Y \subseteq X$ and

(i) for each n-ary $g \in G$ the operations $g^{\mathbb{Y}}$ and $g^{\mathbb{X}}$ agree on Y^n,

(ii) for each n-ary $h \in H$, we have $\mathrm{dom}(h^{\mathbb{Y}}) = \mathrm{dom}(h^{\mathbb{X}}) \cap Y^n$, and $h^{\mathbb{Y}}$ agrees with $h^{\mathbb{X}}$ on this set,

(iii) for each n-ary $r \in R$, we have $r^{\mathbb{Y}} = r^{\mathbb{X}} \cap Y^n$, and

(iv) $\tau^{\mathbb{Y}}$ is the relative topology obtained from $\tau^{\mathbb{X}}$.

A topological structure $\mathbb{M}$ is then said to be *standard* if every Boolean model of the quasi-equational theory of $\mathbb{M}$ is isomorphic to a closed substructure of a non-empty product of $\mathbb{M}$, that is,

$$\mathcal{X} := \mathbb{IS}_c\mathbb{P}^+\mathbb{M} = \mathrm{Mod}_\tau\,\mathrm{Th}_{qe}(\mathbb{M}).$$

For topological structures, in order to ensure the correct description of the members of $\mathcal{X}$, Theorem 1.2 and Theorem 1.3 are modified. The *Topological Separation Theorem* characterizes the existence of separation morphisms into $\mathbb{M}$ and shows that any structure meeting a particular description is in $\mathcal{X}$. The *Preservation Theorem* shows that every topological structure in $\mathcal{X}$ is a model of the quasi-equational theory of $\mathbb{M}$.

Theorem 1.4 (Topological Separation Theorem). [3] *Let $\mathbb{M} = \langle M; G, H, R, \tau \rangle$ be a finite structure, let $\mathcal{X} := \mathbb{IS}_c\mathbb{P}^+\mathbb{M}$, and let $\mathbb{X}$ be a compact topological structure of the same type as $\mathbb{M}$. Then $\mathbb{X} \in \mathcal{X}$ if and only if there is at least one morphism from $\mathbb{X}$ to $\mathbb{M}$ and the following conditions hold:*

(i) *for each $x, y \in X$ where $x \neq y$, there is an $\alpha : \mathbb{X} \to \mathbb{M}$ such that $\alpha(x) \neq \alpha(y)$,*

(ii) *for each n-ary $h \in H$ and $(x_1, x_2, \ldots, x_n) \in X^n \backslash \operatorname{dom}(h^{\mathbf{X}})$, there is an α : $\mathbb{X} \to \mathbb{M}$ such that $(\alpha(x_1), \alpha(x_2), \ldots, \alpha(x_n)) \notin \operatorname{dom}(h^{\mathbf{M}})$,*

(iii) *for each n-ary $r \in R$ and $(x_1, x_2, \ldots, x_n) \in X^n \backslash r^{\mathbf{X}}$, there is an $\alpha : \mathbb{X} \to \mathbb{M}$ such that $(\alpha(x_1), \alpha(x_2), \ldots, \alpha(x_n)) \notin r^{\mathbf{M}}$.*

Theorem 1.5 (Preservation Theorem). [3] *Let $\mathbb{M} = \langle M; G, H, R, \mathcal{T} \rangle$ be a finite structure and let $\mathcal{X} := \mathbb{IS}_c \mathbb{P}^+ \mathbb{M}$. Then every member of $\mathcal{X}$ is a Boolean model of the quasi-equational theory of $\mathbb{M}$, in symbols,*

$$\mathcal{X} \subseteq \operatorname{Mod}_{\mathcal{T}} \operatorname{Th}_{qe}(\mathbb{M}).$$

Since unary algebras have only unary operations, that is, there are no partial operations or relations, Theorem 1.4 and Theorem 1.5 are modified so that $\mathbb{M} = \langle M; G, \mathcal{T} \rangle$ and in Theorem 1.4, (*i*) holds and (*ii*) and (*iii*) do not apply.

Theorem 1.6 (Separation Theorem for Unary Algebras). *Let $\mathbb{M} = \langle M; G, \mathcal{T} \rangle$ be a finite structure, let $\mathcal{X} := \mathbb{IS}_c \mathbb{P}^+ \mathbb{M}$, and let $\mathbb{X}$ be a compact topological structure of the same type as $\mathbb{M}$. Then $\mathbb{X} \in \mathcal{X}$ if and only if there is at least one morphism from $\mathbb{X}$ to $\mathbb{M}$, and for each $x, y \in X$ where $x \neq y$, there is an $\alpha : \mathbb{X} \to \mathbb{M}$ such that $\alpha(x) \neq \alpha(y)$*

Theorem 1.7 (Preservation Theorem for Unary Algebras). *Let $\mathbb{M} = \langle M; G, \mathcal{T} \rangle$ be a finite structure and let $\mathcal{X} := \mathbb{IS}_c \mathbb{P}^+ \mathbb{M}$. Then every member of $\mathcal{X}$ is a Boolean model of the quasi-equational theory of $\mathbb{M}$, in symbols,*

$$\mathcal{X} \subseteq \operatorname{Mod}_{\mathcal{T}} \operatorname{Th}_{qe}(\mathbb{M}).$$

Chapter 2

Standard Topological Structures

2.1 Standardness Theorems

This research studies three-element unary topological algebras to determine which of them are standard. See Table 1 for a list of the three-element unary algebras.

The general method is as follows: Given $a, b, \in \mathcal{X} := \mathbb{IS}_c \mathbb{P}^+ \mathbb{M}$, the topological quasi-variety, if $\mathcal{X}$ can be partitioned into clopen sets with a and b in different sets and the clopen sets manipulated so that they behave like elements of an algebra in the quasi-variety, that is, the clopen sets are isomorphic to a finite subalgebra of a power of $\mathbb{M}$, then the algebra is *standard.* The following material shows that this algorithm is valid.

The following Lemma from [3] says that to show $\mathbb{X}$ is standard it is sufficient to show for any $X \in \mathbb{X}$ that there is a finite substructure, $\mathbb{Y} \in \mathbb{IS}_c \mathbb{P}^+ \mathbb{M}$ that satisfies the quasi-equational theory of $\mathbb{M}$ with the required separating morphism from $\mathbb{X} \to \mathbb{Y}$.

Lemma 2.1. [3] *Let* $\mathbb{M}$ *be a finite structure. Then* $\mathcal{X} := \mathbb{IS}_c \mathbb{P}^+ \mathbb{M}$ *is standard provided that, for every* $\mathbb{X} \in \operatorname{Mod}_\tau \operatorname{Th}_{qe}(\mathbb{M})$,

(i) *for each* $x, y \in X$ *where* $x \neq y$*, there is a finite* $\mathbb{Y} \in \mathcal{X}$ *and an* $\alpha \colon \mathbb{X} \to \mathbb{Y}$ *such* that $\alpha(x) \neq \alpha(y)$,

(ii) *for each n-ary $h \in H$ and $(x_1, x_2, \ldots, x_n) \in X^n \setminus \operatorname{dom}(h^{\mathbb{X}})$, there is a finite $\mathbb{Y} \in \mathcal{X}$ and an $\alpha\colon \mathbb{X} \to \mathbb{Y}$ such that $(\alpha(x_1), \alpha(x_2), \ldots, \alpha(x_n)) \notin \operatorname{dom}(h^{\mathbb{Y}})$,*

(iii) *for each n-ary $r \in R$ and $(x_1, x_2, \ldots, x_n) \in X^n \setminus r^{\mathbb{X}}$, there is a finite $\mathbb{Y} \in \mathcal{X}$ and an $\alpha\colon \mathbb{X} \to \mathbb{Y}$ such that $(\alpha(x_1), \alpha(x_2), \ldots, \alpha(x_n)) \notin r^{\mathbb{Y}}$.*

Again, since the applications of this research are for total algebras, $\mathbb{M} = \langle M; G, \tau \rangle$, with no partial operations or relations, items (ii) and (iii) of Lemma 2.1 are not necessary:

Corollary 2.2. [3] *Let $\mathbb{M}$ be a finite total structure. Then $\mathcal{X} := \mathbb{IS}_c\mathbb{P}^+\mathbb{M}$ is standard provided that, for every $\mathbb{X} \in \operatorname{Mod}_\tau \operatorname{Th}_{qe}(\mathbb{M})$, and for each $x, y \in X$ where $x \neq y$, there is a finite $\mathbb{Y} \in \mathcal{X}$ and an $\alpha\colon \mathbb{X} \to \mathbb{Y}$ such that $\alpha(x) \neq \alpha(y)$.*

Before stating the standardness theorem for unary algebras, we require the following lemma.

Lemma 2.3. *Let $\mathbb{X}$ be a Boolean model of the quasi-equational theory of $\mathbb{M}$ and assume $\mathbb{Y}$ is a partition of $\mathbb{X}$ into finitely many clopen sets where U_x denotes the clopen set in $\mathbb{Y}$ containing x. Let $f \in \mathcal{F}$. If for all $x \in \mathbb{X}$ we have $f^{\mathbb{X}}(U_x) \subseteq U_{f^{\mathbb{X}}(x)}$, then $f^{\mathbb{Y}}\colon \mathbb{Y} \to \mathbb{Y}$ defined by $f^{\mathbb{Y}}(U_x) = U_{f^{\mathbb{X}}(x)}$ is a well defined map.*

Proof. Assume $U_x = U_y$ for some $x, y \in \mathbb{X}$. We have $x, y \in U_x$ so $f^{\mathbb{X}}(y) \in f^{\mathbb{X}}(U_x) \subseteq U_{f^{\mathbb{X}}(x)}$. Thus $U_{f^{\mathbb{X}}(y)} = U_{f^{\mathbb{X}}(x)}$, that is, $f^{\mathbb{Y}}(U_y) = f^{\mathbb{Y}}(U_x)$. Therefore $f^{\mathbb{Y}}\colon \mathbb{Y} \to \mathbb{Y}$ is well defined. □

The algorithm required is then summarized by the following theorem.

Theorem 2.4 (Standardness Theorem for Unary Algebras). *If for every Boolean model $\mathbb{X}$ of the quasi-equational theory of $\mathbb{M}$, and every pair of elements $a, b \in X$ with $a \neq b$, there is a partition $\mathbb{Y}$ of $\mathbb{X}$ into finitely many clopen sets, such that*

(i) *for all* $x \in \mathbb{X}$ *the clopen set containing* x *is* U_x*,*

(ii) $U_a \cap U_b = \emptyset$,

(iii) $f^{\mathbb{X}}(U_x) \subseteq U_{f^{\mathbb{X}}(x)}$ *for all* $x \in \mathbb{X}$,

(iv) $f^{\mathbb{Y}}(U_x) := U_{f^{\mathbb{X}}(x)}$ *for all* $x \in \mathbb{M}$ *and* $f \in \mathcal{F}$*, and*

(v) $\langle Y, \mathcal{F} \rangle$ *satisfies the quasi-equations of* $\mathbb{M}$,

then $\mathbb{M}$ *is standard.*

Proof. Let $\mathbb{Y}$ be the finite partition of clopen sets chosen above with $U_a \neq U_b$. The map $\alpha \colon \mathbb{X} \to \mathbb{Y}$ is defined by $\alpha(x)$ is the clopen set U_x containing x. For all $x \in \mathbb{X}$ the map α is well defined since for each x, there is one clopen set containing x. The map α is operation preserving because for all $f \in F, f(\alpha(x)) = f(U_x) = U_{f(x)} = \alpha(f(x))$. Therefore, α is a morphism. The map α is continuous because Y is finite and has the discrete topology and $\alpha^{-1}(\{U_y\}) = U_y$, a clopen set.

By Lemma 2.3 each $f^{\mathbb{Y}}$ is well defined.

By Theorem 1.2, there is a separating morphism $\beta \colon \langle Y, F \rangle \to \langle \mathbb{M}, F \rangle$ such that $\beta(x) \neq \beta(y)$.

Since $\mathbb{Y}$ is a finite structure and $\langle Y, F \rangle$ satisfies the quasi-equations of $\mathbb{M}$, then by Theorem 1.3, $\langle Y, F \rangle \in \mathbb{ISP}(\langle M, F \rangle)$, but the finite elements of $\mathbb{ISP}(\langle M, F \rangle)$ are in $\mathbb{IS}_c\mathbb{P}^+\mathbb{M}$ when endowed with the discrete topology. As β is continuous when Y and M have the discrete topology, by Theorem 1.6, $\mathbb{Y} \in \mathbb{IS}_c\mathbb{P}^+\mathbb{M}$. Therefore $\langle Y, F, \tau \rangle \in \mathbb{IS}_c\mathbb{P}^+\mathbb{M}$, and by Corollary 2.2, $\mathbb{M}$ is standard. $\square$

Some results are known and are shown in Table 2.1. For example, Clark, Davey, Haviar, Pitkethly and Talukder [3] showed that finite mono-unary topological algebras are standard. Also, Hyndman and Pitkethly [5] have shown two algebras are standard using compactness and one is standard and injective in the topological equational class, and three algebras are non-standard using an inverse limit technique. Chapters 3 and 4 provide proofs that two additional algebras are standard.

The three-element unary algebras, up to isomorphism, are listed in Table 2.1. That this list is complete must be confirmed.

	Generators	Kernels	Elements	Knowledge
1 generated monoids	002	1		Standard - mono-unary
	112	1		Standard - mono-unary
	001	1	000	Standard - mono-unary
	110	1	111	Standard - mono-unary
	220	1	002	Standard - mono-unary
	221	1	112	Standard - mono-unary
2 generated monoids	002, 112	1		
	001, 002	1	000	
	110, 112	1	111	
	001, 110	1	000, 111	Standard - needs compactness
	001, 112	1	000, 111	Standard - injective in the topological equational class
	110, 002	1	000, 111	
	002, 221	1	112, 220	Standard - currently uses compactness
	001, 220	1	000, 111, 222, 110, 002	
	001, 221	1	000, 111, 222, 110, 112	
	010, 011	2		**V** non-standard - inverse limits
	010, 110	2	000, 111	**L** non-standard - inverse limits
	101, 220	2	000, 111, 222, 010, 002	**D** non-standard - inverse limits
3 generated monoids	001, 002, 111	2	000, 111	
	001, 110, 112	2	000, 111	
	001, 002, 112	1	000, 111	
	002, 110, 112	2	000, 111	
	001, 002, 221	1	ALL	
4 generated monoids	001, 002, 110, 112	1	000, 111	

Table 2.1: Three-element Unary Algebras

The proofs in Chapters 3 and 4 frequently require the following lemmas which follow directly from the topology of Hausdorff spaces.

Lemma 2.5. *Let $\mathbb{X}$ be a Boolean model of the quasi-equational theory of $\mathbb{M}$. Given $a, b \in \mathbb{X}$, and $a \neq b$, there exist clopen sets U, V with $a \in U$, $b \in V$ and $U \cap V = \emptyset$. In addition, given $\{c, d\}$ disjoint from $\{a, b\}$, we may assume $c, d \notin U$ and $c, d \notin V$.*

Lemma 2.6. *Given $A \subseteq \mathbb{X}$ closed and $b \in \mathbb{X}$ such that $b \notin A$, there exist clopen sets U, V with $U \cap V = \emptyset$, $A \subseteq U$ and $b \in V$.*

Lemma 2.7. *Given A and B closed and disjoint with $B \subseteq U_o$ and U_o clopen, there exists a clopen set U with $B \subseteq U \subseteq U_o$ and $U \cap A = \emptyset$.*

The next two chapters provide two examples of topological structures that are standard. These are new results.

Chapter 3

Example 1

The first example we consider is the structure $\mathbb{M} = \langle\{0,1,2\};\ \mathcal{F};\ \tau\rangle$ where $\mathcal{F} = \{002, 110\}$ and τ is the discrete topology. When g denotes the operation 002 and p denotes the operation 110, the structure $\mathbb{M}$ is shown in Figure 3.1.

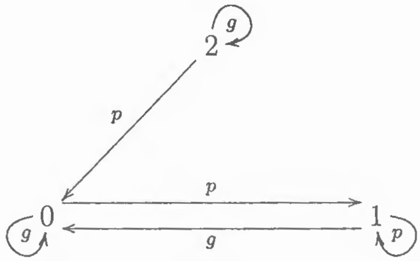

Figure 3.1: The Structure of $\mathbb{M}$

Applying p and g to the 27 elements of $\mathbb{M}^3$ yields Figure 3.2. Knowing the structure of $\mathbb{M}^3$ assists in understanding three variable quasi-equations.

The first result on standardness of unary algebras is the next theorem.

Theorem 3.1. *The structure* $\mathbb{M} = \langle 0,1,2;\ \mathcal{F};\ \tau\rangle$ *where* $\mathcal{F} = \{002, 110\}$ *and* τ *is the discrete topology, is standard.*

The proof of Theorem 3.1 uses Theorem 2.4. To show that a particular structure $\mathbb{M}$ is standard, we construct a finite cover of clopen sets of each Boolean model

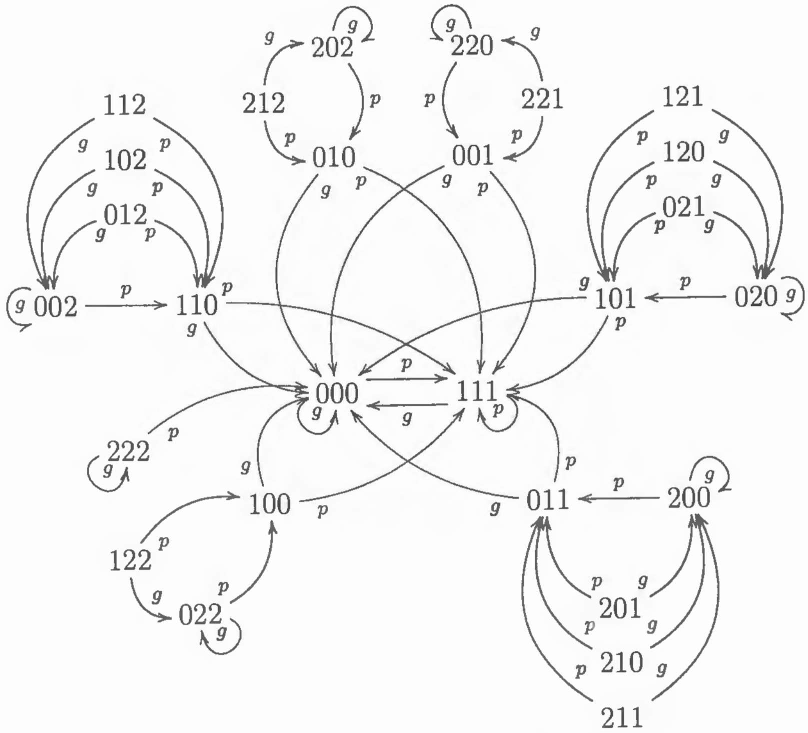

Figure 3.2: The Structure of $\mathbb{M}^3$

$\mathbb{X}$ of the quasi-equational theory of $\mathbb{M}$ in which particular points are separated, and the clopen sets satisfy the quasi-equational theory of $\mathbb{M}$. Choose $\mathbb{X}$ such that $\mathbb{X} \in \mathrm{Mod}_\tau \, \mathrm{Th}_{qe}(\mathbb{M})$, that is, $\mathbb{X}$ is a Boolean model of the quasi-equational theory of $\mathbb{M}$, and then show that $\mathbb{X} \in \mathbb{IS}_c \mathbb{P}^+ \mathbb{M}$. There are a number of cases to consider. Theorem 3.4 provides the basis for the choice of cases. Note that $\mathbb{M}$ has two constant valued fuctions, 0 and 1.

The equations and quasi-equations satisfied by $\mathbb{M}$ include:

$$p^n(x) \approx 1, n > 1 \tag{3.1}$$

$$g^n(x) \approx g(x), n \geq 1 \tag{3.2}$$

$$g^n p(x) \approx 0, n \geq 1 \tag{3.3}$$

$$gp^n(x) \approx 0, n \geq 1 \tag{3.4}$$

$$pg(x) \approx p(x) \tag{3.5}$$

$$p^n g(x) \approx 1, n > 1 \tag{3.6}$$

$$p(x) \approx 1 \iff g(x) \approx 0 \tag{3.7}$$

$$p(x) \approx 0 \Rightarrow g(x) \approx x \tag{3.8}$$

$$g(x) \approx g(y) \iff p(x) \approx p(y) \tag{3.9}$$

$$p(x) \approx x \Rightarrow x \approx 1 \tag{3.10}$$

That these quasi-equations hold can be seen from Table 3.1.

x	p	g	p^2	g^2	pg	gp	pg^2	p^2g	gp^2	pgp	gpg
0	1	0	1	0	1	0	1	1	0	1	0
1	1	0	1	0	1	0	1	1	0	1	0
2	0	2	1	2	0	0	0	1	0	1	0

Table 3.1: Operations of $\mathbb{M}$

Throughout the remainder of Section 3.1, $\mathbb{X}$ is a Boolean model of the quasi-equational theory of $\mathbb{M}$.

The following lemmas provide useful properties of subsets of $\mathbb{X}$.

Lemma 3.2. *Assume $A, B \subseteq \mathbb{X}$.*

(i) *If $A \subseteq p^{-1}(B)$, then $g(A) \subseteq p^{-1}(B)$.*

(ii) *If $A \subseteq g^{-1}(B)$, then $g(A) \subseteq g^{-1}(B)$.*

(iii) *If $A \subseteq p^{-1}(B)$ and $1 \in B$, then $p(A) \subseteq p^{-1}(B)$.*

(iv) *If $A \subseteq g^{-1}(B)$ and $0 \in B$, then $p(A) \subseteq g^{-1}(B)$.*

Proof. For property (i) assume $A \subseteq p^{-1}(B)$, so $p(A) \subseteq B$. By Quasi-equation 3.5, $p(A) = pg(A)$. This implies $pg(A) \subseteq B$ and $g(A) \subseteq p^{-1}(B)$.

Similarly, for property (ii), if $A \subseteq g^{-1}(B)$ then $g(A) \subseteq B$. Then Quasi-equation 3.2 implies $g^2(A) \subseteq B$, so we have $g(A) \subseteq g^{-1}(B)$.

For property (iii), assume $A \subseteq p^{-1}(B)$. If $1 \in B$, then Quasi-equation 3.1 gives $p^n(x) \approx 1$, which implies $\mathbb{X} = p^{-2}(B)$ and $p(\mathbb{X}) \subseteq p^{-1}(B)$. Therefore, $p(A) \subseteq p^{-1}(B)$.

Finally, for property (iv), if $A \subseteq g^{-1}(B)$ and $0 \in B$, then Quasi-equation 3.4 gives $gp(x) \approx 0$, which implies $\mathbb{X} = p^{-1}g^{-1}(0) \subseteq p^{-1}g^{-1}(B)$. Thus $A \subseteq p^{-1}g^{-1}(B)$, and we have $p(A) \subseteq g^{-1}(B)$. $\square$

Lemma 3.3. *Assume $A, B \subseteq \mathbb{X}$ and $A \cap B = \emptyset$. Then $p^{-1}(A) \cap p^{-1}(B) = \emptyset$.*

Proof. Let $d \in p^{-1}(A) \cap p^{-1}(B)$. Then $p(d) \in A$ and $p(d) \in B$, but $A \cap B = \emptyset$, a contradiction. Therefore $p^{-1}(A) \cap p^{-1}(B) = \emptyset$. $\square$

For each $a \in \mathbb{X}$, there are six possibilities and these are described in the next Theorem.

Theorem 3.4. *For each $a \in \mathbb{X}$, a Boolean model of the quasi-equational theory of $\mathbb{M}$, one of the following cases holds.*

(i) $a = 1$; *or*

(ii) $a = 0$; *or*

(iii) $a \notin \{0, 1\}$, $p(a) = 1$, $g(a) \neq a$; *or*

(iv) $\{a, p(a)\}$ *disjoint from* $\{0, 1\}$, $g(a) = a$; *or*

(v) $\{a, p(a)\}$ *disjoint from* $\{0, 1\}$, $g(a) \neq a$; *or*

(vi) $a \notin \{0, 1\}$, $p(a) = 0$, $g(a) = a$.

Moreover, $p(a) \neq 1$ when $a \neq 1$.

Proof. If $a = 1$, then Case (i) holds. If $a = 0$, then Case (ii) holds.

Now assume $a \neq 0$ and $a \neq 1$. It is sufficient to consider the cases when $p(a) = 1$ or $p(a) \neq 1$. If $p(a) = 1$, then Quasi-equation 3.7 gives $g(a) = 0$, which is Case (iii). If $p(a) = 0$, then from Quasi-equation 3.8 we know that $g(a) = a$, which is Case (vi).

Now assume $p(a) \neq 0$ and $p(a) \neq 1$. It is sufficient to consider the cases when $g(a) = a$ or $g(a) \neq a$. The former condition gives Case (iv) and the latter gives Case (v).

Finally, if $p(a) = 1$, Quasi-Equation 3.10 gives $a = 1$. □

Based on the six cases in Theorem 3.4, for distinct a and b, there are potentially 36 cases. By symmetry, we need consider 21 cases, which further reduces to 18 cases that can be described by one of four partitions.

The possible partitions are:

$$\begin{array}{ll}
\textit{Partition } 1: & \mathbb{X} = Z \,\dot\cup\, W \,\dot\cup\, A \,\dot\cup\, B \,\dot\cup\, K \,\dot\cup\, p^{-1}(A) \,\dot\cup\, p^{-1}(B) \,\dot\cup\, p^{-1}(K) \,\dot\cup\, p^{-1}(Z). \\
\textit{Partition } 2: & \mathbb{X} = Z \,\dot\cup\, W \,\dot\cup\, L \,\dot\cup\, p^{-1}(Z) \,\dot\cup\, R \,\dot\cup\, (p^{-1}(L)\backslash R) \\
\textit{Partition } 3: & \mathbb{X} = Z \,\dot\cup\, W \,\dot\cup\, L \,\dot\cup\, C \,\dot\cup\, D \,\dot\cup\, R \,\dot\cup\, p^{-1}(Z) \\
\textit{Base Partition } 4: & \mathbb{X} = Z \,\dot\cup\, W \,\dot\cup\, L \,\dot\cup\, p^{-1}(L) \,\dot\cup\, p^{-1}(Z).
\end{array}$$

Each of the 18 cases is covered by one of the four partitions. This is shown in Table 3.2.

The following lemmas show the construction of clopen sets that separate the constant valued functions 0 and 1. The illustration of these sets is shown in Figure 3.3. The partition illustrated in Figure 3.3 underlies all other partitions constructed.

	b					
a	1	2	3	4	5	6
1	-	4	4	4	4	4
2	-	-	4	4	4	4
3	-	-	1	4	1,4	1,4
4	-	-	-	1,2	1	4
5	-	-	-	-	1,3	4
6	-	-	-	-	-	-

Table 3.2: Six cases for a and b, give rise to 18 situations (22 including subcases) using four partitions.

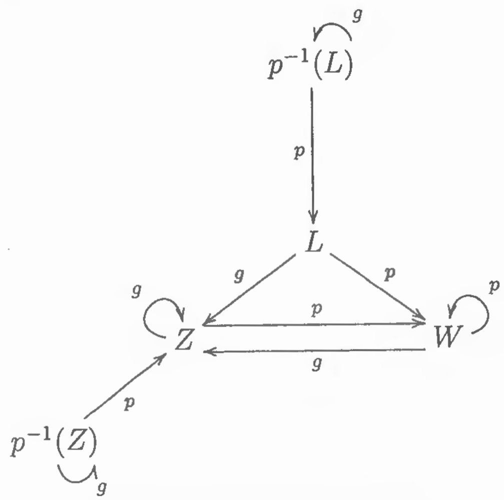

Figure 3.3: Clopen Sets Separating 0 and 1 with a Partition of $\mathbb{X}$

Lemma 3.5. *There are clopen sets U and V with $0 \in U$, $1 \in V$ and $U \cap V = \emptyset$. In addition, given $a \notin \{0,1\}$ and $b \notin \{0,1\}$, we may assume $\{a,b\} \notin U$ and $\{a,b\} \notin V$.*

Moreover, the clopen sets Z_o and W_o, defined as

$$Z_o = U \cap p^{-1}(V) \cap g^{-1}(U) \cap p^{-1}(\mathbb{X}\backslash U) \cap g^{-1}(\mathbb{X}\backslash V)$$
$$W_o = V \cap p^{-1}(V) \cap g^{-1}(U) \cap p^{-1}(\mathbb{X}\backslash U) \cap g^{-1}(\mathbb{X}\backslash V).$$

satisfy the following containments:

1. $0 \in Z_o$, *and* $1 \in W_o$;

2. $Z_o \cap W_o = \emptyset$;

3. $g(Z_o) \subseteq Z_o$;

4. $g(W_o) \subseteq Z_o$;

5. $p(Z_o) \subseteq W_o$;

6. $p(W_o) \subseteq W_o$.

Moreover,

$$Z_o = Z_o \cap p^{-1}(W_o) \cap g^{-1}(Z_o) \cap p^{-1}(\mathbb{X}\backslash Z_o) \cap g^{-1}(\mathbb{X}\backslash W_o)$$
$$W_o = W_o \cap p^{-1}(W_o) \cap g^{-1}(Z_o) \cap p^{-1}(\mathbb{X}\backslash Z_o) \cap g^{-1}(\mathbb{X}\backslash W_o)$$

Proof. The existence of U and V follow from Lemma 2.5. Since Z_o is contained in each of $p^{-1}(V)$, $g^{-1}(U)$, $p^{-1}(\mathbb{X}\backslash U)$ and $g^{-1}(\mathbb{X}\backslash V)$, Lemma 3.2 gives $g(Z_o) \subseteq p^{-1}(V)$, $g^{-1}(U)$, $p^{-1}(\mathbb{X}\backslash U)$ and $g^{-1}(\mathbb{X}\backslash V)$.

Similarly, since W_o is contained in each of $p^{-1}(V)$, $g^{-1}(U)$, $p^{-1}(\mathbb{X}\backslash U)$ and $g^{-1}(\mathbb{X}\backslash V)$, Lemma 3.2 gives $g(W_o) \subseteq p^{-1}(V)$, $g^{-1}(U)$, $p^{-1}(\mathbb{X}\backslash U)$ and $g^{-1}(\mathbb{X}\backslash V)$. Thus we have $g(Z_o) \subseteq Z_o$ and $g(W_o) \subseteq Z_o$, as $Z_o \subseteq g^{-1}(U)$ and $W_o \subseteq g^{-1}(U)$.

By Lemma 3.2, $p(W_o)$ is contained in each of $p^{-1}(V)$, $g^{-1}(U)$, $p^{-1}(\mathbb{X}\backslash U)$, and $g^{-1}(\mathbb{X}\backslash V)$, and $p(Z_o)$ is contained in each of $p^{-1}(V)$, $g^{-1}(U)$, $p^{-1}(\mathbb{X}\backslash U)$, and $g^{-1}(\mathbb{X}\backslash V)$. Thus we have $p(W_o) \subseteq W_o$ and $p(Z_o) \subseteq W_o$ as $W_o \subseteq p^{-1}(V)$ and $Z_o \subseteq p^{-1}(V)$.

The last two equalities of the Lemma hold because $W_o \subseteq V \subseteq \mathbb{X}\backslash U \subseteq \mathbb{X}\backslash Z_o$ and $Z_o \subseteq U \subseteq \mathbb{X}\backslash V \subseteq \mathbb{X}\backslash W_o$.

□

$$\begin{tikzcd} Z_o \arrow[r, shift left, "p"] \arrow[loop left, "g"] & W_o \arrow[l, shift left, "g"] \arrow[loop right, "p"] \end{tikzcd}$$

Figure 3.4: The Elements Z_o and W_o

The behaviour of g and p on Z_o and W_o is illustrated in Figure 3.4.

If $\mathbb{X} = Z_o \cup W_o$, then $\mathbb{X}$ has been partitioned by clopen sets as required by Theorem 2.4. Now assume $Q = \mathbb{X}\backslash(Z_o \cup W_o)$ is non-empty. We need to construct more clopen sets that cover $\mathbb{X}\backslash(Z_o \cup W_o)$. Consider $g^{-1}(Z_o)$, $g^{-1}(W_o)$, $p^{-1}(Z_o)$ and $p^{-1}(W_o)$.

Lemma 3.6. *There exists W a clopen subset of W_o, and Z a clopen subset of Z_o such that*

$$p(p^{-1}(W_o)\backslash g^{-1}(Z_o)) \cap W = \emptyset;$$
$$g(g^{-1}(Z_o)\backslash p^{-1}(W_o)) \cap Z = \emptyset;$$
$$p(p^{-1}(W_o) \cap g^{-1}(Z_o)) \subseteq W;$$
$$g(g^{-1}(Z_o) \cap p^{-1}(W_o)) \subseteq Z.$$

Moreover, the sets Z and W satisfy the following containments:

1. $0 \in Z$ *and* $1 \in W$;

2. $Z \cap W = \emptyset$;

3. $g(Z) \subseteq Z$;

4. $g(W) \subseteq Z$;

5. $p(Z) \subseteq W$;

6. $p(W) \subseteq W$;

7. $p^{-1}(W) = g^{-1}(Z)$.

Proof. The existence of W and Z satisfying the displayed equations, is guaranteed by Lemma 2.7.

Given $1 \in p(p^{-1}(W_o)\backslash g^{-1}(Z_o))$, this implies there exists an $a \in p^{-1}(W_o)\backslash g^{-1}(Z_o)$ with $p(a) = 1$. With $g(0) = 0$ and $0 \in Z_o$ and with $p(0) = 1$ and $1 \in W_o$, we have $0 \in g^{-1}(Z_o) \cap p^{-1}(W_o)$. Therefore $1 = p(0) \in p(g^{-1}(Z_o) \cap p^{-1}(W_o)) \subseteq W$ and $0 = g(0) \in g(g^{-1}(X_o) \cap p^{-1}(W_o)) \subseteq Z$. It then follows with $Z_o \cap W_o = \emptyset$, that $Z \cap W = \emptyset$.

We have $Z \subseteq Z_o \subseteq g^{-1}(Z_o) \cap p^{-1}(W_o)$, and this implies $g(Z) \subseteq g(Z_o) \subseteq g(g^{-1}(Z_o) \cap p^{-1}(W_o)) \subseteq Z$. Therefore, $g(Z) \subseteq Z$. Similarly, $Z \subseteq p^{-1}(W_o)$, so $p(Z) \subseteq W$.

With $W \subseteq W_o \subseteq g^{-1}(Z_o) \cap p^{-1}(W_o)$, we have $g(W) \subseteq g(W_o) \subseteq g(g^{-1}(Z_o) \cap p^{-1}(W_o)) \subseteq Z$. Therefore $g(W) \subseteq Z$. Similarly, we have $p(W) \subseteq W$.

In order to prove the remaining containments, we first show that $g^{-1}(Z) = g^{-1}(Z_o) \cap p^{-1}(W_o) = p^{-1}(W)$.

By choice of Z, we have $g^{-1}(Z_o) \cap p^{-1}(W_o) \subseteq g^{-1}(Z)$. To show $g^{-1}(Z) = g^{-1}(Z_o) \cap p^{-1}(W_o)$, consider that $a \in g^{-1}(Z)$. Then $g(a) \in Z \subseteq Z_o$ and $a \in g^{-1}(Z) \subseteq g^{-1}(Z_o)$. We need to show $a \in p^{-1}(W_o)$. If $a \notin p^{-1}(W_o)$, then $a \in g^{-1}(Z_o)\backslash p^{-1}(W_o)$ and $g(a) \in g(g^{-1}(Z_o)\backslash p^{-1}(W_o))$. But $g(g^{-1}(Z_o)\backslash p^{-1}(W_o)) \cap Z = \emptyset$, a contradiction. Therefore $a \in p^-1(W_o)$. So we have $g^{-1}(Z) \subseteq g^{-1}(Z_o) \cap p^{-1}(W_o)$.

Similarly, claim $p^{-1}(W) = g^{-1}(Z_o) \cap p^{-1}(W_o)$. Certainly $p^{-1}(W) \subseteq p^{-1}(W_o)$. Then let $a \in p^{-1}(W)$. We need to show $a \in g^{-1}(Z_o)$. If $a \notin g^{-1}(Z_o)$, then $a \in p^{-1}(W_o)\backslash g^{-1}(Z_o)$, which implies $p(a) \in p(p^{-1}(W_o)\backslash g^{-1}(Z_o))$. But $p(p^{-1}(W_o)\backslash g^{-1}(Z_o)) \cap W = \emptyset$. So we have $a \in g^{-1}(Z_o)$. □

Lemma 3.7. *For Z and W with the properties described in Lemma 3.6, the sets satisfy:*

(i) $g^{-1}(W) = \emptyset$;

(ii) $p^{-1}(W) \neq \emptyset$ *and* $g^{-1}(Z) \neq \emptyset$.

Proof. Suppose $g^{-1}(W) \neq \emptyset$, then there is some element $c \in g^{-1}(W)$, that is, $g(c) \in W$. Then by Quasi-equation 3.2, $g(c) = g^2(c) \in g(W)$. By Lemma 3.5, $g(W) \subseteq Z$, so $g(c) \in g(W) \subseteq Z$. Therefore, $g(c) \in Z$ which implies that $g(c) \in W \cap Z$, a contradiction since $W \cap Z = \emptyset$. Therefore, $g^{-1}(W) = \emptyset$.

Now by Lemma 3.5, we have $1 \in W$ and $0 \in Z$. It follows from Quasi-equation 3.1, that $p(0) \approx 1$ and $1 \in W$. So $0 \in p^{-1}(1) \subseteq p^{-1}(W)$. Similarly, $g(1) \approx 0$ and $0 \in Z$. Then $g^{-1}(1) \subseteq g^{-1}(Z)$ and we have $1 \in g^{-1}(Z)$. Therefore $p^{-1}(W)$ and $g^{-1}(Z)$ are non-empty. □

An additional set that is part of the clopen cover of $\mathbb{X}$ is $L = p^{-1}(W) \backslash (Z \cup W)$.

Lemma 3.8. *The sets L, Z and W satisfy*

(i) $p(L) \subseteq W$ *and* $g(L) \subseteq Z$;

(ii) $p(p^{-1}(L)) \subseteq L$ *and* $g(p^{-1}(L)) \subseteq p^{-1}(L)$;

(iii) $g^{-1}(L) = \emptyset$.

Proof. We have $L \subseteq p^{-1}(W) = g^{-1}(Z)$, so $L \subseteq p^{-1}(W)$ and $L \subseteq g^{-1}(Z)$. Then it follows that $p(L) \subseteq W$ and $g(L) \subseteq Z$. As $p^{-1}(L) \subseteq p^{-1}(L)$, by Lemma 3.2, $g(p^{-1}(L)) \subseteq p^{-1}(L)$.

Now, assume $e \in g^{-1}(L)$. Then $g(e) \in L$ and we have $g(e) \in p^{-1}(W)$ and $g(e) \notin (Z \cup W)$. But $p^{-1}(W) = g^{-1}(Z)$, so $g(e) \in g^{-1}(Z)$ and $g^2(e) \in Z$. Quasi-equation 3.2 implies $g(e) \in Z$, a contradiction since $g(e) \notin (Z \cup W)$. Therefore, $g^{-1}(L) = \emptyset$. □

Lemma 3.9. *Z satisfies, $p(p^{-1}(Z)) \subseteq Z$ and $g(p^{-1}(Z)) \subseteq p^{-1}(Z)$.*

In addition,

(i) $p^{-1}(Z) \cap (Z \cup W) = \emptyset$;

(ii) $p^{-1}(Z) \cap L = \emptyset$.

Proof. If $p^{-1}(Z) = \emptyset$, then the claims are vacuously true, so assume $p^{-1}(Z) \neq \emptyset$. As $p^{-1}(Z) \subseteq p^{-1}(Z)$, it follows that $p(p^{-1}(Z)) \subseteq Z$, and by part (i) of Lemma 3.2, $g(p^{-1}(Z)) \subseteq p^{-1}(Z)$. By Quasi-equation 3.5, $pg(p^{-1}(Z)) = p(p^{-1}(Z)) \subseteq Z$.

Since $p(p^{-1}(Z)) \subseteq Z$ and we know $p(Z) \subseteq W$ and $p(W) \subseteq W$, then $p(Z \cup W) \subseteq W$. But this implies $p(p^{-1}(Z) \cap (Z \cup W)) \subseteq p(p^{-1}(Z)) \cap p(Z \cup W) \subseteq Z \cap W$, but $Z \cap W = \emptyset$. Therefore $p^{-1}(Z) \cap (Z \cup W) = \emptyset$ as claimed.

Now $L = p^{-1}(W) \backslash (Z \cup W)$. With $Z \cap W = \emptyset$, by Lemma 3.3, we get $p^{-1}(Z) \cap p^{-1}(W) = \emptyset$. □

Accordingly, there are disjoint clopen sets around the elements 0 and 1 in which particular points are separated from 0 and 1, such that

$$\mathbb{X} = p^{-1}(L) \mathbin{\dot\cup} L \mathbin{\dot\cup} Z \mathbin{\dot\cup} W \mathbin{\dot\cup} p^{-1}(Z).$$

and the clopen sets satisfy the quasi-equational theory of $\mathbb{M}$.

The next four sections illustrate the construction of the four partitions of $\mathbb{X}$.

3.1 Partition 1

In this section the first non-trivial partition of $\mathbb{X}$ into clopen sets is constructed. This partition is larger than the base partition and is illustrated in Figure 3.5.

Theorem 3.10. *Assume $a, b \in \mathbb{X}$ with $\{a, b\}$ disjoint from $\{0, 1\}$ and $a \neq b$, and a, b satisfy one of 1, 2, or 3, below. Then there exist disjoint clopen sets, A, B, Z, W and L, such that Z, W and L satisfy the properties of Lemma 3.6 to Lemma 3.9, and $\mathbb{X}$ can be partitioned as*

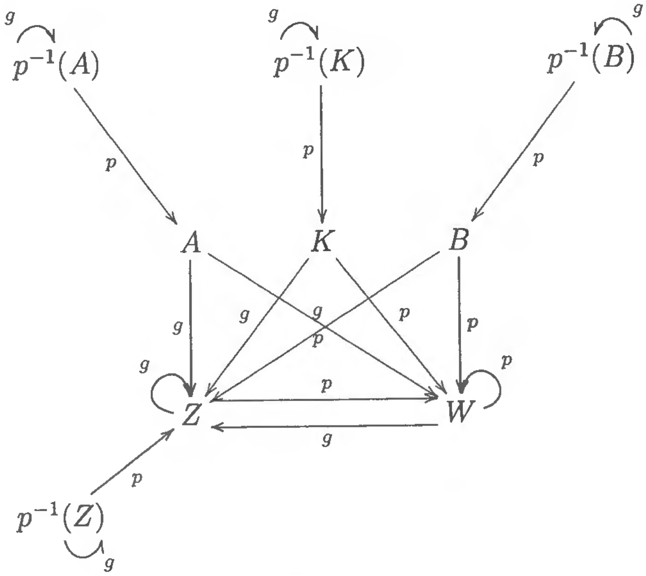

Figure 3.5: Partition 1

$$\mathbb{X} = Z \,\dot\cup\, W \,\dot\cup\, A \,\dot\cup\, B \,\dot\cup\, K \,\dot\cup\, p^{-1}(A) \,\dot\cup\, p^{-1}(B) \,\dot\cup\, p^{-1}(K) \,\dot\cup\, p^{-1}(Z),$$

where the partition separates $a, b, 0$ and 1, and the sets satisfy the quasi-equations of $\mathbb{M}$.

1. $p(a) = 1$ *and* $g(a) \neq a$, *with either*

   (a) $p(b) = 1$ *and* $g(b) \neq b$; *or*

   (b) $p(b) \notin \{0, 1, a\}$, $g(b) = b$; *or*

   (c) $p(b) \notin \{0, 1, a\}$, $g(b) \neq b$.

2. $p(a) \notin \{0, 1\}$, $g(a) = a$ *and* $p(a) \neq p(b)$, *with either*

   (a) $p(b) \notin \{0, 1\}$, $g(b) = b$; *or*

   (b) $p(b) \notin \{0, 1\}$, $g(b) \notin \{a, b\}$.

3. $p(a) \notin \{0, 1\}$ *and* $g(a) \neq a$, *with* $p(b) \notin \{0, 1\}, g(b) \neq b$ *and* $p(a) \neq p(b)$.

Cases 1, 2, and 3 set out in Theorem 3.10 can be rephrased in terms of whether $a \in L$, $p(a) \in L$, $b \in L$ or $p(b) \in L$.

Throughout the remainder of this section, we may assume $a \neq b$. The proof of Theorem 3.10 requires the construction of clopen sets A and B, as follows.

Lemma 3.11. *If $\{a, p(a)\} \cap L \neq \emptyset$ and $\{b, p(b)\} \cap L \neq \emptyset$ then $\mathbb{X}$ can be partitioned as Partition 1, that is,*

$$\mathbb{X} = Z \mathbin{\dot\cup} W \mathbin{\dot\cup} A \mathbin{\dot\cup} B \mathbin{\dot\cup} K \mathbin{\dot\cup} p^{-1}(A) \mathbin{\dot\cup} p^{-1}(B) \mathbin{\dot\cup} p^{-1}(K) \mathbin{\dot\cup} p^{-1}(Z)$$

where each set is clopen and where a and b are in distinct sets in this partition.

Proof. We construct Z, W and L as described in Lemmas 3.5 to 3.9 starting with the assumption $a, b \notin Z_o \cup W_o$, so $a, b \notin Z \cup W$. If $a \in L$ then $p(a) \in p(L) \subseteq W$. Since $L \cap W \neq \emptyset$, the set $\{a, p(a)\} \cap L$ has exactly one element. Similarly, $\{b, p(b)\} \cap L$ has exactly one element. Let $c \in \{a, p(a)\} \cap L$ and $d \in \{b, p(b)\} \cap L$.

There exist disjoint clopen sets $A \subseteq L$, and $B \subseteq L$ with $c \in A$ and $d \in B$. By Lemma 3.8, $L \cap (Z \cup W) = \emptyset$. Then since $A \subseteq L$ and $B \subseteq L$, we have $A \cap Z = \emptyset$, $A \cap W = \emptyset$, $B \cap Z = \emptyset$ and $B \cap W = \emptyset$.

Let $K = L \backslash (A \cup B)$, then $p^{-1}(L) = p^{-1}(A) \mathbin{\dot\cup} p^{-1}(B) \mathbin{\dot\cup} p^{-1}(K)$. The sets A, B and K satisfy the following:

1. $p(A) \subseteq W$;

2. $g(A) \subseteq Z$;

3. $p(B) \subseteq W$;

4. $g(B) \subseteq Z$;

5. $p^{-1}(A) \cap p^{-1}(B) = \emptyset$;

6. $p^{-1}(A) \subseteq p^{-1}(L)$ and $p^{-1}(B) \subseteq p^{-1}(L)$;

7. $p(p^{-1}(A)) \subseteq A$ and $p(p^{-1}(B)) \subseteq B$;

8. $g(p^{-1}(A)) \subseteq p^{-1}(A)$ and $g(p^{-1}(B)) \subseteq p^{-1}(B)$;

9. $g^{-1}(A) = \emptyset$ and $g^{-1}(B) = \emptyset$;

10. $p^{-1}(Z) \cap (A \cup B) = \emptyset$;

11. $p(K) \subseteq W$ and $g(K) \subseteq Z$;

12. $p(p^{-1}(K)) \subseteq K$ and $g(p^{-1}(K)) \subseteq p^{-1}(K)$;

13. $g^{-1}(K) = \emptyset$.

The proof of these thirteen claims now follows.

As $A \subseteq L = p^{-1}(W)\backslash(Z \cup W)$ implies $A \subseteq p^{-1}(W)$, it follows that $p(A) \subseteq W$. Also, $A \subseteq p^{-1}(W) = g^{-1}(Z)$ implies $A \subseteq g^{-1}(Z)$. Thus $g(A) \subseteq Z$. Similarly, $p(B) \subseteq W$ and $g(B) \subseteq Z$.

Since $A \cap B = \emptyset$, Lemma 3.3 gives $p^{-1}(A) \cap p^{-1}(B) = \emptyset$. Given $A \subseteq L$ and $B \subseteq L$, it follows that $p^{-1}(A) \subseteq p^{-1}(L)$ and $p^{-1}(B) \subseteq p^{-1}(L)$. Then As $p^{-1}(A) \subseteq p^{-1}(A)$ we have $p(p^{-1}(A)) \subseteq A$ and by Lemma 3.2, $g(p^{-1}(A)) \subseteq p^{-1}(A)$. Similarly, $p(p^{-1}(B)) \subseteq B$ and $g(p^{-1}(B)) \subseteq p^{-1}(B)$.

Now, if $g^{-1}(A) \neq \emptyset$, then there is an $e \in g^{-1}(A)$ with $g(e) \in A \subseteq L$. So $g(e) \in L = g^{-1}(Z)\backslash(Z \cup W)$, and $g^2(e) \in Z$. Then Quasi-equation 3.2 implies $g(e) \in Z$, which is a contradiction, since $Z \cap L = \emptyset$. Similarly, we get $g^{-1}(B) = \emptyset$.

By Lemma 3.9, $p^{-1}(Z) \cap L = \emptyset$, and with $A \subseteq L$ and $B \subseteq L$, then $p^{-1}(Z) \cap (A \cup B) = \emptyset$, and further, $p^{-1}(Z)$ satisfies all of the properties of Lemma 3.9.

Using $K \subseteq p^{-1}(W) = g^{-1}(Z)$ gives $K \subseteq p^{-1}(W)$ and $K \subseteq g^{-1}(Z)$. Then it follows that $p(K) \subseteq W$ and $g(K) \subseteq Z$. As $p^{-1}(K) \subseteq p^{-1}(K)$, then $p(p^{-1}(K)) \subseteq K$

and by Lemma 3.2, $g(p^{-1}(K)) \subseteq p^{-1}(K)$. With $K \subseteq L$, by Lemma 3.8, $g^{-1}(L) = \emptyset$, and then we get $g^{-1}(K) = \emptyset$.

The element a is either in A or in $p^{-1}(A)$, and the element b is either in B or in $p^{-1}(B)$ and these are all pairwise disjoint as required. □

We have partitioned $\mathbb{X}$ into clopen sets as shown in Figure 3.5 and we can now complete the proof of Theorem 3.10. Recall that Theorem 3.10 states the following:

Theorem 3.10. Assume $a, b \in \mathbb{X}$ with $\{a, b\}$ disjoint from $\{0, 1\}$ and $a \neq b$, and a, b satisfy one of 1, 2, or 3, below. Then there exist disjoint clopen sets, A, B, Z, W and L, such that Z, W and L satisfy the properties of Lemma 3.6 to Lemma 3.9, and $\mathbb{X}$ can be partitioned as

$$\mathbb{X} = Z \mathbin{\dot\cup} W \mathbin{\dot\cup} A \mathbin{\dot\cup} B \mathbin{\dot\cup} K \mathbin{\dot\cup} p^{-1}(A) \mathbin{\dot\cup} p^{-1}(B) \mathbin{\dot\cup} p^{-1}(K) \mathbin{\dot\cup} p^{-1}(Z),$$

where the partition separates $a, b, 0$ and 1, and the sets satisfy the quasi-equations of $\mathbb{M}$.

1. $p(a) = 1$ and $g(a) \neq a$, with either
   (a) $p(b) = 1$ and $g(b) \neq b$; or
   (b) $p(b) \notin \{0, 1, a\}$, $g(b) = b$; or
   (c) $p(b) \notin \{0, 1, a\}$, $g(b) \neq b$.
2. $p(a) \notin \{0, 1\}$, $g(a) = a$ and $p(a) \neq p(b)$, with either
   (a) $p(b) \notin \{0, 1\}$, $g(b) = b$; or
   (b) $p(b) \notin \{0, 1\}$, $g(b) \notin \{a, b\}$.
3. $p(a) \notin \{0, 1\}$ and $g(a) \neq a$, with $p(b) \notin \{0, 1\}, g(b) \neq b$ and $p(a) \neq p(b)$.

Proof. Recall that W and Z may be chosen so that $1 \in W$ and $0 \in Z$. For $c \in \{a, b, p(a), p(b)\}$, with $c \neq 1$, then $c \notin W$, and for $d \in \{a, b, g(a), g(b)\}$ with $d \neq 0$, then $d \notin Z$. We have assumed $a, b \notin \{0, 1\}$ and $a \neq b$. The proof of Theorem 3.10 relies on showing $\{a, p(a)\}$ and $\{b, p(b)\}$ intersect L. Then Lemma 3.11 may be used.

In part 1(a), we have $p(a) = 1$, by Quasi-equation 3.7 $g(a) = 0$, so $g(a) \neq a$. Then $p(a) = 1$ implies $p(a) \in W$ and $a \in p^{-1}(W)$, so $a \in L$. Similarly, if $p(b) = 1$, and $g(b) \neq b$, we have $b \in L$.

In parts 1(b) and 1(c) when $p(a) = 1$ and $b \notin \{0, 1\}$, we have $a \in L$. But $b \notin L$ since if $b \in L$, then we have $b \in p^{-1}(W)$, giving $p(b) \in W$, which is a contradiction. Now $p^2(b) = 1$ and $1 \in W$, so this implies $p(b) \in p^{-1}(1) \subseteq p^{-1}(W)$, which implies $p(b) \in p^{-1}(W) \backslash (Z \cup W) = L$. If $g(b) = b$ as in part 1(b), or $g(b) \neq b$ as in part 1(c), with $b \in p^{-1}(L)$, we have $g(b) \in g(p^{-1}(L)) \subseteq p^{-1}(L)$ by Lemma 3.8. Then $a \in L$ and $p(b) \in L$.

In parts 2(a) and 2(b), if $p(a) = p(b)$, by Quasi-equation 3.9, $g(a) = g(b)$. But this implies $a = g(a) = g(b) = b$, a contradiction. Therefore $p(a) \neq p(b)$. With $g(a) = a$, and $g(b) = b$ in part 2(a), or $g(b) \neq b$ in part 2(b), we choose $a \in p^{-1}(A)$ and $b \in p^{-1}(B)$. By Lemma 3.11, $p^{-1}(A) \subseteq p^{-1}(L)$ and $p^{-1}(B) \subseteq p^{-1}(L)$. Accordingly, $p(a) \in L$ and $p(b) \in L$.

In part 3, we have $p(a) \neq p(b)$ as in part 2, and we again choose $a \in p^{-1}(A)$ and $b \in p^{-1}(B)$. Then $p(a) \in L$ and $p(b) \in L$ and Lemma 3.11 may be applied.

Accordingly, $\mathbb{X}$ can be partitioned into disjoint clopent sets in which particular points are separated, such that,

$$\mathbb{X} = Z \,\dot{\cup}\, W \,\dot{\cup}\, A \,\dot{\cup}\, B \,\dot{\cup}\, K \,\dot{\cup}\, p^{-1}(A) \,\dot{\cup}\, p^{-1}(B) \,\dot{\cup}\, p^{-1}(K) \,\dot{\cup}\, p^{-1}(Z)$$

and the clopen sets satisfy the quasi-equational theory of $\mathbb{M}$.

□

3.2 Partition 2

This section defines the second partition of $\mathbb{X}$ into clopen sets, which is illustrated in Figure 3.6.

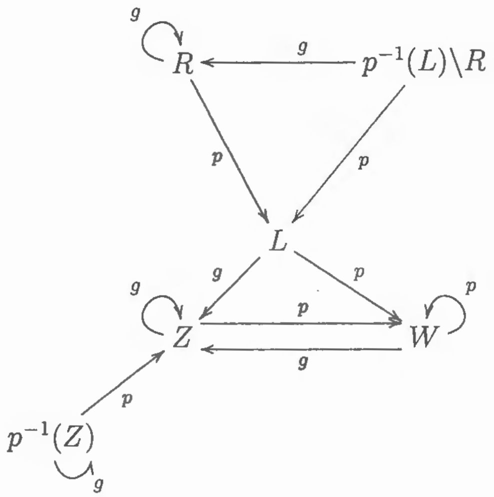

Figure 3.6: Partition 2

Theorem 3.12. *For $a, b \in \mathbb{X}$ with $\{a, b\}$ disjoint from $\{0, 1\}$ and $a \neq b$, when $p(a) \notin \{0,1\}$, $g(a) = a$, $p(b) \notin \{0,1\}$, $g(b) = b$ and $p(a) = p(b)$, there exist disjoint clopen sets Z, W, L and R, such that Z, W and L satisfy the properties of Lemmas 3.6 to Lemma 3.9, and $a, b \notin Z \cup W$, and $\mathbb{X}$ can be partitioned as Partition 2, that is,*

$$\mathbb{X} = Z \mathbin{\dot\cup} W \mathbin{\dot\cup} L \mathbin{\dot\cup} p^{-1}(Z) \mathbin{\dot\cup} R \mathbin{\dot\cup} (p^{-1}(L)\backslash R)$$

where these sets separate $0, 1, a$ and b and satisfy the quasi-equations of $\mathbb{M}$.

The proof of this Theorem requires the construction of a clopen set R as in the following Lemma.

Lemma 3.13. *There exists a clopen set $R \subseteq p^{-1}(L)$ such that $g(p^{-1}(L)) \subseteq R \subseteq p^{-1}(L)$ and $b \notin R$.*

Proof. By Lemma 3.8, $g(p^{-1}(L)) \subseteq p^{-1}(L)$. If $b \in g(p^{-1}(L))$ then $b = g(c)$, where $c \in p^{-1}(L)$. Thus $a = g(b) = g^2(c) = g(c) = b$, a contradiction. Since $\mathbb{X}$ is a totally disconnected Hausdorff space, there is a clopen set R with $g(p^{-1}(L)) \subseteq R \subseteq p^{-1}(L)$ and $b \notin R$. ☐

Lemma 3.14. *If $p(a) \in L$ and $p(b) \in L$, and $g(a) = a$, then $\mathbb{X}$ can be partitioned as*

$$\mathbb{X} = Z \mathbin{\dot{\cup}} W \mathbin{\dot{\cup}} L \mathbin{\dot{\cup}} p^{-1}(Z) \mathbin{\dot{\cup}} R \mathbin{\dot{\cup}} (p^{-1}(L)\backslash R).$$

Proof. We have disjoint clopen sets Z, W, L and $p^{-1}(Z)$ satisfying the properties of Lemma 3.6 to Lemma 3.9, specifically, $p(L) \subseteq W$, $g(L) \subseteq Z$, $p(p^{-1}(Z)) \subseteq Z$ and $g(p^{-1}(Z)) \subseteq p^{-1}(Z)$. Further, the sets R and $p^{-1}(L)\backslash R$ satisfy the following:

1. $p(R) \subseteq L$;

2. $g(R) \subseteq R$;

3. $g(p^{-1}(L)\backslash R) \subseteq R$;

4. $p(p^{-1}(L)\backslash R) \subseteq L$.

The set $R \subseteq p^{-1}(L)$, and it follows that $p(R) \subseteq p(p^{-1}(L)) \subseteq L$, and $g(R) \subseteq g(p^{-1}(L) \subseteq R$. Since $g(p^{-1}(L)) \subseteq R$ and $p(p^{-1}(L)) \subseteq L$, it also follows that $g(p^{-1}(L)\backslash R) \subseteq R$ and $p(p^{-1}(L)\backslash R) \subseteq L$.

☐

We will now state the proof of Theorem 3.12. Recall that Theorem 3.12 states:

Theorem 12. For $a, b \in \mathbb{X}$ with $\{a, b\}$ disjoint from $\{0, 1\}$ and $a \neq b$, when $p(a) \notin \{0, 1\}$, $g(a) = a$, $p(b) \notin \{0, 1\}$, $g(b) = b$ and $p(a) = p(b)$, there exist disjoint clopen sets Z, W, L and R, such that Z, W and L satisfy the properties of Lemmas 3.6 to Lemma 3.9, and $a, b \notin Z \cup W$, and $\mathbb{X}$ can be partitioned as Partition 2, that is,

$$\mathbb{X} = Z \,\dot\cup\, W \,\dot\cup\, L \,\dot\cup\, p^{-1}(Z) \,\dot\cup\, R \,\dot\cup\, (p^{-1}(L) \backslash R)$$

where these sets separate $0, 1, a$ and b and satisfy the quasi-equations of $\mathbb{M}$.

Proof. Given that $g(b) = a$, then $pg(b) = p(a)$ and by Quasi-equation 3.5, $pg(b) = p(b)$, so we have $p(a) = p(b)$. Now $p^2(a) = 1 \in W$, so $p(a) \in p^{-1}(W) \backslash (Z \cup W) = L$ and $a \in p^{-1}(L)$. As $p(a) = p(b)$, it follows that $b \in p^{-1}(L)$. Since $a \in p^{-1}(L)$, then $g(a) \in g(p^{-1}(L))$ and with $g(a) = a$, then $a \in g(p^{-1}(L) \subseteq R$. If $b \in p^{-1}(L)$, then $g(b) \in g(p^{-1}(L)$, but $g(b) \neq b$, so $b \notin R$, and $b \in p^{-1}(L) \backslash R$. Accordingly, $p(a) \in L$ and $p(b) \in L$ and Lemma 3.14 can be applied to obtain Partition 2 as a partition of $\mathbb{X}$.

Accordingly, $\mathbb{X}$ can be partitioned into disjoint clopen sets in which particular points are separated, such that

$$\mathbb{X} = Z \,\dot\cup\, W \,\dot\cup\, L \,\dot\cup\, p^{-1}(Z) \,\dot\cup\, R \,\dot\cup\, (p^{-1}(L) \backslash (R).$$

and the clopen sets satisfy the quasi-equational theory of $\mathbb{M}$. □

3.3 Partition 3

The third partition of clopen sets of $\mathbb{X}$ is constructed in this section and is illustrated in Figure 3.7.

Theorem 3.15. *For $a, b \in \mathbb{X}$ with $\{a, b\}$ disjoint from $\{0, 1\}$ and $a \neq b$, when $p(a) \notin \{0, 1\}$, $g(a) \neq a$ and $p(b) \notin \{0, 1\}$, $g(b) \neq b$ and $p(a) = p(b)$, there exists disjoint clopen sets Z, W, L, C, D and R, such that Z, W and L satisfy the*

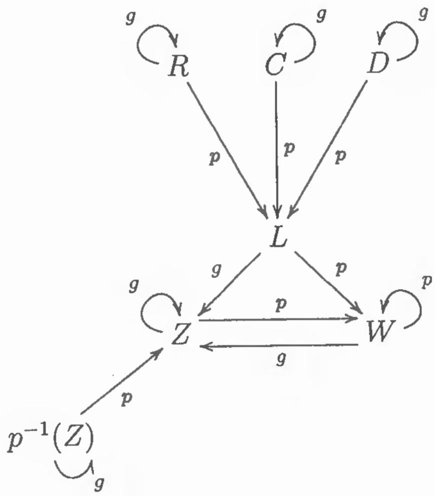

Figure 3.7: Partition 3

properties of Lemma 3.6 to Lemma 3.9, and $a, b \notin Z \cup W$*, and* $\mathbb{X}$ *can be partitioned as Partition 3, that is,*

$$\mathbb{X} = Z \,\dot{\cup}\, W \,\dot{\cup}\, L \,\dot{\cup}\, C \,\dot{\cup}\, D \,\dot{\cup}\, R \,\dot{\cup}\, p^{-1}(Z)$$

where these sets separate $0, 1, a$ *and* b *and satisfy the quasi-equations of* $\mathbb{M}$.

The proof of this Theorem follows the construction of the clopen sets C and D.

By Lemma 2.5, there are clopen sets C' and D' with $a \in C'$, $b \in D'$ and $C' \cap D' = \emptyset$. Then the case set out in Theorem 3.17 can be rephrased in terms of whether or not $a \in p^{-1}(L)$ and $b \in p^{-1}(L)$.

Lemma 3.16. *If* $p(a) \in L$ *and* $p(b) \in L$*, and* $g(a) \neq a$*, then* $\mathbb{X}$ *can be partitioned as*

$$\mathbb{X} = Z \,\dot{\cup}\, W \,\dot{\cup}\, L \,\dot{\cup}\, C \,\dot{\cup}\, D \,\dot{\cup}\, R \,\dot{\cup}\, p^{-1}(Z).$$

Proof. Choose the sets Z, W, L and $p^{-1}(Z)$ satisfying the properties of Lemma 3.6 to Lemma 3.9, more specifically, $p(L) \subseteq W$, $g(L) \subseteq Z$, $p(p^{-1}(Z)) \subseteq Z$ and $g(p^{-1}(Z))$ $\subseteq p^{-1}(Z)$.

By Lemma 3.5, we may assume $p(a) \notin Z \cup W$. This implies $p(a) \in L$. Given that $g(a) \neq a$, and $g(a) \in g(p^{-1}(L))$, then $a \notin g(p^{-1}(L))$, since if $a = g(a)$ with $a \in p^{-1}(L)$, then we have $p(a) \in L$. This gives $a = g(a) \in g(l) \subseteq Z$, a contradiction. Accordingly, there exists a clopen set $R \subseteq p^{-1}(L)$ with $g(p^{-1}(L)) \subseteq R$ and $a, b \notin R$.

In addition, there exist disjoint clopen sets C and D defined as $C = C' \cap p^{-1}(L)$ and $D = D' \cap p^{-1}(L)$, with $a \in C$ and $b \in D$, so $C \subseteq p^{-1}(L)\backslash R$ and $D = p^{-1}(L)\backslash(R \cup C)$. The sets C, D and R satisfy the following:

(i) $p(C) \subseteq L$ and $p(D) \subseteq L$;

(ii) $g(C) \subseteq g(p^{-1}(L)) \subseteq R$ and $g(D) \subseteq g(p^{-1}(L)) \subseteq R$;

(iii) $p(R) \subseteq L$;

(iv) $g(R) \subseteq g(p^{-1}(L)) \subseteq R$.

Since $C \subseteq p^{-1}(L)$, $p(C) \subseteq L$. Then $g(C) \subseteq g(p^{-1}(L)) \subseteq R$. Similarly, $p(D) \subseteq L$ and $g(D) \subseteq g(p^{-1}(L)) \subseteq R$. Now, $R \subseteq p^{-1}(L)$, so $p(R) \subseteq p(p^{-1}(L)) \subseteq L$, and $g(R) \subseteq g(p^{-1}(L)) \subseteq R$. □

We can now proceed to the proof of Theorem 3.15.

Proof. Given $g(a) \neq a$ and $g(b) \neq b$ and $p(a) = p(b)$, Quasi-equation 3.9, implies that $g(a) = g(b)$. We have disjoint clopen sets C and D, so we choose $a \in C$ and $b \in D$. Since $a \in C \subseteq p^{-1}(L)$, and $b \in D \subseteq p^{-1}(L)$, we have $a \in p^{-1}(L)$, so $p(a) \in L$ and $b \in p^{-1}(L)$. Therefore $p(b) \in L$ and Lemma 3.16 can be applied to obtain Partition 3 as a partition of $\mathbb{X}$.

Accordingly, $\mathbb{X}$ can be partitioned into disjoint clopen sets, in which particular points are separated, such that

$$\mathbb{X} = Z \,\dot{\cup}\, L \,\dot{\cup}\, C \,\dot{\cup}\, D \,\dot{\cup}\, R \,\dot{\cup}\, p^{-1}(Z)$$

and the clopen sets satisfy the quasi-equational theory of $\mathbb{M}$. $\square$

3.4 Base Partition

The base partition of $\mathbb{X}$ is illustrated in Figure 3.8. All cases not consider in previous partitions use this partition.

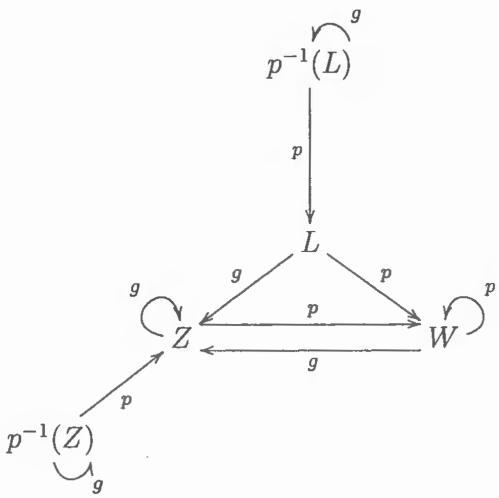

Figure 3.8: Base Partition

Theorem 3.17. *For $a, b \in \mathbb{X}$ and $a \neq b$, when a and b are as set out in the following cases, $\mathbb{X}$ can be partitioned as follows:*

$$\mathbb{X} = Z \,\dot\cup\, W \,\dot\cup\, L \,\dot\cup\, p^{-1}(L) \,\dot\cup\, p^{-1}(Z).$$

1. $a = 1$, $p(a) = 1$, $g(a) \neq a$, *with either*

   (a) $b = 0$, $p(b) = 1$ *and* $g(b) = b$*; or*

   (b) $b \notin \{0, 1\}$, $p(b) = 1$, $g(b) \neq b$*; or*

   (c) $b \notin \{0, 1\}$, $p(b) \notin \{0, 1\}$, $g(b) = b$*; or*

   (d) $b \notin \{0, 1\}$, $p(b) \notin \{0, 1\}$, $g(b) \neq b$*; or*

(e) $b \notin \{0,1\}$, $p(b) = 0$, $g(b) = b$.

2. $a = 0$, $p(a) = 1$, $g(a) = a$, *and* $b \notin \{0,1\}$, *with either*

   (a) $p(b) = 1$, $g(b) \neq b$; *or*

   (b) $p(b) \notin \{0,1\}$, $g(b) = b$; *or*

   (c) $p(b) \notin \{0,1\}$, $g(b) \neq b$; *or*

   (d) $p(b) = 0$, $g(b) = b$.

3. $a \notin \{0,1\}$ $p(a) = 1$, $g(a) \neq a$ *and* $b \notin \{0,1\}$, *with either*

   (a) $p(b) \notin \{0,1\}$, $p(b) = a$, $g(b) = b$; *or*

   (b) $p(b) \notin \{0,1\}$, $p(b) = a$, $g(b) \neq b$; *or*

   (c) $p(b) = 0$, $g(b) = b$.

4. $a \notin \{0,1\}$, $p(a) \notin \{0,1\}$ *and* $g(a) = a$, *with* $b \notin \{0,1\}$, $p(b) = 0$ *and* $g(b) = b$;

5. $a \notin \{0,1\}$, $p(a) \notin \{0,1\}$ *and* $g(a) \neq a$, *with* $b \notin \{0,1\}$, $p(b) = 0$, $g(b) = b$ *and* $p(a) \neq p(b)$;

Moreover, Z, W and L, such that Z, W and L satisfy the properties of Lemma 3.6 to Lemma 3.9, and $a, b \notin Z \cup W$, and these sets separate a and b and satisfy the quasi-equations of $\mathbb{M}$.

Proof. In each case, it is sufficient to show that a and b are in distinct sets in the collection Z, W, L, $p^{-1}(Z)$, $p^{-1}(L)$.

In part 1 we have $1 = a \in W$. In part 1(a), we have $0 = b$, so $0 \in Z$. In 1(b), 1(c) and 1(d), $b \notin \{0,1\}$ and by Lemma 3.2, we may assume $b \notin W$.

In part 2, we have $0 = a \in Z$. For 2(a), 2(b) and 2(c), $b \notin \{0,1\}$ so by Lemma 3.2 $b \notin Z$.

In part 3, we have $a \notin \{0,1\}$ which implies $a \notin Z \cup W$. We also have $p(a) = 1$ and $1 \in W$, so $a \in p^{-1}(W)$. Then by Lemma 3.8, $a \in L$. In part 3(a), we have $b \notin \{0,1\}$ but $p(b) = a \in L$, so $p^{-1}(p(b)) \in p^{-1}(L)$, which implies $b \in p^{-1}(L)$. In 3(b), $p(b) = 0 \in Z$ implies $p^{-1}(p(b)) \in p^{-1}(Z)$, so $b \in p^{-1}(Z)$.

In part 4, $a \notin \{0,1\}$ and $p(a) \notin \{0,1\}$ but $p(b) = 0$. Therefore $b \in p^{-1}(Z)$ and $a \notin p^{-1}(Z)$.

Similarly, in part 5, $b \in p^{-1}(Z)$ but $a \notin p^{-1}(Z)$.

□

3.5 The Proof of Theorem 3.1

In this section we complete the first proof of standardness, that is, the proof of Theorem 3.1. First, recall Theorem 3.1 states:

Theorem 3.1 The structure $\mathbb{M} = \langle 0, 1, 2; \mathcal{F}; \tau \rangle$ where $\mathcal{F} = \{002, 110\}$ and τ is the discrete topology, is standard.

Proof. By Theorem 3.4, there are six possible conditions for a and for b giving rise to 36 possible cases. By symmetry, we need consider 15 situations where a and b come from distinct cases. Of the six situations where a and b satisfy the same property, two are immediately excluded because they imply $a = b$. A third situation also reduces to $a = b$. Thus there are 18 non-trivial situations to be considered.

Recall the six possibilities for a:

(i) $a = 1$;

(ii) $a = 0$;

(iii) $a \notin \{0,1\}, p(a) = 1, g(a) \neq a$;

(iv) $\{a, p(a)\}$ disjoint from $\{0,1\}, g(a) = a$;

(v) $\{a, p(a)\}$ disjoint from $\{0,1\}, g(a) \neq a$;

(vi) $a \notin \{0,1\}, p(a) = 0, g(a) = a$.

In Case (i) for a and case (ii), (iii), (iv), (v) or (vi) for b, these cases are covered by Theorem 3.17, in parts 1(a) - 1(e). In Case (ii) for a and case (iii), (iv), (v) or (vi) for b, this is the situation covered by parts 2(a) - 2(d) of Theorem 3.17.

Now consider Case (iii) for a. Case (iii) for b is covered by part 1(a) of Theorem 3.10 and Case (vi) for b is covered by part 3(c) of Theorem 3.17. For Cases (iv) and (v) for b, there are two subcases that are independent of whether $g(b) = b$ or $g(b) \neq b$. If $p(b) \neq a$, Cases (iv) and (v) for b are covered by part 1(b) and (c) of Theorem 3.10, and if $p(b) = a$, then these cases are covered by parts 3(a) and (b) of Theorem 3.17.

Case (iv) for a and Case (iv) for b also has two subcases independent of whether $g(b) = b$ or $g(b) \neq b$. The first subcase, when $p(a) \neq p(b)$ is covered by part 2(a) of Theorem 3.10 and the second subcase when $p(a) = p(b)$ is covered by Theorem 3.12. Then Case (v) for b is covered by part 2(b) of Theorem 3.10 and Case (vi) for b is covered by part 4 of Theorem 3.17.

Similarly, in Case (v) for a and Case (v) for b, when $p(a) \neq p(b)$, then this case is covered by part 3 of Theorem 3.10, and when $p(a) = p(b)$, this is covered by Theorem 3.15. Then Case (vi) for b is covered by part 5 of Theorem 3.17.

Finally, consider Case (vi) for a and case (vi) for b. With $0 = p(a) = p(b)$, we have $g(a) = g(b)$, which gives $a = b$, and this case is not possible. $\square$

Given the algebra $\mathbb{M} = \langle \{0,1,2\}; \mathcal{F}; \tau \rangle$ where $\mathcal{F} = \{002, 110\}$ and τ is the discrete topology, there is a finite cover of disjoint clopen sets in which the particular points a and b are separated, that satisfies the quasi-equational theory of $\mathbb{M}$. Therefore, $\mathbb{M}$ is standard.

Chapter 4

Example 2

The second example that we show is standard is the structure $\mathbb{M}_2 = \langle\{0,1,2\}; \mathcal{F}; \tau\rangle$ where $\mathcal{F} = \{001, 002\}$ and τ is the discrete topology. When g denotes the operation 002 and r denotes the operation 001, the structure $\mathbb{M}_2$ is shown in Figure 4.1.

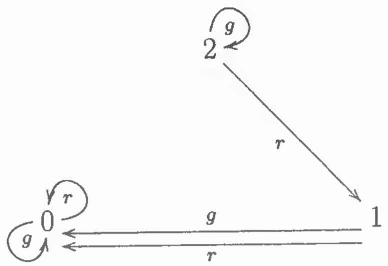

Figure 4.1: The Structure of $\mathbb{M}_2$

Applying g and r to the 27 elements of ${\mathbb{M}_2}^3$ yields Figure 4.2

The second result on standardness of unary algebras is the following theorem.

Theorem 4.1. *The structure* $\mathbb{M}_2 = \langle 0,1,2;\ \mathcal{F};\ \tau\rangle$ *where* $\mathcal{F} = \{001, 002\}$ *and* τ *is the discrete topology, is standard.*

The proof of Theorem 4.1 uses Theorem 2.4 to show that $\mathbb{M}_2$ is standard by showing that for each $\mathbb{X} \in \mathrm{Mod}_\tau \mathrm{Th}_{qe}(\mathbb{M})_2$, and each pair a, $b \in \mathbb{X}$ with $a \neq b$

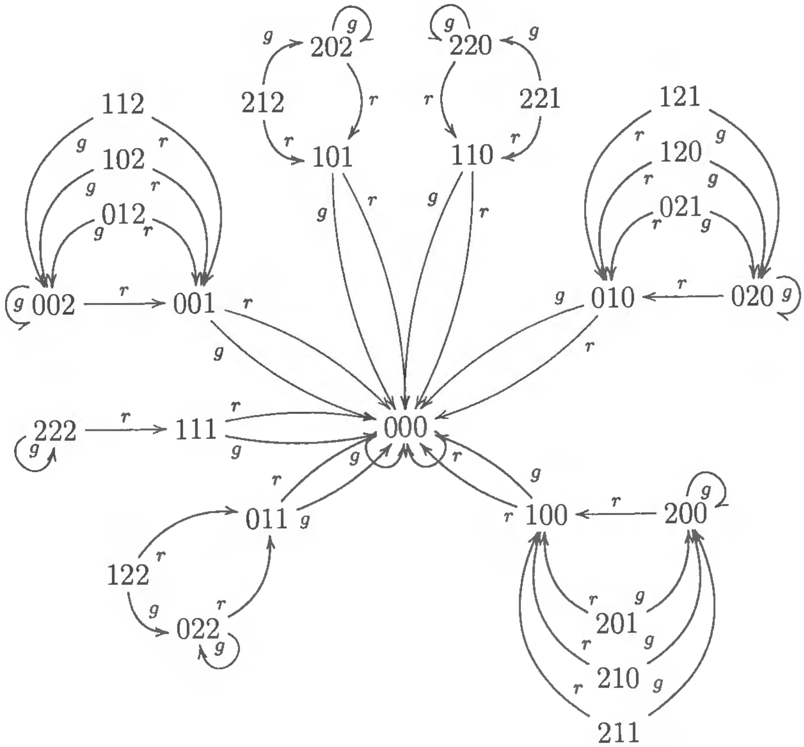

Figure 4.2: The Structure of $\mathbb{M}_2{}^3$

there is a partition of clopen sets that separates a and b such that the blocks of the partition satisfy the quasi-equations of $\mathbb{M}_2$.

Note that $\mathbb{M}_2$ has one constant valued function, 0.

The equations and quasi-equations satisfied by $\mathbb{M}_2$ include

$$r^n(x) \approx 0,\ n > 1 \tag{4.1}$$

$$g^n(x) \approx g(x),\ n > 1 \tag{4.2}$$

$$g^n r(x) \approx 0,\ n > 0 \tag{4.3}$$

$$rg(x) \approx r(x) \tag{4.4}$$

$$r^n g(x) \approx 0,\ n > 1 \tag{4.5}$$

$$r(x) \approx 0 \iff g(x) \approx 0 \tag{4.6}$$

$$g(x) \approx g(y) \iff r(x) \approx r(y) \tag{4.7}$$

$$g(x) \approx r(y) \Rightarrow g(x) \approx 0 \tag{4.8}$$

That these quasi-equations hold can be seen from Table 4.1.

x	r	g	r^2	g^2	rg	gr	rg^2	r^2g	g^2r	gr^2
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	1	2	0	2	1	0	1	0	0	0

Table 4.1: Operations of $\mathbb{M}_2$

Throughout the remainder of Chapter 4, $\mathbb{X}$ is a Boolean model of the quasi-equational theory of $\mathbb{M}_2$.

The proof of Theorem 4.1 requires Lemma 2.5, Lemma 2.6 and Lemma 2.7, together with the following Lemmas which provide useful properties of subsets of $\mathbb{X}$.:

Lemma 4.2. *Assume $A, B \subseteq \mathbb{X}$.*

(i) *If $A \subseteq r^{-1}(B)$, then $g(A) \subseteq r^{-1}(B)$;*

(ii) *If $A \subseteq g^{-1}(B)$, then $g(A) \subseteq g^{-1}(B)$;*

(iii) *If $A \subseteq r^{-1}(B)$ and $0 \in B$, then $r(A) \subseteq r^{-1}(B)$;*

(iv) *If $A \subseteq g^{-1}(B)$ and $0 \in B$, then $r(A) \subseteq g^{-1}(B)$.*

Proof. For property (i), assume $A \subseteq r^{-1}(B)$, so $r(A) \subseteq B$. By Quasi-equation 4.4, $r(A) = rg(A)$ so $rg(A) \subseteq B$. This implies $r^{-1}(rg(A)) \subseteq r^{-1}(B)$, and $g(A) \subseteq r^{-1}(A)$.

Similarly, for property (ii), if $A \subseteq g^{-1}(B)$ then $g(A) \subseteq B$ and by Quasi-equation 4.2, $g^2(A) \subseteq B$ and $g(A) \subseteq g^{-1}(B)$.

Now for property (iii), assume $A \subseteq r^{-1}(B)$ and $0 \in B$. By Quasi-equation 4.1, $r^n(x) \approx 0$. Then $\mathbb{X} = r^{-2}(B)$ and $A \subseteq r^{-2}(B)$. It then follows that $r(A) \subseteq r^{-1}(B)$.

Finally, for property (iv), if $A \subseteq g^{-1}(B)$ and $0 \in B$, by Quasi-equation 3.4, we have $gr(x) \approx 0$, so $\mathbb{X} = r^{-1}g^{-1}(0)$. Then $A \subseteq r^{-1}g^{-1}(B)$, which implies $r(A) \subseteq g^{-1}(B)$. □

Lemma 4.3. *Assume A, $B \in \mathbb{X}$ and $A \cap B = \emptyset$. Then $r^{-1}(A) \cap r^{-1}(B) = \emptyset$.*

Proof. Let $d \in r^{-1}(A) \cap r^{-1}(B)$. Then $r(d) \in A$ and $r(d) \in B$, but $A \cap B = \emptyset$, a contradiction. Therefore $r^{-1}(A) \cap r^{-1}(B) = \emptyset$. □

For each $a \in \mathbb{X}$, there are four possibilities and these are described in the next Theorem.

Theorem 4.4. *For each $a \in \mathbb{X}$, a Boolean model of the quasi-equational theory of $\mathbb{M}_2$, one of the following cases holds.*

(i) $a = 0$, $r(a) = 0$, $g(a) = 0$;

(ii) $a \neq 0$, $r(a) = 0$, $g(a) = 0$;

(iii) $a \neq 0$, $r(a) \neq 0$, $g(a) = a$;

(iv) $a \neq 0$, $r(a) \neq 0$, $g(a) \neq 0$, $g(a) \neq a$.

Proof. If $a = 0$, then Case (i) holds. Now assume $a \neq 0$. By Quasi-equation 4.1, $r^2(a) = 0$, so it is sufficient to consider the cases when $r(a) = 0$ or $r(a) \neq 0$. If $r(a) = 0$, then Quasi-equation 4.6 gives $g(a) = 0$, which is Case (ii). Now assume $r(a) \neq 0$. By Quasi-equation 4.6, $g(a) \neq 0$. Then by Quasi-equation 4.2,

$g^2(a) = g(a)$, so it is now sufficient to consider the cases when $g(a) = a$ or $g(a) \neq a$. The former condition gives Case (iii) and the latter gives Case (iv). $\square$

Based on the four cases in Theorem 4.4, for $a, b \in \mathbb{X}$ and $a \neq b$, there are potentially 16 cases. By symmetry, we need consider 9 cases that can be described by one of four partitions.

The possible partitions are:

$$Partition\ \ 1: \quad \mathbb{X} = Z \dot\cup A \dot\cup B \dot\cup K \dot\cup r^{-1}(A) \dot\cup r^{-1}(B) \dot\cup r^{-1}(K)$$

$$Partition\ \ 2: \quad \mathbb{X} = Z \dot\cup L \dot\cup R \dot\cup r^{-1}(L) \backslash R$$

$$Partition\ \ 3: \quad \mathbb{X} = Z \dot\cup L \dot\cup C \dot\cup D \dot\cup R$$

$$Partition\ \ 4: \quad \mathbb{X} = Z \dot\cup L \dot\cup r^{-1}(L)$$

Each of the 9 cases is covered by one of the four partitions. The cases and their partitions are shown in Table 4.2

	b			
a	1	2	3	4
1	-	4	4	4
2	-	1	1,4	1,4
3	-	-	1	1,2
4	-	-	-	1,3

Table 4.2: Four cases for a and b give rise to 9 situations (13 including subcases) using four partitions.

The following lemmas show the construction of clopen sets around the constant valued function 0. The illustration of these sets is shown in Figure 4.3.

The partition illustrated in Figure 4.3 underlies all other partitions constructed in this Section.

$$
\begin{tikzcd}
r^{-1}(L) \arrow[loop above, "g"] \arrow[d, "r"] \\
L \arrow[d, bend right, "g"'] \arrow[d, bend left, "r"] \\
Z \arrow[loop left, "r"] \arrow[loop right, "g"]
\end{tikzcd}
$$

Figure 4.3: Clopen Sets Around 0

Given $b \neq 0$, by Lemma 2.5 there are clopen sets U and V, with $0 \in U$, $b \in V$, and $U \cap V = \emptyset$. Construct Z_1 clopen with $0 \in Z_1$ and $b \notin Z_1$. If there is an element $a \in \mathbb{X}$ with $a \neq 0$, we may assume $\{a, b\}$ and Z_1 are disjoint.

Now define the set Z_2 as:

$$Z_2 = Z_1 \cap r^{-1}(Z_1) \cap g^{-1}(Z_1)$$

Lemma 4.5. *Z_2 satisfies $r(Z_2) \subseteq Z_2$ and $g(Z_2) \subseteq Z_2$.*

Proof. We have $r(Z_2) \subseteq r(r^{-1}(Z_1))$, so $r(Z_2) \subseteq Z_1$. Then since $0 \in Z_1$, we have $\mathbb{X} = r^{-2}(0) \subseteq r^{-2}(Z_1)$, so $Z_2 \subseteq r^{-2}(Z_1)$, that is, $r(Z_2) \subseteq r^{-1}(Z_1)$. By Quasi-equation 4.3, $gr(x) = 0 \in Z_1$. Therefore $gr(Z_2) \subseteq Z_1$, so $r(Z_2) \subseteq g^{-1}(Z_1)$, and accordingly we have $r(Z_2) \subseteq Z_2$.

Now $g(Z_2) \subseteq g(g^{-1}(Z_1)) \subseteq Z_1$. Therefore, $g(Z_2) \subseteq Z_1$, and $Z_2 \subseteq g^{-1}(Z_1)$. By part (ii) of Lemma 4.2 $g(Z_2) \subseteq g^{-1}(Z_1)$. By quasi-equation 4.4, $rg(Z_2) = r(Z_2) \subseteq Z_2$. Then $g(Z_2) \subseteq r^{-1}(Z_2) \subseteq r^{-1}(Z_1)$. Therefore $g(Z_2) \subseteq Z_2$. $\square$

Define the sets S and T as follows:

$$S = r(r^{-1}(Z_2)\backslash g^{-1}(Z_2))$$
$$T = g(r^{-1}(Z_2)\backslash g^{-1}(Z_2))$$

Lemma 4.6. *The sets $S \cup T$ and $g(g^{-1}(Z_2)) \cup r(g^{-1}(Z_2))$ are disjoint.*

Proof. Assume $a \in S \cap r(g^{-1}(Z_2))$. This implies $a = r(b)$ with $b \in r^{-1}(Z_2)\backslash g^{-1}(Z_2)$. Thus, $r(b) \in Z_2$ and $g(b) \notin Z_2$. However, $a = r(c)$ for some $c \in g^{-1}(Z_2)$, that is, $g(c) \in Z_2$. Since $r(b) = r(c)$, Quasi-equation 4.7 implies $g(b) = g(c)$. This is a contradiction as $g(b) \notin Z_2$ and $g(c) \in Z_2$.

Assume $a \in T \cap r(g^{-1}(Z_2))$. Now, $a \in T$ implies $a = g(b)$ for some $b \in r^{-1}(Z_2)\backslash g^{-1}(Z_2)$, with $r(b) \in Z_2$ and $g(b) \notin Z_2$. But $a = r(c)$ for some $c \in g^{-1}(Z_2)$, that is, $g(c) \in Z_2$ and we have $a = g(b) = r(c)$. Then Quasi-equation 4.8 implies $g(b) = r(c) = 0$. Since $g(b) \notin Z_2$, this is a contradiction.

Assume $a \in S \cap g(g^{-1}(Z_2))$. Again $a \in S$ implies $a = r(b)$ with $r(b) \in Z_2$ and $g(b) \notin Z_2$. But $a = g(c)$ for some $c \in g^{-1}(Z_2)$ so $g(c) \in Z_2$. By Quasi-equation 4.8 we have $a = r(b) = g(c) = 0$, a contradiction as this gives $a = 0$.

Assume $a \in T \cap g(g^{-1}(Z_2))$. Again $a \in T$ implies $a = g(b)$ for some b with $r(b) \in Z_2$ and $g(b) \notin Z_2$. But $a = g(c)$ with $c \in g^{-1}(Z_2)$, and $g(c) \in Z_2$. We have $a = g(b) = g(c)$ with $g(b) \notin Z_2$ and $g(c) \in Z_2$, a contradiction. □

Lemma 4.7. *Claim $g^{-1}(Z_2) \subseteq r^{-1}(Z_2)$*

Proof. Let $c \in g^{-1}(Z_2)$. Then $g(c) \in Z_2$ and $rg(c) \in r(Z_2) \subseteq Z_2$. By Quasi-equation 4.4, $rg(c) = r(c) \in r(Z_2) \subseteq Z_2$. So $r(c) \in Z_2$, which implies $c \in r^{-1}(Z_2)$. □

Since the sets referred to in Lemma 4.6 are all closed we may use Lemma 2.7 to find Z clopen such that $Z \subseteq Z_2$; Z and $S \cup T$ are disjoint; and

$$g(g^{-1}(Z_2)) \cup r(g^{-1}(Z_2)) \subseteq Z.$$

Note that 0 is in Z.

Lemma 4.8. *The following properties hold for Z:*

(i) $r(Z) \subseteq Z$;

(ii) $g(Z) \subseteq Z$; *and*

(iii) $g^{-1}(Z) = r^{-1}(Z)$.

Proof. Since $Z \subseteq Z_2$ and $Z_2 \subseteq g^{-1}(Z_2)$, we have $r(Z) \subseteq r(g^{-1}(Z_2))$. But by choice of Z, the latter set is contained in Z. Similarly $g(Z) \subseteq g(g^{-1}(Z_2))$ and the choice of Z implies $g(Z) \subseteq Z$.

Let $c \in g^{-1}(Z)$. Then $g(c) \in Z$ and $rg(c) \in r(Z) \subseteq Z$. By Quasi-equation 4.4, $rg(c) = r(c) \in r(Z) \subseteq Z$. So $r(c) \in Z$, which implies $c \in r^{-1}(Z)$. Therefore, $g^{-1}(Z) \subseteq r^{-1}(Z)$.

Notice that $g(g^{-1}(Z_2)) \subseteq Z$ implies $g^{-1}(Z_2) \subseteq g^{-1}(Z)$. However, $Z \subseteq Z_2$ implies $g^{-1}(Z) \subseteq g^{-1}(Z_2)$. Thus $g^{-1}(Z) = g^{-1}(Z_2)$.

To show $r^{-1}(Z) \subseteq g^{-1}(Z)$ assume there is an $a \in r^{-1}(Z) \backslash g^{-1}(Z)$. As $Z \subseteq Z_2$ and by the previous statement, we have $a \in r^{-1}(Z_2) \backslash (g^{-1}(Z_2)$. Hence, $r(a) \in S$ which is disjoint from Z. But the choice of a gives $r(a) \in Z$. This contradiction gives $r^{-1}(Z) \subseteq g^{-1}(Z)$ and we have $r^{-1}(Z) = g^{-1}(Z)$.

□

The behaviour of g and r on Z is illustrated in Figure 4.4.

Figure 4.4: The Element Z

If $\mathbb{X} = Z$ then $\mathbb{X}$ has been partitioned by clopen sets as required by Theorem 2.4. Now assume $Q = \mathbb{X} \backslash Z$ is non-empty. We need to construct more clopen sets that cover $\mathbb{X} \neq Z$. We may assume Z is chosen so that $r^{-1}(Z) = g^{-1}(Z)$.

Lemma 4.9. *Let $L = r^{-1}(Z) \backslash Z$, then*

(i) $g(L) \subseteq Z$ *and* $r(L) \subseteq Z$;

(ii) $g(r^{-1}(L)) \subseteq r^{-1}(L)$ *and* $r(r^{-1}(L)) \subseteq L$;

(iii) $g^{-1}(L) = \emptyset$.

Proof. We have $L \subseteq r^{-1}(Z) = g^{-1}(Z)$ so $L \subseteq r^{-1}(Z)$ and $L \subseteq g^{-1}(Z)$. Then it follows that $r(L) \subseteq Z$ and $g(L) \subseteq Z$. As $r^{-1}(L) \subseteq r^{-1}(L)$, by Lemma 4.2, $g(r^{-1}(L)) \subseteq r^{-1}(L)$.

Now assume $e \in g^{-1}(L)$. Then $g(e) \in L$, so $g(e) \in r^{-1}(Z)$ and $g(e) \notin Z$. But $r^{-1}(Z) = g^{-1}(Z)$, which implies $g(e) \in g^{-1}(Z)$ and $g^2(e) \in Z$. Quasi-equation 4.2 implies $g(e) \in Z$, a contradiction. Therefore, $g^{-1}(L) = \emptyset$. □

Accordingly, there is a disjoint clopen set around the element 0, such that

$$\mathbb{X} = r^{-1}(L) \,\dot{\cup}\, L \,\dot{\cup}\, Z$$

as previously illustrated in Figure 4.3.

Corollary 4.10. *For an element a, exactly one of the following hold.*

(i) a, $r(a) \in Z$; *or*

(ii) $a \in L$ *and* $r(a) \in Z$; *or*

(iii) $a \in r^{-1}(L)$ *and* $r(a) \in L$.

4.1 Partition 1

In this section the first non-trivial partition of $\mathbb{X}$ into clopen sets is constructed. This partition is larger than the base partition and is illustrated in Figure 4.5.

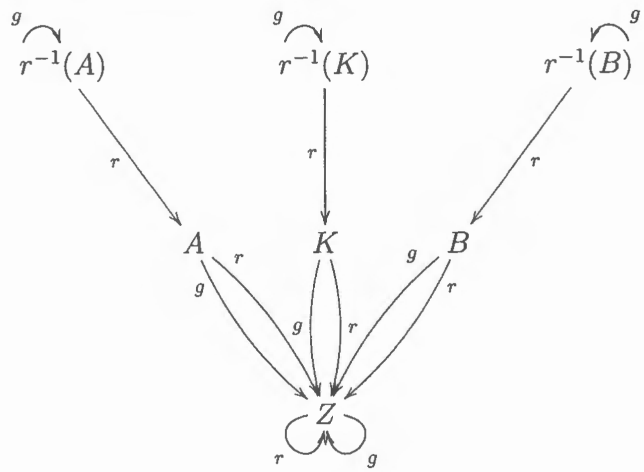

Figure 4.5: Partition 1

Theorem 4.11. *Assume $a, b \in \mathbb{X}$ with $\{a, b\}$ disjoint from $\{0\}$ and $a \neq b$, and a, b satisfying one of 1, 2 or 3 below. Then there exist disjoint clopen sets, A, B, Z, L and K, such that Z and L satisfy the properties of Lemmas 4.8 and 4.9, and $\mathbb{X}$ can be partitioned as,*

$$\mathbb{X} = Z \,\dot{\cup}\, A \,\dot{\cup}\, B \,\dot{\cup}\, K \,\dot{\cup}\, r^{-1}(A) \,\dot{\cup}\, r^{-1}(B) \,\dot{\cup}\, r^{-1}(K)$$

where the partition separates a, b and 0, and the sets satisfy the quasi-equations of $\mathbb{M}_2$.

1. $r(a) = 0$ *and* $g(a) = 0$, *with either*

(a) $r(b) = 0$ *and* $g(b) = 0$*; or*

(b) $r(b) \notin \{0, a\}$, $g(b) = b$ *and* $r(a) \neq r(b)$*; or*

(c) $r(b) \notin \{0, a\}$, $g(b) \notin \{0, b\}$ *and* $r(a) \neq r(b)$.

2. $r(a) \neq 0, g(a) \neq 0$, $g(a) = a$ *and* $r(a) \neq r(b)$, *with either*

(a) $r(b) \neq 0$, $g(b) = b$*; or*

(b) $r(b) \neq 0$ *and* $g(b) \notin \{0, a, b\}$.

3. $r(a) \neq 0$, $g(a) \notin \{0, a\}$, *with* $r(b) \neq 0$, $g(b) \notin \{0, b\}$ *and* $r(a) \neq r(b)$.

Throughout the remainder of this section, we may assume a, b and 0 are distinct. Fix Z and L that satisfy Lemmas 4.8 to 4.9, such that $a, b \notin Z$. The proof of this Theorem requires the construction of clopen sets A and B below.

The cases set out in Theorem 4.11 can be rephrased in terms of whether $a \in L$, $r(a) \in L$, $b \in L$ or $r(b) \in L$.

Lemma 4.12. *If* $\{a, r(a)\} \cap L \neq \emptyset$*, and* $\{b, r(b)\} \cap L \neq \emptyset$*, then* $\mathbb{X}$ *can be partitioned as Partition* 1*, that is*

$$\mathbb{X} = Z \mathbin{\dot\cup} A \mathbin{\dot\cup} B \mathbin{\dot\cup} K \mathbin{\dot\cup} r^{-1}(A) \mathbin{\dot\cup} r^{-1}(B) \mathbin{\dot\cup} r^{-1}(K).$$

with r *and* g *acting on the sets as illustrated in Figure 4.5 and* a *and* b *in distinct sets.*

Proof. Since $a \neq 0$, $a \notin Z$, and by Corollary 4.10 either $a \in L$ or $r(a) \in L$, but not both, we let $c \in \{a, r(a)\} \cap L$ and let $d \in \{b, r(b)\} \cap L$. There is a clopen set $A \subseteq L$ with $c \in A$ and $d \notin A$. There is also a clopen set $B \subseteq L$ with $d \in B$ and $c \notin B$ and $A \cap B = \emptyset$. By Lemma 4.9, $L \cap Z = \emptyset$. Therefore, we have $A \cap Z = \emptyset$, and $B \cap Z = \emptyset$.

Let $K = L\backslash(A \cup B)$, then $r^{-1}(L) = r^{-1}(A) \dot\cup r^{-1}(B) \dot\cup r^{-1}(K)$. The sets A, B and K satisfy the following:

1. $r(A) \subseteq Z$ and $g(A) \subseteq Z$;

2. $r(B) \subseteq Z$ and $g(B) \subseteq Z$;

3. $r(K) \subseteq Z$ and $g(K) \subseteq Z$;

4. $r^{-1}(A)$, $r^{-1}(B)$ and $r^{-1}(K)$ are pairwise disjoint;

5. $r^{-1}(A) \subseteq r^{-1}(L)$, $r^{-1}(B) \subseteq r^{-1}(L)$ and $r^{-1}(K) \subseteq r^{-1}(L)$;

6. $r(r^{-1}(A)) \subseteq A$, $r(r^{-1}(B)) \subseteq B$ and $r(r^{-1}(K)) \subseteq K$;

7. $g(r^{-1}(A)) \subseteq r^{-1}(A)$, $g(r^{-1}(B)) \subseteq r^{-1}(B)$ and $g(r^{-1}(K)) \subseteq r^{-1}(K)$;

8. $g^{-1}(A) = \emptyset$, $g^{-1}(B) = \emptyset$ and $g^{-1}(K) = \emptyset$.

The proof that the above eight claims are true follows.

As $A \subseteq r^{-1}(Z)\backslash Z$ implies $A \subseteq r^{-1}(Z)$, it follows that $r(A) \subseteq Z$. Since $A \subseteq r^{-1}(Z) = g^{-1}(Z)$, then $A \subseteq g^{-1}(Z)$ and $g(A) \subseteq Z$. Similarly, we have $r(B) \subseteq Z$, $g(B) \subseteq Z$, $r(K) \subseteq Z$ and $g(K) \subseteq Z$.

Since A, B and K are pairwise disjoint, Lemma 4.3 gives $r^{-1}(A)$, $r^{-1}(B)$ and $r^{-1}(K)$ are pairwise disjoint. Given $A \subseteq L$, $B \subseteq L$ and $K \subseteq L$, it follows that $r^{-1}(A) \subseteq r^{-1}(L)$, $r^{-1}(B) \subseteq r^{-1}(L)$ and $r^{-1}(K) \subseteq r^{-1}(L)$.

As $r^{-1}(A) \subseteq r^{-1}(A)$, then $r(r^{-1}(A)) \subseteq A$ and by Lemma 4.2, $g(r^{-1}(A)) \subseteq r^{-1}(A)$. Similarly, $r(r^{-1}(B)) \subseteq B$, $g(r^{-1}(B)) \subseteq r^{-1}(B)$, $r(r^{-1}(K)) \subseteq K$, and $g(r^{-1}(K)) \subseteq r^{-1}(K)$.

Now if $g^{-1}(A) \neq \emptyset$, then there is a $c \in g^{-1}(A)$ with $g(c) \in A \subseteq L$. So $g(c) \in L = g^{-1}(Z)\backslash Z$ and $g^2(c) \in Z$. Then by Quasi-equation 4.2 $g(c) \in Z$, a contradiction since $Z \cap L = \emptyset$. Therefore, $g^{-1}(A) = \emptyset$. Similarly, $g^{-1}(B) = \emptyset$ and $g^{-1}(K) = \emptyset$.

$\square$

We have partitioned $\mathbb{X}$ into clopen sets as shown in Figure 4.5 and can now complete the Proof of Theorem 4.11

Recall that Theorem 4.11 states:

Theorem 4.11. Assume $a, b \in \mathbb{X}$ with $\{a, b\}$ disjoint from $\{0\}$ and $a \neq b$, and a, b satisfying one of 1, 2 or 3 below. Then there exist disjoint clopen sets, A, B, Z, L and K, such that Z and L satisfy the properties of Lemmas 4.8 and 4.9, and $\mathbb{X}$ can be partitioned as,

$$\mathbb{X} = Z \,\dot{\cup}\, A \,\dot{\cup}\, B \,\dot{\cup}\, K \,\dot{\cup}\, r^{-1}(A) \,\dot{\cup}\, r^{-1}(B) \,\dot{\cup}\, r^{-1}(K)$$

where the partition separates a, b and 0, and the sets satisfy the quasi-equations of $\mathbb{M}_2$.

1. $r(a) = 0$ and $g(a) = 0$, with either
   (a) $r(b) = 0$ and $g(b) = 0$; or
   (b) $r(b) \notin \{0, a\}$, $g(b) = b$ and $r(a) \neq r(b)$; or
   (c) $r(b) \notin \{0, a\}$, $g(b) \notin \{0, b\}$ and $r(a) \neq r(b)$.
2. $r(a) \neq 0, g(a) \neq 0$, $g(a) = a$ and $r(a) \neq r(b)$, with either
   (a) $r(b) \neq 0$, $g(b) = b$; or
   (b) $r(b) \neq 0$ and $g(b) \notin \{0, a, b\}$.
3. $r(a) \neq 0$, $g(a) \notin \{0, a\}$, with $r(b) \neq 0$, $g(b) \notin \{0, b\}$ and $r(a) \neq r(b)$.

Proof. Recall that Z may be chosen so that $0 \in Z$ and for $c \in \{a, b, r(a), r(b)\}$, with $c \neq 0$, then $c \notin Z$. We have assumed $a \neq 0$, $b \neq 0$ and $a \neq b$. The proof of

Theorem 4.11 relies on showing $\{a, r(a)\}$ and $\{b, r(b)\}$ intersect L so that Lemma 4.12 may be used.

Since $a \neq 0$ and $b \neq 0$, we have $a, b \notin Z$. When $r(a) = 0$, by Corollary 4.10, $a \in L$ as $r(a) \in Z$. When $r(a) \neq 0$, by assumption $r(a) \notin Z$, so by Corollary 4.10, $a \in r^{-1}(L)$ and $r(a) \in L$. Similarly for b.

In part 1, $r(a) = 0$, so $a \in L$. The three subcases of part 1 rely on b. In part 1(a), $r(b) = 0$, so $b \in L$. In both parts 1(b) and 1(c), we have $r(b) \neq 0$, so by the above, $r(b) \in L$.

In part 2 and part 3, $r(a) \neq 0$, so $r(a) \in L$, and $r(b) \neq 0$ so $r(b) \in L$.

Accordingly, $\mathbb{X}$ can be partitioned as

$$\mathbb{X} = Z \mathbin{\dot\cup} A \mathbin{\dot\cup} B \mathbin{\dot\cup} K \mathbin{\dot\cup} r^{-1}(A) \mathbin{\dot\cup} r^{-1}(B) \mathbin{\dot\cup} r^{-1}(K).$$

By Lemma 4.12, the operations on the blocks behave as indicated in Figure 4.5. □

4.2 Partition 2

The second partition of $\mathbb{X}$ is constructed in this section and is illustrated in Figure 4.6.

Theorem 4.13. *For $a, b \in \mathbb{X}$ with $\{a, b\}$ disjoint from $\{0\}$ and $a \neq b$, when $r(a) \neq 0$, $g(a) = a$, $r(b) \neq 0$, $g(b) = a$ and $r(a) = r(b)$, there exist disjoint clopen sets, Z, L and R, such that Z and L satisfy the properties of Lemmas 4.8 to Lemma 4.9, and $a, b \notin Z$, and $\mathbb{X}$ can be partitioned as,*

$$\mathbb{X} = Z \mathbin{\dot\cup} L \mathbin{\dot\cup} R \mathbin{\dot\cup} r^{-1}(L) \backslash R$$

where the sets separate a, b and 0 and the sets satisfy the quasi-equations of $\mathbb{M}_2$.

$$
\begin{tikzcd}
R \arrow[loop left, "g"] \arrow[dr, "r"'] & & r^{-1}(L)\backslash R \arrow[ll, "g"'] \arrow[dl, "r"] \\
& L \arrow[d, bend right, "g"'] \arrow[d, bend left, "r"] & \\
& Z \arrow[loop below, "r"'] \arrow[loop below, "g"] &
\end{tikzcd}
$$

Figure 4.6: Partition 2

Throughout this section, assume $g(a) = a$ and $g(b) = a$, and $a, b \notin Z$. The proof of this Theorem requires the construction of the clopen set R in the following Lemma.

Lemma 4.14. *There exists a clopen set $R \subseteq r^{-1}(L)$ such that $a \in g(r^{-1}(L)) \subseteq R$ and $b \notin R$.*

Proof. Given that $g(a) = a$ and $a \notin Z$, then $a \in g(r^{-1}(L))$ and by Lemma 4.9, $g(r^{-1}(L)) \subseteq r^{-1}(L)$. If $b = g(c)$ for some c, then $g(b) = g^2(c)$, but $g^2(c) = g(c)$ by Quasi-equation 4.2, so $a = g(b) = g(c) = b$, a contradiction, as $a \neq b$. Thus $b \notin g(r^{-1}(L))$. Since $\mathbb{X}$ is a totally disconnected Hausdorff space, there is a clopen set R with $g(r^{-1}(L)) \subseteq R \subseteq r^{-1}(L)$ and $b \in r^{-1}(L)\backslash R$. Therefore $b \notin R$. $\square$

Lemma 4.15. *If $r(a) = r(b)$ and $r(a), r(b) \in L$, then $\mathbb{X}$ can be partitioned as*

$$\mathbb{X} = Z \;\dot\cup\; L \;\dot\cup\; R \;\dot\cup\; (r^{-1}(L)\backslash(R).$$

Proof. We have disjoint clopen sets Z, and L satisfying the properties of Lemmas 4.8 to Lemma 4.9, specifically, $r(L) \subseteq Z$, and $g(L) \subseteq Z$. Further, the sets R and $r^{-1}(L)\backslash R$ set out in Lemma 4.14 satisfy the following:

1. $r(R) \subseteq L$;

2. $g(R) \subseteq R$;

3. $g(r^{-1}(L)\backslash R) \subseteq R$;

4. $r(r^{-1}(L)\backslash R) \subseteq L$.

Since $R \subseteq r^{-1}(L)$, it follows that $r(R) \subseteq r(r^{-1}(L)) \subseteq L$, and $g(R) \subseteq g(r^{-1}(L)$ $\subseteq R$. It also follows that with $g(r^{-1}(L)) \subseteq R$ and $r(r^{-1}(L)) \subseteq L$, then $g(r^{-1}(L)\backslash R)$ $\subseteq R$ and $r(r^{-1}(L)\backslash R) \subseteq L$. □

We have partitioned $\mathbb{X}$ into clopen sets as illustrated in Figure 4.6, and now show the proof of Theorem 4.13.

Proof. Given $g(b) = a$, $rg(b) = r(a)$ and by Quasi-equation 4.4, $rg(b) = r(b)$, then $r(a) = r(b)$. Now $r^2(a) = 0 \in Z$, so this implies $r(a) \in r^{-1}(Z)\backslash Z = L$ and $a \in r^{-1}(L)$. With $r(a) = r(b)$, we have $b \in r^{-1}(L)$ and since $a \in r^{-1}(L)$, this implies $g(a) \in g(r^{-1}(L))$. Then, given $g(a) = a$, we have $a \in g(r^{-1}(L)) \subseteq R$. If $b \in r^{-1}(L)$, this implies $g(b) \in g(r^{-1}(L))$, but $g(b) \neq b$, so that means $b \notin R$, and $b \in r^{-1}(L)\backslash R$. Accordingly, $r(a) \in L$ and $r(b) \in L$ and Lemma 4.15 can be applied to obtain Partition 2 as a partition of $\mathbb{X}$. Therefore, $\mathbb{X}$ can be partitioned as:

$$\mathbb{X} = Z \,\dot{\cup}\, L \,\dot{\cup}\, R \,\dot{\cup}\, (r^{-1}(L)\backslash(R)$$

□

4.3 Partition 3

The third partition of $\mathbb{X}$ in this section is shown in Figure 4.7.

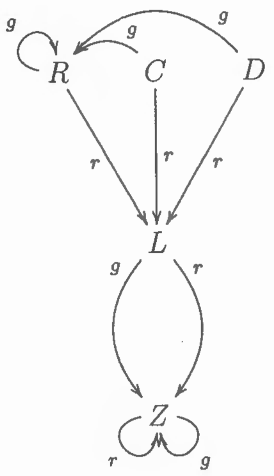

Figure 4.7: Partition 3

Theorem 4.16. *For $a, b \in \mathbb{X}$ with $\{a, b\}$ disjoint from $\{0\}$ and $a \neq b$, when $r(a) \neq 0$, $g(a) \notin \{0, a\}$, and $r(b) \neq 0$, and $g(b) \notin \{0, b\}$ and $r(a) = r(b)$, there exists disjoint clopen sets Z, L, C, D and R, such that Z, and L satisfy the properties of Lemma 4.8 and Lemma 4.9, and $\mathbb{X}$ can be partitioned as,*

$$\mathbb{X} = Z \mathbin{\dot\cup} L \mathbin{\dot\cup} C \mathbin{\dot\cup} D \mathbin{\dot\cup} R.$$

where these sets separate a, b and 0 and satisfy the quasi-equations of $\mathbb{M}_2$.

The proof of this Theorem requires the construction of the clopen sets C and D, as follows.

By Lemma 2.5, there are clopen sets C' and D' with $a \in C'$, $b \in D'$ and $C' \cap D' = \emptyset$. Then the case set out in Theorem 4.16 can be rephrased in terms of whether a and b are in $r^{-1}(L)$.

Lemma 4.17. *If $r(a) \in L$ and $r(b) \in L$, and $g(a) \neq a$ and $g(b) \neq b$, then $\mathbb{X}$ can be partitioned as*

$$\mathbb{X} = Z \mathbin{\dot\cup} L \mathbin{\dot\cup} C \mathbin{\dot\cup} D \mathbin{\dot\cup} R.$$

Proof. The sets Z and L satisfy the properties of Lemma 4.8 to Lemma 4.9, more specifically, $r(L) \subseteq Z$, and $g(L) \subseteq Z$. By Lemma 2.5, we may assume $r(a) \notin Z$. This implies $r(a) \in L$. Since $g(a) \neq a$, and $g(a) \in g(r^{-1}(L))$, this implies $a \notin g(r^{-1}(L))$, because if $a = g(c)$ with $c \in r^{-1}(L)$, then $g(a) = g^2(c) = g(c)$, by Quasi-equation 4.2, that is $g(a) = a$, a contradiction. Similarly, $b \notin g(r^{-1}(L))$.

Accordingly, there exists a clopen set $R \subseteq r^{-1}(L)$ with $g(r^{-1}(L)) \subseteq R$ and $\{a, b\}$ disjoint from R. In addition, there exist disjoint clopen sets C and D defined as $C = C' \cap r^{-1}(L)$ and $D = D' \cap r^{-1}(L)$, with $a \in C$ and $b \in D$. So we have $C \subseteq r^{-1}(L) \backslash R$ and $D \subseteq r^{-1}(L) \backslash (R \cup C)$. The sets C, D and R satisfy the following:

1. $r(C) \subseteq L$ and $r(D) \subseteq L$;

2. $g(C) \subseteq g(r^{-1}(L)) \subseteq R$ and $g(D) \subseteq g(r^{-1}(L)) \subseteq R$;

3. $r(R) \subseteq L$;

4. $g(R) \subseteq g(r^{-1}(L)) \subseteq R$.

Since $C \subseteq r^{-1}(L)$, then $r(C) \subseteq L$. This implies $g(C) \subseteq g(r^{-1}(L)) \subseteq R$. Similarly, $r(D) \subseteq L$ and $g(D) \subseteq g(r^{-1}(L)) \subseteq R$. Now $R \subseteq r^{-1}(L)$, so $r(R) \subseteq r(r^{-1}(L)) \subseteq L$, and $g(R) \subseteq g(r^{-1}(L)) \subseteq R$. □

We have partitioned $\mathbb{X}$ into clopen sets as shown in Figure 4.7 and can now complete the Proof of Theorem 4.16

Proof. Given that $r(a) = r(b)$, we have $g(a) = g(b)$ by Quasi-equation 4.7 with $g(a) \neq a$ and $g(b) \neq b$. Now we have disjoint clopen sets C and D, so we choose $a \in C$ and $b \in D$. Since $a \in C \subseteq r^{-1}(L)$, and $b \in D \subseteq r^{-1}(L)$, we have $a \in r^{-1}(L)$, so $r(a) \in L$ and $b \in r^{-1}(L)$. Therefore $b \in L$ and Lemma 4.17 can again be applied. Accordingly, $\mathbb{X}$ can be partitioned as follows.

$$\mathbb{X} = Z \dot{\cup} L \dot{\cup} C \dot{\cup} D \dot{\cup} R$$

with a and b in distinct sets. $\square$

4.4 Base Partition

The final partition for this example is constructed here and shown in Figure 4.8.

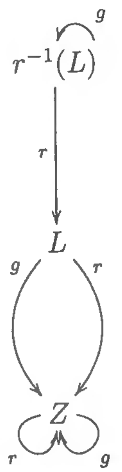

Figure 4.8: Base Partition

Theorem 4.18. *For $a, b \in \mathbb{X}$ with $\{a, b\}$ disjoint from $\{0\}$ and $a \neq b$, when a and b are as set out in the following cases, $\mathbb{X}$ can be partitioned as follows:*

$$\mathbb{X} = Z \dot{\cup} L \dot{\cup} r^{-1}(L)$$

1. $a = 0$, $r(a) = 0$, $g(a) = 0$, *with either*

   (a) $b \neq 0$, $r(b) = 0$ *and* $g(b) = 0$*; or*

   (b) $b \neq 0$, $r(b) \neq 0$, $g(b) \neq 0$, $g(b) = b$*; or*

   (c) $b \neq 0$, $r(b) \neq 0$, $g(b) \notin \{0, b\}$.

2. $a \neq 0$, $r(a) = 0$, $g(a) = 0$, $r(b) = a$ *with either*

   (a) $b \neq 0$, $r(b) \neq 0$, *and* $g(b) = b$; *or*

   (b) $b \neq 0$, $r(b) \neq 0$, $g(b) \notin \{0, b\}$.

Moreover, the disjoint clopen sets satisfy the properties of Lemma 4.8 and Lemma 4.9, and $a, b \notin Z$, *and these sets separate* a *and* b *and satisfy the quasi-equations of* $\mathbb{M}_2$.

Proof. In each case, it is sufficient to show that a and b are in distinct sets in the collection Z, L, and $r^{-1}(L)$.

In part 1 we have $0 = a \in Z$ and in 1(a), $r(b) = 0 \in Z$, which implies $b \in r^{-1}(Z)$. In part 1(b) and 1(c), $b \neq 0$ and by Lemma 4.2, we may assume $b \notin Z$.

In part 2, $a \neq 0$, so $a \notin Z$, but $r(a) = 0 \in Z$, so this implies $a \in r^{-1}(Z)$ and by Lemma 4.9, we have $a \in L$. Then we have $b \neq 0$, but $r(b) = a \in L$, so $r^{-1}(r(b))$ $\in r^{-1}(L)$, which gives $b \in r^{-1}(L)$. □

4.5 Proof of Theorem 4.1

In this section we complete the second proof of standardness, that is, the proof of Theorem 4.1. Recall Theorem 4.1 states the following:

Theorem 4.1 The structure $\mathbb{M}_2 = \langle 0, 1, 2; \mathcal{F}; \tau \rangle$ where $\mathcal{F} = \{001, 002\}$ and τ is the discrete topology, is standard.

Proof. By Theorem 4.4, there are four possible conditions for a and for b giving rise to 16 possible cases. By symmetry, we need only consider 9 situations where a and b come from distinct cases.

Recall the four possibilities for a:

(i) $a = 0$, $r(a) = 0$, $g(a) = 0$;

(ii) $a \neq 0$, $r(a) = 0$, $g(a) = 0$;

(iii) $a \neq 0$, $r(a) \neq 0$, $g(a) = a$;

(iv) $a \neq 0$, $r(a) \neq 0$, $g(a) \notin \{0, a\}$.

In Case (i) for a and Case (ii), (iii) or (iv) for b, these cases are covered by part 1(a), (b) and (c) of Theorem 4.18.

In Case (ii) for a and Case (ii) for b, this is the situation covered by part 1(a) of Theorem 4.11. In Case (ii) for a and Case (iii) for b, there are two subcases. If $r(b) = a$, this case is covered by part 2(a) of Theorem 4.18, and if $r(b) \neq a$, this case is covered by part 1(b) of Theorem 4.11. Similarly in Case (ii) for a and Case (iv) for b, there are two subcases. If $r(b) = a$, this case is covered by part 2(b) of Theorem 4.18, and if $r(b) \neq a$, this case is covered by part 1(c) of Theorem 4.11.

In Case (iii) for a and Case (iii) for b, this situation is covered by part 2(a) of Theorem 4.11. In Case (iii) for a and Case (iv) for b there are two subcases. When $g(b) = a$, this case is covered by Theorem 4.13, and when $g(b) \neq a$, this case is covered by part 2(b) of Theorem 4.11.

Finally, in Case (iv) for a and Case (iv) for b, there are two subcases. The case when $r(a) = r(b)$ is covered by Theorem 4.16, and the case when $r(a) \neq r(b)$ is covered by part 3 of Theorem 4.11.

Each of the four partitions are isomorphic to subalgebras of $\mathbb{M}_2$, so they do satisfy the quasi-equations of $\mathbb{M}_2$. $\square$

Given the algebra $\mathbb{M}_2 = \langle \{0, 1, 2\}; \mathcal{F}; \tau \rangle$ where $\mathcal{F} = \{001, 002\}$ and τ is the discrete topology, for any $\mathbb{X} \in \mathrm{Mod}_\tau \, \mathrm{Th}_{qe} (\mathbb{M})_2$, and any pair of elements $a, b \in \mathbb{X}$, we have constructed a finite cover of clopen sets in which the particular points a and b are separated, and the clopen sets satisfy the quasi-equational theory of $\mathbb{M}_2$. Therefore, $\mathbb{M}_2$ is standard.

The proof of Theorem 4.1 uses Theorem 2.4 to show that $\mathbb{M}_2$ is standard by showing that for each $\mathbb{X} \in \mathrm{Mod}_\tau \mathrm{Th}_{qe}(\mathbb{M})_2$, and each pair a, $b \in \mathbb{X}$ with $a \neq b$ there is a partition of clopen sets that separates a and b such that the blocks of the partition satisfy the quasi-equations of $\mathbb{M}_2$.

Chapter 5

Two More Examples

5.1 Example 3

The third example for standardness is the structure $\mathbb{M}_3 = \langle\{0, 1, 2\}; \mathcal{F}; \tau\rangle$ where $\mathcal{F} = \{110, 112\}$ and τ is the discrete topology. When p denote the operation 110 and b denote the operation 112, the structure $\mathbb{M}_3$ is shown in Figure 5.1.

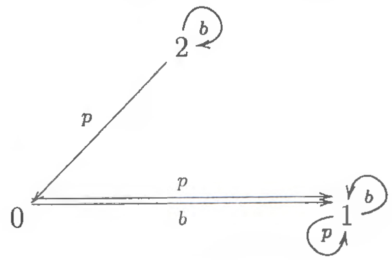

Figure 5.1: The Structure of $\mathbb{M}_3$

Applying p and b to the 27 elements of $\mathbb{M}_3{}^3$ yields Figure 5.2.

This algebra is isomorphic to $\mathbb{M}_2 = \langle\{0, 1, 2\}; \mathcal{F}; \tau\rangle$ where $\mathcal{F} = \{001, 002\}$, and is therefore standard.

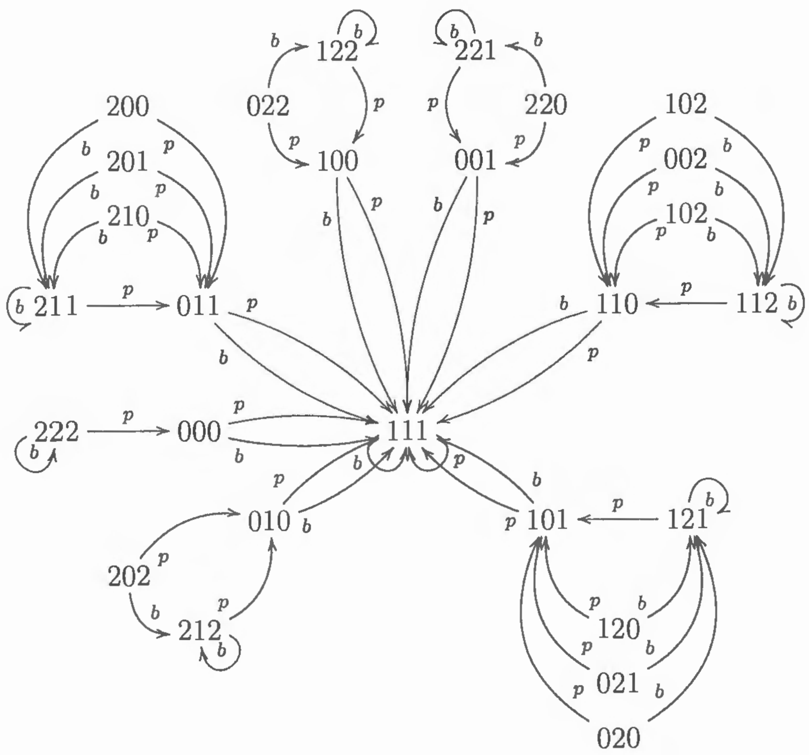

Figure 5.2: The Structure of $\mathbb{M}_3{}^3$

5.2 Example 4

This is the fourth example in which it was intended to determine whether or not the particular structure described below was standard. This structure, however, does not have any constant valued functions. Accordingly, using the same method as that in the previous three examples, was more difficult that what was expected and this example remains incomplete.

What follows is the beginning of the work that was completed for this example.

The fourth example is the structure $\mathbb{M}_4 = \langle\{0,1,2\}; \mathcal{F}; \tau\rangle$ where $\mathcal{F} = \{002, 112\}$ and τ is the discrete topology. When g denote the operation 002 and s denote the operation 112, the structure $\mathbb{M}_4$ is shown in Figure 5.3.

Applying g and s to the 27 elements of $\mathbb{M}_4{}^3$ yields Figure 5.4.

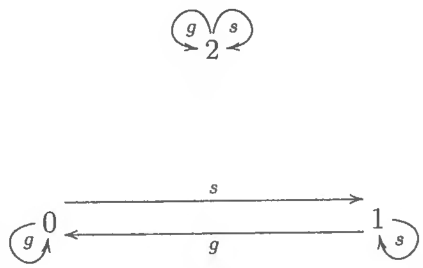

Figure 5.3: The Structure of $\mathbb{M}_4$

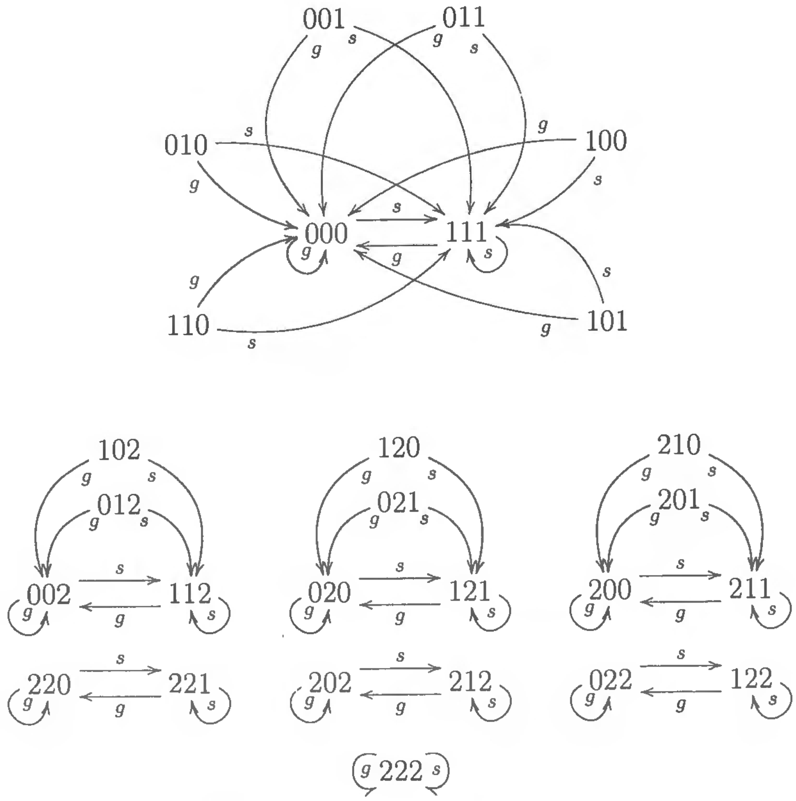

Figure 5.4: The Structure of $\mathbb{M}_4^3$

Conjecture 5.0.1. *The structure* $\mathbb{M}_4 = \langle 0, 1, 2;\ \mathcal{F};\ \tau\rangle$ *where* $\mathcal{F} = \{002, 112\}$ *and* τ *is the discrete topology, is standard.*

The proof that $\mathbb{M}_4$ is standard would require the use of Theorem 2.4 and would show that if $\mathbb{M}_4$ is standard there will be a finite cover of clopen sets of each Boolean model of the quasi-equational theory of $\mathbb{M}_4$ in which a and b are separated.

The equations and quasi-equations satisfied by $\mathbb{M}_4{}^3$ include

(1) $g^n(x) \approx g(x), n \geq 1$

(2) $s^n(x) \approx s(x), n \geq 1$

(3) $sg(x) \approx s(x)$

(4) $gs(x) \approx g(x)$

(5) $g(x) \approx g(y) \iff s(x) \approx s(y)$

(6) $g(x) \approx s(y) \Rightarrow x \approx y$

(7) $g(x) \approx x$ and $s(x) \approx x$, and $g(y) \approx y$ and $s(y) \approx y \Rightarrow x \approx y$.

That these quasi-equations hold can be seen from Table 5.1.

x	g	s	g^2	s^2	sg	gs	sg^2	s^2g	gs^2	g^2s
0	0	1	0	1	1	0	1	1	0	0
1	0	1	0	1	1	0	1	1	0	0
2	2	2	2	2	2	2	2	2	2	2

Table 5.1: Operations of $\mathbb{M}_4$

Conjecture 5.0.2. *For each $a \in \mathbf{X}$, a Boolean model of the quasi-equational theory of $\langle\{0, 1, 2\}; g, b\rangle$ where $g = 002$ and $s = 112$, one of the following cases holds.*

(i) $g(a) = s(a) = a$;

(ii) $g(a) = a, s(a) \neq a$;

(iii) $g(a) \neq a, s(a) = a$;

(iv) $g(a) \neq a, s(a) \neq a$.

Proof. By Quasi-equation 1, $g^2(a) = g(a)$ and by Quasi-equation 2, $s^2(a) = s(a)$, so we consider the cases where $g(a) = a$ or $g(a) \neq a$ with $s(a) \neq a$ or $s(a) = a$. If $g(a) = a$, and $s(a) = a$, this is Quasi-equation 6 and we get Case i. Then if $s(a) \neq a$ we get Case ii. The remaining two cases are when $g(a) \neq a$ and we need only consider when $s(a) = a$ or $s(a) \neq a$. The former condition gives Case iii and the latter Case iv. $\square$

The proof that $\mathbb{M}_4$ is standard will frequently require the application of Lemmas 2.5 and 2.6 and the following Lemmas which provide useful properties of subsets of $\mathbb{X}$.

Lemma 5.1. *Given A closed and $b \in \mathbb{X}$ such that $b \notin A$, there exist clopen sets U, V with $U \cap V = \emptyset$, $A \subseteq U$ and $b \in V$.*

Lemma 5.2. *Assume $A, B \in \mathbb{X}$.*

(i) If $A \subseteq s^{-1}(B)$, then $g(A) \subseteq s^{-1}(B)$

(ii) If $A \subseteq g^{-1}(B)$, then $g(A) \subseteq g^{-1}(B)$

(iii) If $A \subseteq s^{-1}(B)$, then $s(A) \subseteq s^{-1}(B)$

(iv) If $A \subseteq g^{-1}(B)$, then $s(A) \subseteq g^{-1}(B)$.

Proof. Assume $A, B \in \mathbb{X}$. If $A \subseteq s^{-1}(B)$, then $s(A) \subseteq B$. By quasi equation 3, $s(A) = sg(A)$, so $sg(A) \subseteq B$, and $g(A) \subseteq s^{-1}(B)$. If $A \subseteq g^{-1}(B)$, then $g(A) \subseteq B$. Quasi-equation 1 says $g(A) = g^2(A)$ so $g^2(A) \subseteq B$. Then $g(A) \subseteq g^{-1}(B)$. If $A \subseteq s^{-1}(B), s(A) \subseteq B$. By quasi-equation 2, $s(A) = s^2(A) \subseteq B$. Then $s(A) \subseteq s^{-1}(B)$. If $A \subseteq g^{-1}(B)$, then $g(A) \subseteq B$, and by quasi-equation 4, $g(A) = gs(A) \subseteq B$, so $s(A) \subseteq g^{-1}B$. $\square$

Lemma 5.3. *Assume $A, B \in \mathbb{X}$ and $A \cap B = \emptyset$. Then $s^{-1}(A) \cap s^{-1}(B) = \emptyset$.*

Proof. Let $d \in s^{-1}(A) \cap s^{-1}(B)$. Then $s(d) \in A$ and $s(d) \in B$, but $A \cap B = \emptyset$, a contradiction. Therefore $s^{-1}(A) \cap s^{-1}(B) = \emptyset$. $\square$

This ends the work completed for this example.

Chapter 6

Summary

In this work I proved that Example 1 and Example 2 are standard. Example 3 is also standard. In order to complete the proof that Example 4 is standard requires some more work. Because there is no constant valued function for Example 4, the partitions were more complicated than expected.

There may be other techniques used including the application of the following theorem.

Theorem 6.1. *If* $\mathbb{M}$ *is a* 3*-element unary structure with* $\mathbb{V}, \mathbb{L}$ *or* $\mathbb{D}$ *as an iso-reduct then* $\mathbb{M}$ *is non-standard.*

The proof of this theorem uses an inverse limit technique. This is a technique that I have not learned. The algebras that are known to be non-standard using this technique are shown in Table 2.1.

If all remaining structures are found to be standard, the following Conjecture would hold.

Conjecture 6.1.1. *If* $\mathbb{M}$ *is a* 3*-element unary structure without* $\mathbb{V}, \mathbb{L}$ *or* $\mathbb{D}$ *as an iso-reduct then* $\mathbb{M}$ *is standard.*

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