SEMIDISTRIBUTIVE LATTICES, MEET-SEMILATTICES, AND CONGRUENCE HEREDITY

by

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B.Sc., University of Northern British Columbia, 2003

THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE
in
MATHEMATICAL, COMPUTER, AND PHYSICAL SCIENCES
(MATHEMATICS)

THE UNIVERSITY OF NORTHERN BRITISH COLUMBIA
August 2006

Your file Votre référence
ISBN: 978-0-494-28412-4
Our file Notre référence
ISBN: 978-0-494-28412-4

APPROVAL

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Degree: Master of Science

Thesis Title: Semidistributive Lattices, Meet-Semilattices, And Congruence Heredity

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Abstract

The congruence lattice $\mathbf{Con}(\mathbf{A})$ of a finite algebra $\mathbf{A}$ is power-hereditary if, for all n, every 0-1 sublattice of $(\mathbf{Con}(\mathbf{A}))^n$ is the congruence lattice of an algebra on the universe of $\mathbf{A}^n$. In this thesis, I provide necessary and sufficient conditions for congruence lattice representations of $\mathbf{S}_7$, the smallest meet-semidistributive lattice not satisfying the join-semidistributive law, and its dual to be power-hereditary. I then show that the congruence lattice of $(\mathbf{2}_\wedge)^2$, the meet-semilattice on 2^2, is power-hereditary. Consequently, I prove that every finite lattice in the variety of $\mathbf{S}_7$ is isomorphic to the congruence lattice of an algebra with a meet operation. A natural question — whether the congruence lattice of $(\mathbf{2}_\wedge)^n$, the meet-semilattice on 2^n, is power-hereditary for all $n \geq 3$ — leads to a proof that every automorphism of $\mathbf{Con}((\mathbf{2}_\wedge)^n)$ is carried by an automorphism of $(\mathbf{2}_\wedge)^n$. I discuss the implications of this and conclude with some open questions.

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Acknowledgement

Thank you to my committee members for reviewing and revising this thesis, especially David Casperson for the extra pre-submission revisions. Thank you to John Snow for his very helpful suggestions and wealth of results to tap from as well as the opportunity to present my research at the meetings of the AMS. Thank you to my partner and family for not being upset at being neglected for lengthy periods of time during the writeup of this thesis and to Tracy and Erin for sharing in the mathematics. Most of all, though, thank you to my supervisor and mentor, Jennifer Hyndman, for the brilliant guidance during the last two years of this degree. Moreover, in the last eight years, thank you to Jennifer for fostering in me a love for algebras of every form, be it groups, fields, rings, or lattices.

Chapter 1

Introduction and Background

1.1 Introduction

Lattices show up frequently in mathematics — almost anywhere where there is an ordered structure. They are seen in families of topologies, logical structures, and the ordering of the integers when written as the products of primes. But most importantly, they show up in Universal Algebra, whether it be in the ordering of varieties or, as in this thesis, as the congruence lattices of various algebras.

This research is motivated by the Finite Lattice Congruence Representation Problem: "Is every finite lattice representable by a finite algebra?" In other words, "Is every lattice $\mathbf{L}$ with a finite number of elements isomorphic to the congruence lattice of an algebra $\mathbf{A}$ with a finite number of elements?" The problem is described by Pálfy in [7] as "perhaps the most famous unsolved problem in Universal Algebra" and, as of today, the complete answer remains unknown. Many advances have been made towards solving it, though, and, in the last few years, interest has returned to the subject.

Many classes of lattices have been proven to be representable including all finite distributive lattices and every finite lattice in the varieties of $\mathbf{M}_3$ and $\mathbf{N}_5$, the

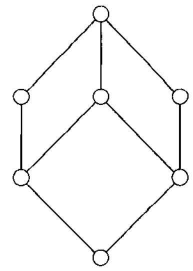

Figure 1.1: The lattice $\mathbf{S}_7$

smallest modular non-distributive and non-modular lattices, respectively. Recently, thanks to some new and unique proof techniques used by John Snow in [18], Péter Pálfy and Pál Hegedűs introduced the notion of hereditary and power-hereditary congruence lattices [7]. These definitions opened up a whole new range of problems pertaining to the representation problem and provided the impetus for the questions posed and answered in this thesis. The major question asked is, "do there exist congruence lattice representations of $\mathbf{S}_7$ and its dual and, if so, are they power-hereditary?" See Figure 1.1.

I begin with some important definitions that are needed to understand the arguments and results in this and the following chapters. Following the definitions, I give a history of some of the previous results pertaining to the Finite Lattice Congruence Representation Problem. Many of the results mentioned in this section are used as tools for proofs in the subsequent chapters and so are referred to frequently.

In Chapter 2, two lattices are considered — $\mathbf{S}_7$, the smallest meet-semidistributive lattice not satisfying the join-semidistributive law, and its dual, $\mathbf{S}_7{}^*$. Theorem 2.14 provides a necessary and sufficient condition for a congruence lattice representation of $\mathbf{S}_7$ to be power-hereditary. Theorem 2.18 describes a similar condition for a representation of $\mathbf{S}_7{}^*$.

The thesis question is then partially answered in Chapter 3 where I describe $(\mathbf{2}_\wedge)^2$, the four-element Boolean meet-semilattice, and show that its congruence lattice is isomorphic to $\mathbf{S}_7$. I then proceed to prove the main theorem of this chapter:

Theorem 3.7. *The congruence lattice of* $(\mathbf{2}_\wedge)^2$ *is a power-hereditary representation of the lattice* $\mathbf{S}_7$.

Included, in addition to my original proof presented, is an alternate proof of this theorem suggested to me by John Snow. The chapter concludes with an application of Theorem 3.7:

Theorem 3.15. *Every finite lattice in the variety generated by* $\mathbf{S}_7$ *is representable by the congruence lattice of a finite algebra with a meet operation.*

The algebra $(\mathbf{2}_\wedge)^2$ is just a specific finite Boolean meet-semilattice. In Chapter 4, I begin the process of generalizing Theorem 3.7 to $(\mathbf{2}_\wedge)^n$ for all n. I motivate the proofs in the chapter with a discussion of another specific case ($n = 3$) and then proceed to prove in Theorem 4.16 that, for all n, every automorphism of the congruence lattice of $(\mathbf{2}_\wedge)^n$ is, in fact, carried by an automorphism of $(\mathbf{2}_\wedge)^n$. Finally in Chapter 5, I discuss the implications of Theorem 4.16 and how I will attempt to proceed in proving the congruence lattice of $(\mathbf{2}_\wedge)^n$ is power-hereditary for all n. The thesis concludes with a discussion of some questions that have arisen as a result of the research done.

1.2 Basic definitions

These definitions follow the standard form used in [1], [2], and [24].

Let L be a set of elements combined with a *partial order* $\leq$ that is *reflexive* ($a \leq a$ for every a in L), *transitive* (for all a, b, and c in L, if $a \leq b$ and $b \leq c$, then $a \leq c$), and *antisymmetric* (for all a and b in L, if $a \leq b$ then $b \nleq a$). Let x, y, and z be in L. The element z is the *least upper bound* of x and y if $x \leq z$ and $y \leq z$ and, for all a in L, whenever $x \leq a$ and $y \leq a$, then $z \leq a$. If, for every x and y in L, there exists a least upper bound z of x and y, then say the *join* of x and y exists and is equal to z, in symbols, $x \vee y = z$. Similarly, if the greatest lower bound z of x and y exists, then call z the *meet* of x and y, in symbols, $x \wedge y = z$. A partially ordered set $\mathbf{L}$ is a *lattice* whenever, for all x and y in L, the join and meet of x and y both exist and are elements of $\mathbf{L}$. A lattice with a finite set of elements is aptly named a *finite lattice*. In this thesis every lattice discussed is a finite lattice. It follows from the definitions of join and meet that a finite lattice also has a bottom element, denoted 0, and a top element, denoted 1. Also, for any subset $K = \{k_1, \ldots, k_n\}$ of a finite lattice $\mathbf{L}$, define $\bigwedge K = k_1 \wedge \ldots \wedge k_n$. Similarly, $\bigvee K = k_1 \vee \ldots \vee k_n$. I adopt the convention that in a finite lattice $\bigwedge \emptyset = 1$.

A *meet-semilattice* is a partially ordered set of elements where the meet of any two elements exists but the join of any two elements does not necessarily exist. A finite meet-semilattice has a bottom element 0 but does not necessarily have a top element. Partially ordered sets, meet-semilattices, and lattices have the delightful property that their Hasse diagrams can be drawn and their ordering, meets, and joins can all be recovered completely from these diagrams. Two Hasse diagrams appear in Figure 1.2.

A subset $\mathbf{L}'$ of elements of a lattice $\mathbf{L}$ is a *sublattice* if for all a and b in $\mathbf{L}'$, $a \wedge b$ and $a \vee b$ are also in $\mathbf{L}'$. Moreover, $\mathbf{L}'$ is a *0-1 sublattice* if it is a sublattice containing the 0 and 1 elements of $\mathbf{L}$ and this is denoted $\mathbf{L}' \underset{0\text{-}1}{\leq} \mathbf{L}$. A subset C of $\mathbf{L}$ is *convex* if, for all x, y, and z in $\mathbf{L}$ such that $x \leq y \leq z$, whenever x and z are in C, the element y is also in C. If a and b are any two elements in $\mathbf{L}$ such that $a \leq b$, then the *subinterval* $[a, b]$ is the set of all elements x in $\mathbf{L}$ such that $a \leq x \leq b$. A subset A of elements of $\mathbf{L}$ is a *chain* if $x \leq y$ or $y \leq x$ for all x and y in A and A is an *antichain* if $x \leq y$ implies that $x = y$ for every x and y in A. We say an element a *covers* b, written $b \prec a$ if, for a, b, and c in $\mathbf{L}$, when $b \leq c < a$ it follows that $b = c$. An element x in $\mathbf{L}$ is *join-irreducible* if $x = y \vee z$ implies $x = y$ or $x = z$ for all y and z in $\mathbf{L}$. It follows that if x is join-irreducible and $y \prec x$ and $z \prec x$, then

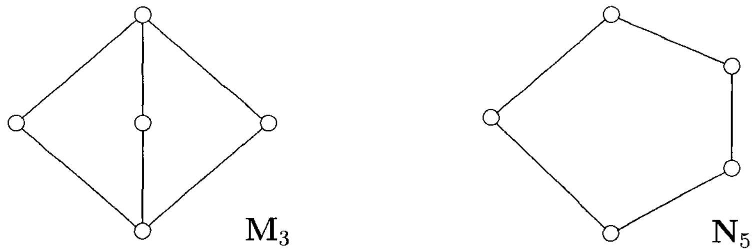

Figure 1.2: The modular but non-distributive lattice $\mathbf{M}_3$ and the non-modular lattice $\mathbf{N}_5$

$y = z$. Similarly, for x, y, and z in $\mathbf{L}$, the element x is *meet-irreducible* if $x = y \wedge z$ implies $x = y$ or $x = z$. Two elements x and y in $\mathbf{L}$ are *incomparable* with each other if $x \nleq y$ and $y \nleq x$ and this is denoted by $x \parallel y$. Finally, an element x of $\mathbf{L}$ is an *atom* if x covers 0 and a *coatom* if 1 covers x. An atom is defined similarly in meet-semilattice $\mathbf{S}$ and a coatom is defined in $\mathbf{S}$ only if $\mathbf{S}$ has a top element.

A lattice $\mathbf{L}$ is *distributive* if every x, y, and z in $\mathbf{L}$ satisfies the equation

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z). \tag{D}$$

$\mathbf{L}$ is *modular* if, for every x, y, and z with $x \leq z$ in $\mathbf{L}$,

$$x \wedge (y \vee z) = (x \wedge y) \vee z. \tag{M}$$

We denote the smallest non-modular lattice by $\mathbf{N}_5$ and the smallest modular but non-distributive lattice by $\mathbf{M}_3$ (see Figure 1.2). It is well known that every lattice that is not modular contains $\mathbf{N}_5$ as a sublattice of it and that every lattice that is not distributive contains the sublattice $\mathbf{M}_3$ or $\mathbf{N}_5$. In addition, say that $\mathbf{L}$ is *join-semidistributive* if, for all x, y, and z in $\mathbf{L}$,

$$x \vee y = x \vee z \text{ implies } x \vee y = x \vee (y \wedge z) \tag{SD$_\vee$}$$

and **L** is *meet-semidistributive* if, for all x, y, and z in **L**,

$$x \wedge y = x \wedge z \text{ implies } x \wedge y = x \wedge (y \vee z). \tag{SD$_\wedge$}$$

Note that $\mathrm{SD}_\vee$ and $\mathrm{SD}_\wedge$ are *dual* statements of each other. Any statement about lattices can be dualized by replacing every $\wedge$ with $\vee$ and vice-versa.

An *algebra* $\mathbf{A} = \langle A; F \rangle$ is a non-empty set of elements, A, called the *universe* of **A** and a set, F, of finitary operations. Each operation f in F is a function from A^n to A for some n and f is an n-ary operation. A common example of an algebra is a *group* $\mathbf{G} = \langle G; \circ, ^{-1}, id \rangle$ with a binary, a unary, and a nullary operation that satisfies the following three identities for all x,y, and z in **G**:

1. $x \circ (y \circ z) = (x \circ y) \circ z$,
2. $x \circ id = id \circ x = x$, and
3. $x \circ x^{-1} = x^{-1} \circ x = id$.

Another well-known example is a *vector space* $\mathbf{V} = \langle V; +, -, 0, \{f_r\}_{r \in R} \rangle$ with a binary operation $+$, a unary operation $-$, a nullary operation 0, and, for each r, a unary operation f_r equal to scalar multiplication by the element r. In fact, a finite lattice $\mathbf{L} = \langle L; \wedge, \vee, 0, 1 \rangle$ is also an algebra with finite universe L and two binary and two nullary operations that satisfies the following equations for all x, y, and z in **L**:

$$(x \vee y) \vee z = x \vee (y \vee z), \quad (x \wedge y) \wedge z = x \wedge (y \wedge z), \qquad \text{(associativity)}$$

$$x \vee y = y \vee x, \quad x \wedge y = y \wedge x, \qquad \text{(commutativity)}$$

$$x \vee x = x, \quad x \wedge x = x, \qquad \text{(idempotency)}$$

$$x \vee (x \wedge y) = x \quad x \wedge (x \vee y) = x. \qquad \text{(absorption)}$$

An algebra $\mathbf{A}$ is a *finite algebra* if the universe of $\mathbf{A}$ is a finite set of elements. In this thesis, every algebra discussed is a finite algebra. An algebra is *trivial* if it has only one element. A *subalgebra* of $\mathbf{A}$ is a subset of A that is closed under the operations of $\mathbf{A}$. Examples of subalgebras include the sublattices of a lattice. A *direct product* of two algebras $\mathbf{A}$ and $\mathbf{B}$ with the same operations F is denoted $\mathbf{A} \times \mathbf{B}$ and any operation f on elements in the universe $A \times B$ is done coordinatewise. If n is some integer greater than 1, then $\mathbf{A}^n$ is the direct product n times of $\mathbf{A}$. Another interesting family of algebras, which I discuss in Chapter 4, are the *Boolean meet-semilattices*, defined for all $n > 0$ to be $(\mathbf{2}_\wedge)^n = \langle 2^n; \wedge \rangle$ where

$$2^n = \{\langle a_1, \dots, a_n \rangle \mid a_i \in \{0, 1\}\}$$

and the meet operation is applied coordinatewise.

A function α mapping a set A to a set B is *one-to-one* if, for all a_1 and a_2 in A, $\alpha(a_1) = \alpha(a_2)$ implies $a_1 = a_2$. The map α is *onto* if, to every b in B, there corresponds an a in A such that $\alpha(a) = b$. A map that is both one-to-one and onto is called a *bijection*. A *permutation* of a set S is a one-to-one and onto function from S to itself. The set of all permutations of n elements form a group of size $n!$ called the *symmetric group* and is denoted $\mathbf{\Sigma}_n$. If α maps an algebra $\mathbf{A}$ to an algebra $\mathbf{B}$ of the same type, then α is *operation preserving* if $\alpha(f(a_1, \dots, a_n)) = f(\alpha(a_1), \dots, \alpha(a_n))$ for all n, all $a_1, \dots, a_n$ in A, and every n-ary operation f in F. A map $\alpha : \mathbf{A} \to \mathbf{B}$ is a *homomorphism* if it is operation preserving. If α is an onto map then $\mathbf{B}$ is a *homomorphic image* of $\mathbf{A}$. If α is a homomorphism mapping $\mathbf{A}$ to itself, then α is an *endomorphism*. We can now define the *variety* of $\mathbf{A}$, or $\mathcal{V}(\{\mathbf{A}\})$, as the set of all homomorphic images of subalgebras of direct products of $\mathbf{A}$.

Suppose $\mathbf{A}$ is a subalgebra of $\prod_{i=1}^{n} \mathbf{A}_i$ and let $\{j_1, \dots, j_m\} \subseteq \{1, \dots, n\}$. Then the mapping $\pi_{j_1,\dots,j_m} : \mathbf{A} \to \prod_{i=1}^{m} \mathbf{A}_{j_i}$, defined by $\pi_{j_1,\dots,j_m}(a_1, \dots, a_n) = (a_{j_1}, \dots, a_{j_m})$, is the *projection* of $\mathbf{A}$ onto $\prod_{i=1}^{m} \mathbf{A}_{j_i}$. If $\pi_i(\mathbf{A}) = \mathbf{A}_i$ for all i, then $\mathbf{A}$ is a *subdirect product* of $\prod_{i=1}^{n} \mathbf{A}_i$ and this is denoted $\mathbf{A} \underset{sd}{\leq} \prod_{i=1}^{n} \mathbf{A}_i$.

A relation R on a set A is said to be *symmetric* if $a_1 \, R \, a_2$ if and only if $a_2 \, R \, a_1$ for all a_1 and a_2 in A. An *equivalence relation* θ on a set A is a subset of A^2 that is reflexive, symmetric, and transitive. The set of all equivalence relations on A form a lattice, the *lattice of equivalence relations* on A, and is denoted $\mathbf{Eq}(A)$. A *congruence*, θ, on $\mathbf{A}$ is an equivalence relation on the universe of $\mathbf{A}$ that is preserved by all of the operations in F; that is, for all n and $\langle a_i, b_i \rangle$ in θ where $i = 1, \ldots, n$, the pair $\langle f(a_1, \ldots, a_n), f(b_1, \ldots, b_n) \rangle$ is in θ for every n-ary operation f. For any a in $\mathbf{A}$ and congruence θ of $\mathbf{A}$, the *congruence class* of a in θ is the the set of all b in $\mathbf{A}$ such that $\langle a, b \rangle$ is in θ and is denoted a/θ. For $a_1, \ldots, a_n$ in an algebra $\mathbf{A}$, the *congruence generated by the set* $\{a_1, \ldots, a_n\}$ is the smallest congruence such that $a_1, \ldots, a_n$ are in the same congruence class and is denoted $Cg(a_1, \ldots, a_n)$. Moreover, for any x and y in $\mathbf{A}$, the congruence $Cg(x, y)$ is called a *principal congruence*. The set of all congruences of $\mathbf{A}$ form a lattice, denoted $\mathbf{Con}(\mathbf{A})$, and 0 and 1 in $\mathbf{Con}(\mathbf{A})$ are often denoted Δ and ∇, respectively. A finite lattice $\mathbf{L}$ is *finitely fermentable* if $\mathbf{L}$ and $\mathbf{Con}(\mathbf{L})$ have the same number of join-irreducible elements and an algebra $\mathbf{A}$ is *congruence-distributive* if $\mathbf{Con}(\mathbf{A})$ is a distributive lattice.

Now, two algebras $\mathbf{A}$ and $\mathbf{B}$ are *isomorphic* to each other if there exists a bijective homomorphism α mapping $\mathbf{A}$ to $\mathbf{B}$ and this is denoted $\mathbf{A} \cong \mathbf{B}$. In addition, α is an *isomorphism*. In the case where $\mathbf{A} = \mathbf{B}$, call α an *automorphism*. The identity automorphism of $\mathbf{A}$ is often denoted $id_{\mathbf{A}}$. In fact, the set of automorphisms of an algebra $\mathbf{A}$ form a group, called the *automorphism group* of $\mathbf{A}$, a result which is given later in Theorem 4.3. An automorphism Φ of $\mathbf{Con}(\mathbf{A})$ is said to be *carried* by a function ϕ if

$$\Phi(\theta) = \{\langle \phi(x), \phi(y) \rangle \in \mathbf{A} \times \mathbf{A} \mid x \, \theta \, y\}$$

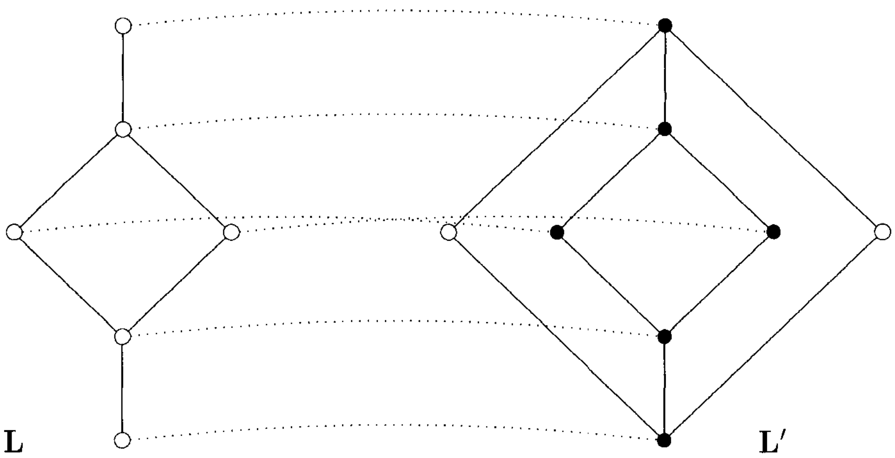

Figure 1.3: **L** embedded as a 0-1 sublattice of **L**′

for all θ in **Con**(**A**). We then define the *graph of an automorphism* ϕ of a lattice **L** to be the subset G of $\mathbf{L} \times \mathbf{L}$ such that $G = \{\langle x, y\rangle \in \mathbf{L} \times \mathbf{L} : \phi(x) = y\}$. We say a lattice **L** can be *embedded* in **L**′ if **L**′ has a sublattice isomorphic to **L** (see Figure 1.3). If, for some finite lattice **L**, there exists a finite algebra **A** such that **L** is isomorphic to the congruence lattice of **A** — that is, $\mathbf{L} \cong \mathbf{Con}(\mathbf{A})$ — then **L** is *representable* by **A**. An example is the lattice $\mathbf{M}_3$ which is representable by the trivial 3-element algebra without any operations. **L** is *strongly representable* if, whenever **L** can be embedded in the lattice of equivalence relations of a set S, **L** can be represented by an algebra on S.

For the following, let $\mathbf{L} = \mathbf{Con}(\mathbf{A})$ for some finite algebra **A**. If every 0-1 sublattice of **L** is the congruence lattice of an algebra on the universe A of **A**, then **L** is *hereditary*. In addition, **L** is *power-hereditary* if every 0-1 sublattice of $\mathbf{L}^n$ is the congruence lattice of an algebra on A^n, the universe of $\mathbf{A}^n$. The congruence lattice **Con**(**A**) is then called a *power-hereditary representation* of **L**.

A few terms from graph theory are needed for the following discussion and are taken from [12]. We consider a *graph* (V, E) as a binary relation E on a set of vertices V. The set E is the *edge set* and is a subset of $V \times V$. We call (a, b) in E an *arc* in the graph. If E is symmetric, then the graph is *undirected*. A function α mapping a graph (V, E) to another graph (U, F) is a *graph isomorphism* if $\alpha\colon V \to U$ is one-to-one, onto, and (a, b) is in E if and only if $(\alpha(a), \alpha(b))$ is in F. A *graph automorphism* is an isomorphism mapping a graph to itself. If $\mathbf{G}$ is a group of graph automorphisms on the undirected graph (V, E), then $\langle E; G\rangle$ is an algebra with unary operations $g \in \mathbf{G}$ where $g : E \to E$. In addition, $\mathbf{G}$ is *vertex-transitive* if, for every pair of vertices v_1 and v_2 in V, there exists a ϕ in $\mathbf{G}$ such that $\phi(v_1) = v_2$. Likewise, $\mathbf{G}$ is *arc-transitive* if it acts similarly on the set of arcs. The *stabilizer* of a point a in V is the set $\mathbf{G}_a = \{g \in \mathbf{G} \mid g(a) = a\}$ and, for a set of points $\{a, b\}$, $\mathbf{G}_{a,b} = \mathbf{G}_a \cap \mathbf{G}_b$ is their *pointwise stabilizer* and

$$\mathbf{G}_{\{a,b\}} = \{g \in \mathbf{G} \mid (g(a) = a \text{ and } g(b) = b) \text{ or } (g(a) = b \text{ and } g(b) = a)\}$$

their *setwise stabilizer*.

Finally, for some set of constant symbols or variables $t_1, \ldots, t_n$ and an n-ary relation symbol r, the formula $r(t_1, \ldots, t_n)$ is an *atomic formula*. A *primitive positive formula* is an existentially quantified conjunction of atomic formulas. That is, a primitive positive formula consists only of existentially quantified variables (i.e. $\exists x_1, x_2$), logical conjunction symbols ($\wedge$), and atomic formulas. In this thesis, the only atomic formulas are used with equivalence relations $s_{i,j}$ and are of the form $r(x_1, x_2)$ defined such that

$$x_1 \; r \; x_2 \quad \text{if and only if} \quad \exists x_3, \ldots, x_n \bigwedge_{1 \le i < j \le n} x_i \; s_{i,j} \; x_j.$$

It is important to note that conjunction, also represented by the symbol $\wedge$, should not be confused with the meet operation. It should be clear from the context which is being used.

1.3 History of the problem

In 1970, Quackenbush and Wolk proved a very important result about finite distributive lattices:

Theorem 1.1. [16] *Every finite distributive lattice is strongly representable.*

Four years later, Pudlák and Tůma described the class of finitely fermentable lattices and proved the following results about them.

Theorem 1.2. [14]

(a) *A lattice* **L** *is finitely fermentable if and only if it is finite and* **L** *and* **Con**(**L**) *have the same number of join-irreducible elements.*

(b) *The class of finitely fermentable lattices is closed under homomorphisms, sublattices, and finite direct products.*

(c) *Every finitely fermentable lattice is representable.*

Thus, a finitely fermentable lattice **L** is not only representable but every lattice in the variety of **L** is also representable!

Then, in 1980, a surprising connection was drawn by Pálfy and Pudlák in the following theorem.

Theorem 1.3. [13] *The following statements are equivalent:*

(a) *Any finite lattice is isomorphic to the congruence lattice of a finite algebra.*

(b) *Any finite lattice is isomorphic to an interval of the subgroup lattice of a finite group.*

Pálfy used this result again, over 20 years later, in the proof of Theorem 1.12.

More recent work has been done on the Finite Lattice Congruence Representation Problem by Snow, Pálfy, and Hegedűs, amongst others. In 2000, through studying the primitive positive formulas on the lattice of equivalence relations of a set, Snow provided in Corollary 2.3 of [18] an important necessary and sufficient condition for a 0-1 sublattice of $\mathbf{Eq}(A)$ to be the congruence lattice of an algebra on the set A.

Lemma 1.4. [18] *Suppose* $\mathbf{L}$ *is a 0-1 lattice of equivalence relations on a finite set* A. *There is an algebra* $\mathbf{A}$ *on* A *with* $\mathbf{Con}(\mathbf{A}) = \mathbf{L}$ *if and only if* $\mathbf{L}$ *contains every equivalence relation* σ *on* A *definable by primitive positive formulas of the form*

$$x_1 \,\sigma\, x_2 \quad \textit{if and only if} \quad \exists x_3, \ldots, x_n \bigwedge_{1 \leq i < j \leq n} x_i \, s_{i,j} \, x_j \tag{C}$$

where $s_{i,j}$ *are in* $\mathbf{L}$. *(That is, every equivalence relation generated in this way is already an element of* $\mathbf{L}$.*)*

As Lemma 1.4 is used frequently in Section 3.3, say that $\mathbf{L}$, a 0-1 lattice of equivalence relations on a finite set A, is *closed under primitive positive formulas* if $\mathbf{L}$ contains every equivalence relation on A definable by primitive positive formulas of the form (C).

In the same paper, Snow developed various methods from which new representable lattices could be constructed from known representable lattices. Some of these methods are presented in the next two lemmas.

Lemma 1.5. [18] *Suppose* $\mathbf{A}$ *is finite algebra and* α *and* β *are equivalence relations on* A. *There is an algebra* $\mathbf{A}'$ *on* A *with*

$$\mathbf{Con}(\mathbf{A}') = \{x \in \mathbf{Con}(\mathbf{A}) : x \leq \alpha \text{ or } x \geq \beta\}.$$

Lemma 1.6. [18] *The class of representable lattices is closed under subintervals.*

The main theorem of [18], which is presented next, is an example of the application of these methods.

Theorem 1.7. [18] *Every finite lattice which contains no three element antichain is representable.*

A few years later in another paper, Snow proved the following about the lattice $\mathbf{M}_3$.

Theorem 1.8. [19] *Every finite lattice in the variety generated by* $\mathbf{M}_3$ *is representable.*

This result led Pálfy and Hegedűs to introduce the idea of hereditary and power-hereditary congruence lattices. They then reformulated Theorem 1.8 in terms of congruence heredity in the next theorem.

Theorem 1.9. [7] $\mathbf{M}_3$ *is a power-hereditary congruence lattice.*

Another valuable result from this paper — a result which provides the "backbone" to the proofs in Chapter 2 of this thesis — is the following theorem about subdirect products. It states that to prove a congruence lattice $\mathbf{L}$ is power-hereditary, I need only consider a subset of the sublattices of $\mathbf{L} \times \mathbf{L}$.

Theorem 1.10. [7] *Let* X *be a finite set and* $\mathbf{L} \subseteq \mathbf{Eq}(X)$ *a 0-1 sublattice. If every subdirect product* $\mathbf{L}'' \subseteq \mathbf{L} \times \mathbf{L} \subseteq \mathbf{Eq}(X^2)$ *containing* $(\{0\} \times \mathbf{L}) \cup (\mathbf{L} \times \{1\})$ *is a congruence lattice, then* $\mathbf{L} \subseteq \mathbf{Eq}(X)$ *is a power-hereditary congruence lattice.*

In response to Pálfy and Hegedűs, Snow used Theorem 1.10 in the proofs of the following results for finite algebras $\mathbf{A}$ and $\mathbf{B}$.

Theorem 1.11. [20]

(a) *If* $\mathbf{Con}(\mathbf{A})$ *is distributive, then every subdirect product of* $\mathbf{Con}(\mathbf{A})$ *and* $\mathbf{Con}(\mathbf{B})$ *is a congruence lattice of an algebra on* $A \times B$*, where* A *and* B *are the universes of* $\mathbf{A}$ *and* $\mathbf{B}$*, respectively.*

(b) *If* $\mathbf{Con}(\mathbf{A})$ *is distributive and* $\mathbf{Con}(\mathbf{B})$ *is power-hereditary, then* $\mathbf{Con}(\mathbf{A}) \times \mathbf{Con}(\mathbf{B})$ *is power-hereditary.*

(c) *If* $\mathbf{Con}(\mathbf{A})$ *is isomorphic to* $\mathbf{N}_5$ *and* $\mathbf{Con}(\mathbf{B})$ *is modular, then every subdirect product of* $\mathbf{Con}(\mathbf{A})$ *and* $\mathbf{Con}(\mathbf{B})$ *is representable.*

(d) *Every congruence lattice representation of* $\mathbf{N}_5$ *is power-hereditary.*

The next result by Pálfy gives a sufficent condition for a representation of $\mathbf{M}_3$ to not be power-hereditary.

Theorem 1.12. [12] *Let* (V, E) *be a finite undirected graph and* $\mathbf{G} \leq \mathrm{Aut}(V, E)$ *a group of graph automorphisms. Let* $(u, v) \in E$ *be an arbitrary arc. Suppose the following:*

(a) *The graph* (V, E) *does not contain any triangle.*

(b) $\mathbf{G}$ *is vertex-transitive.*

(c) $\mathbf{G}$ *is arc-transitive.*

(d) $\mathbf{G}_{u,v}$ *is contained in exactly five subgroups:* $\mathbf{G}_{u,v}$, $\mathbf{G}_u$, $\mathbf{G}_v$, $\mathbf{G}_{\{u,v\}}$, $\mathbf{G}$.

Then $\mathbf{Con}(\langle E; G \rangle) \cong \mathbf{M}_3$ *is not power-hereditary.*

In the same paper, Pálfy then provided an example — the Higman-Sims sporadic simple group — that satisfies Theorem 1.12. This last result states that certain lattices have both power-hereditary and non-power-hereditary representations. Thus, the property is not uniform for every representation of a lattice as Theorem 1.11 (d) might lead some to believe.

Snow's next result involves Day's doubling construction for convex sets in lattices described in [3].

Theorem 1.13. [21] *Suppose* **L** *is a finite lattice obtained from a distributive lattice by doubling a convex set. Every congruence lattice representation of* **L** *is hereditary.*

Finally, during the time that research was being done for this thesis, Snow proved the following result about the congruence lattices of finite vector spaces.

Theorem 1.14. [23] *Suppose that* **V** *is a finite vector space.*

(a) *The congruence lattice of* **V** *is hereditary if and only if either* **V** *is simple and dim* $\mathbf{V} = 1$ *or* **V** *is* ${Z_2}^2$ *or* ${Z_2}^3$.

(b) *The congruence lattice of* **V** *is power-hereditary if and only if either* **V** *is simple and dim* $\mathbf{V} = 1$ *or* **V** *is* ${Z_2}^2$.

Chapter 2

Power-hereditary Representations of $\mathbf{S}_7$ and $\mathbf{S}_7{}^*$

In this chapter, I discuss two lattices and prove necessary and sufficient conditions for congruence lattice representations of these lattices to be power-hereditary. In Section 2.1, I introduce $\mathbf{S}_7$, the smallest meet-semidistributive lattice not satisfying the join-semidistributive law, and $\mathbf{S}_7{}^*$, the smallest join-semidistributive lattice not satisfying the meet-semidistributive law, and provide some useful lemmas. Then, in Section 2.2, I prove a necessary and sufficient condition for a representation of $\mathbf{S}_7$ to be power-hereditary. Finally, I provide a similar condition for $\mathbf{S}_7{}^*$ in Section 2.3.

2.1 The lattices $\mathbf{S}_7$ and $\mathbf{S}_7{}^*$

When I was initially looking for questions to try and answer for my thesis, John Snow suggested I pick my favourite small lattice and find the power-hereditary representations of it. After some discussion, my thesis supervisor, Jennifer Hyndman, and I decided that I would begin my research with the lattices $\mathbf{S}_7$ and its dual $\mathbf{S}_7{}^*$ (see Figure 2.1). In order to discuss why these specific lattices are important, recall

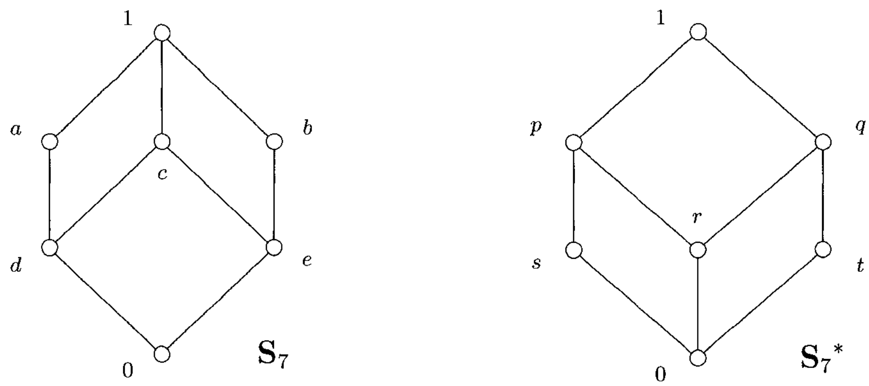

Figure 2.1: The lattices $\mathbf{S}_7$ and $\mathbf{S}_7{}^*$

from Section 1.2 that a lattice $\mathbf{L}$ has the join-semidistributive property if, for all x, y, and z in $\mathbf{L}$,

$$x \vee y = x \vee z \text{ implies } x \vee y = x \vee (y \wedge z) \qquad (\mathrm{SD}_\vee)$$

and a lattice has the meet-semidistributive property if, for all x, y, and z in $\mathbf{L}$,

$$x \wedge y = x \wedge z \text{ implies } x \wedge y = x \wedge (y \vee z). \qquad (\mathrm{SD}_\wedge)$$

The following well-known result concerns $\mathbf{S}_7$ and $\mathbf{S}_7{}^*$.

Theorem 2.1. *The lattice* $\mathbf{S}_7$ *is the smallest lattice satisfying* $\mathrm{SD}_\wedge$ *but not* $\mathrm{SD}_\vee$ *and* $\mathbf{S}_7{}^*$ *is the smallest lattice satisfying* $\mathrm{SD}_\vee$ *but not* $\mathrm{SD}_\wedge$.

In fact, $\mathbf{S}_7$ and $\mathbf{S}_7{}^*$ play quite an important role in Tame Congruence Theory when they show up as sublattices of congruence lattices of finite algebras [8]. For instance, when $\mathbf{S}_7$ can be embedded as the sublattice of a congruence lattice of a finite algebra $\mathbf{A}$, then the congruence lattices of algebras in $\mathcal{V}(\mathbf{A})$ satisfy no lattice identity. Hence, studying the power-hereditary representations of $\mathbf{S}_7$ and $\mathbf{S}_7{}^*$ may have far-reaching consequences in other areas of Universal Algebra.

It has already been proven in [20] that every representation of $\mathbf{N}_5$ is power-hereditary. As $\mathcal{V}(\mathbf{N}_5)$ is covered by varieties generated by $\mathbf{S}_7$ and $\mathbf{S}_7{}^*$ [9], it is reasonable to ask similar questions about these two lattices. Although I am unable to prove as strong a result for $\mathbf{S}_7$ or $\mathbf{S}_7{}^*$ (perhaps because they both may have non-power-hereditary representations), I provide necessary and sufficient conditions in Section 2.2 and 2.3 for representations of either to be power-hereditary.

In order to prove the main theorems of this chapter, I first need some previously known results. This first lemma shows that, when considering the congruences of an algebra, I only need to consider the principal congruences.

Lemma 2.2. [1] *Let* $\mathbf{A}$ *be an algebra, and suppose* $a_1, b_1, \ldots, a_n, b_n \in \mathbf{A}$ *and* $\theta \in \mathbf{Con}(\mathbf{A})$. *Then*

(a) $Cg(a_1, b_1) = Cg(b_1, a_1)$

(b) $Cg(\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle) = Cg(a_1, b_1) \vee \ldots \vee Cg(a_n, b_n)$

(c) $Cg(a_1, \ldots, a_n) = Cg(a_1, a_2) \vee Cg(a_2, a_3) \vee \ldots \vee Cg(a_{n-1}, a_n)$

(d) $\theta = \bigcup\{Cg(a, b) \colon \langle a, b\rangle \in \theta\} = \bigvee\{Cg(a, b) \colon \langle a, b\rangle \in \theta\}$.

The next lemma says that certain nicely defined sublattices of the square of a congruence lattice of an algebra $\mathbf{A}$ are congruence lattices of algebras on the underlying set of $\mathbf{A} \times \mathbf{A}$.

Lemma 2.3. [20] *Suppose that* $\mathbf{K}$ *is the congruence lattice of a finite algebra* $\mathbf{A}$ *and let* $\mathbf{L} = \{\langle u, v\rangle \in \mathbf{K} \times \mathbf{K} : u \leq v\}$. *Then* $\mathbf{L}$ *is the congruence lattice of an algebra on the universe of* $\mathbf{A} \times \mathbf{A}$.

The following lemma states that the intersection of the congruence lattices of two algebras on the same universe is also the congruence lattice of an algebra on that universe.

Lemma 2.4. [18] *Suppose* $\mathbf{A}$ *and* $\mathbf{B}$ *are algebras on a set* A. *There is an algebra* $\mathbf{C}$ *on* A *so that* $\mathbf{Con}(\mathbf{C}) = \mathbf{Con}(\mathbf{A}) \cap \mathbf{Con}(\mathbf{B})$.

Finally, I need a lemma first stated in correspondence with John Snow. I provide the necessary proof.

Lemma 2.5. *If* $\mathbf{L}$ *is the congruence lattice of an algebra* $\mathbf{A}$*, then* $\mathbf{L} \times \mathbf{L}$ *is the congruence lattice of an algebra on the universe of* $\mathbf{A} \times \mathbf{A}$.

Proof. Let $\mathbf{L} = \mathbf{Con}(\mathbf{A})$. By Lemma 1.4, the lattice $\mathbf{L}$ is closed under primitive positive formulas of the form

$$\exists x_3, \ldots, x_n \bigwedge_{1 \leq i < j \leq n} x_i \; s_{i,j} \; x_j$$

where $s_{i,j}$ are in $\mathbf{L}$.

Now consider $\mathbf{L} \times \mathbf{L}$, the lattice of equivalence relations on the set $A \times A$ of the form $\langle \alpha, \beta \rangle$ where α and β are in $\mathbf{L}$ and, for $\langle x_1, y_1 \rangle$ and $\langle x_2, y_2 \rangle$ in $A \times A$,

$$\langle x_1, y_1 \rangle \; \langle \alpha, \beta \rangle \; \langle x_2, y_2 \rangle$$

if and only if

$$x_1 \; \alpha \; x_2 \text{ and } y_1 \; \beta \; y_2.$$

Let σ be an equivalence relation on $A \times A$ defined by a primitive positive formula as

$$\langle x_1, y_1 \rangle \; \sigma \; \langle x_2, y_2 \rangle \Leftrightarrow \exists \langle x_3, y_3 \rangle, \ldots, \langle x_n, y_n \rangle \bigwedge_{1 \leq i < j \leq n} \langle x_i, y_i \rangle \; \langle s_{i,j}, t_{i,j} \rangle \; \langle x_j, y_j \rangle$$

where each $\langle s_{i,j}, t_{i,j} \rangle$ is in $\mathbf{L} \times \mathbf{L}$. Since each $\langle s_{i,j}, t_{i,j} \rangle$ applies coordinatewise, it follows that $\sigma = \langle \alpha, \beta \rangle$ where

$$x_1 \; \alpha \; x_2 \Leftrightarrow \exists x_3, \ldots, x_n \bigwedge_{1 \leq i < j \leq n} x_i \; s_{i,j} \; x_j$$

and

$$y_1 \,\beta\, y_2 \Leftrightarrow \exists y_3, \ldots, y_n \bigwedge_{1 \le i < j \le n} y_i \, t_{i,j} \, y_j.$$

However, α and β must both be in $\mathbf{L}$ as it is closed under primitive positive formulas. Hence $\sigma = \langle \alpha, \beta \rangle$ is an element of $\mathbf{L} \times \mathbf{L}$. Thus $\mathbf{L} \times \mathbf{L}$ is closed under primitive positive formulas of the form

$$\exists \langle x_3, y_3 \rangle, \ldots, \langle x_n, y_n \rangle \bigwedge_{1 \le i < j \le n} \langle x_i, y_i \rangle \, \langle s_{i,j}, t_{i,j} \rangle \, \langle x_j, y_j \rangle$$

and, by Lemma 1.4, is the congruence lattice of an algebra on $A \times A$. ■

2.2 A necessary and sufficient condition for a representation of $\mathbf{S}_7$ to be power-hereditary

I now provide a necessary and sufficient condition for a congruence lattice representation of $\mathbf{S}_7$ to be power-hereditary. For any algebra $\mathbf{A}$ such that $\mathbf{Con}(\mathbf{A}) \cong \mathbf{S}_7$, I show that all but five subdirect products of $\mathbf{Con}(\mathbf{A}) \times \mathbf{Con}(\mathbf{A})$ containing $(\{0\} \times \mathbf{Con}(\mathbf{A})) \cup (\mathbf{Con}(\mathbf{A}) \times \{1\})$ are congruence lattices of algebras on the universe of $\mathbf{A} \times \mathbf{A}$. Then, in Theorem 2.14, I prove that if $\mathbf{L}_0$, a particular one of those five subdirect products, is a congruence lattice, then so are the other four. Hence by Theorem 1.10, $\mathbf{Con}(\mathbf{A})$ is power-hereditary if $\mathbf{L}_0$ is the congruence lattice of an algebra on the universe of $\mathbf{A} \times \mathbf{A}$.

Before I proceed with the proof, though, I define some useful notation. For this section, let $\mathbf{A}$ be an algebra such that $\mathbf{Con}(\mathbf{A})$ is isomorphic to $\mathbf{S}_7$. Note that such an $\mathbf{A}$ exists since $\mathbf{S}_7 \cong \mathbf{Con}((\mathbf{2}_\wedge)^2)$ as is discussed in Section 3.1. Notationally then, when $\mathbf{S}_7$ is referred to, it is as the lattice of congruences on $\mathbf{A}$. Suppose that $\mathbf{L}$ is the subdirect product of $\mathbf{K} \times \mathbf{M}$ for some finite lattices $\mathbf{K}$ and $\mathbf{M}$. Then, for any y

in $\mathbf{M}$, define

$$y_{\mathbf{L}}^{\Uparrow} = \bigvee\{x \in \mathbf{K} : \langle x, y\rangle \in \mathbf{L}\} \text{ and}$$
$$y_{\mathbf{L}}^{\Downarrow} = \bigwedge\{x \in \mathbf{K} : \langle x, y\rangle \in \mathbf{L}\}.$$

As $\langle y_{\mathbf{L}}^{\Uparrow}, y\rangle = \bigwedge\{x \in \mathbf{M} : \langle x, y\rangle \in \mathbf{L}\}$, it follows that $\langle y_{\mathbf{L}}^{\Uparrow}, y\rangle$ is in $\mathbf{L}$. Similarly $\langle y_{\mathbf{L}}^{\Downarrow}, y\rangle$ is in $\mathbf{L}$. Note that $y^{\Uparrow}$ is written for $y_{\mathbf{L}}^{\Uparrow}$ where the context makes the lattice $\mathbf{L}$ clear. The following useful lemma relates to $y^{\Uparrow}$ and $y^{\Downarrow}$. The proof is provided for completeness.

Lemma 2.6. *Let* $\mathbf{K}$ *and* $\mathbf{M}$ *be finite lattices and let* $\mathbf{L}$ *be a subdirect product of* $\mathbf{K}$ *and* $\mathbf{M}$*.*

(a) *For any* $x \in \mathbf{K}$ *and* $y \in \mathbf{M}$*,* $\langle x, y\rangle \in \mathbf{L}$ *if and only if* $y^{\Downarrow} \leq x \leq y^{\Uparrow}$*.*

(b) *The map* $y \mapsto y^{\Downarrow}$ *is a join homomorphism.*

(c) *The map* $y \mapsto y^{\Uparrow}$ *is a meet homomorphism.*

Proof. (a): Suppose $\langle x, y\rangle$ is in $\mathbf{L}$. Then x is in $\{x \in \mathbf{K} : \langle x, y\rangle \in \mathbf{L}\}$. Thus $x \leq \bigvee\{x \in \mathbf{K} : \langle x, y\rangle \in \mathbf{L}\}$ and $x \geq \bigwedge\{x \in \mathbf{K} : \langle x, y\rangle \in \mathbf{L}\}$. Therefore, $y^{\Downarrow} \leq x \leq y^{\Uparrow}$.

Now, suppose $y^{\Downarrow} \leq x \leq y^{\Uparrow}$. There exists a in $\mathbf{M}$ such that $\langle x, a\rangle$ is in $\mathbf{L}$ as $\mathbf{L}$ is a subdirect product. Hence $\langle x, a\rangle \wedge \langle y^{\Uparrow}, y\rangle = \langle x, a \wedge y\rangle$ is in $\mathbf{L}$ which implies $\langle y^{\Downarrow}, y\rangle \vee \langle x, a \wedge y\rangle = \langle x, y \vee (a \wedge y)\rangle = \langle x, y\rangle$ is in $\mathbf{L}$.

(b): Let a and b be elements of $\mathbf{M}$. Then $\langle a^{\Downarrow}, a\rangle$ and $\langle b^{\Downarrow}, b\rangle$ are in $\mathbf{L}$ which implies $\langle a^{\Downarrow} \vee b^{\Downarrow}, a \vee b\rangle$ is also an element of $\mathbf{L}$. Thus $(a \vee b)^{\Downarrow} \leq a^{\Downarrow} \vee b^{\Downarrow}$ by (a).

Now, since $\langle a^{\Downarrow}, a\rangle$ and $\langle (a \vee b)^{\Downarrow}, a \vee b\rangle$ are in $\mathbf{L}$ it follows that $\langle a^{\Downarrow} \wedge (a \vee b)^{\Downarrow}, a \wedge (a \vee b)\rangle = \langle a^{\Downarrow} \wedge (a \vee b)^{\Downarrow}, a\rangle$ is also in $\mathbf{L}$. Hence $a^{\Downarrow} \leq a^{\Downarrow} \wedge (a \vee b)^{\Downarrow}$ so $a^{\Downarrow} \leq (a \vee b)^{\Downarrow}$. Similarly, $b^{\Downarrow} \leq (a \vee b)^{\Downarrow}$ so $a^{\Downarrow} \vee b^{\Downarrow} \leq (a \vee b)^{\Downarrow}$.

Finally, $(a \vee b)^{\Downarrow} \leq a^{\Downarrow} \vee b^{\Downarrow}$ and $a^{\Downarrow} \vee b^{\Downarrow} \leq (a \vee b)^{\Downarrow}$ means $a^{\Downarrow} \vee b^{\Downarrow} = (a \vee b)^{\Downarrow}$. Therefore, the map $y \mapsto y^{\Downarrow}$ is a join homomorphism.

(c): This is the dual argument of (b). ■

The following lemma concerns subdirect product of finite lattices $\mathbf{K}$ and $\mathbf{M}$ containing $(\{0\} \times \mathbf{M}) \cup (\mathbf{K} \times \{1\})$.

Lemma 2.7. *If* $\mathbf{L}$ *is a subdirect product of finite lattices* $\mathbf{K}$ *and* $\mathbf{M}$*, then the following are equivalent:*

(a) $\langle 0, 1\rangle$ *is in* $\mathbf{L}$,

(b) $\mathbf{K} \times \{1\} \subseteq \mathbf{L}$,

(c) $\{0\} \times \mathbf{M} \subseteq \mathbf{L}$,

(d) $(\{0\} \times \mathbf{M}) \cup (\mathbf{K} \times \{1\}) \subseteq \mathbf{L}$,

(e) $y^{\Downarrow} = 0$ *for all* y *in* $\mathbf{M}$,

(f) $1^{\Downarrow} = 0$.

Proof. Suppose $\langle 0, 1\rangle$ is in $\mathbf{L}$. Since $\mathbf{L}$ is a subdirect product, for each x in $\mathbf{K}$, there exists some a in $\mathbf{M}$ such that that $\langle x, a\rangle$ is in $\mathbf{L}$. Hence, $\langle x, a\rangle \vee \langle 0, 1\rangle = \langle x, 1\rangle$ is in $\mathbf{L}$ for all x in $\mathbf{K}$. Thus, $\mathbf{K} \times \{1\} \subseteq \mathbf{L}$ and, by a similar argument, $\{0\} \times \mathbf{M} \subseteq \mathbf{L}$ as well. This in turn implies that $(\{0\} \times \mathbf{M}) \cup (\mathbf{K} \times \{1\}) \subseteq \mathbf{L}$. In addition, $\langle 0, 1\rangle$ in $\mathbf{L}$ implies $1^{\Downarrow} = 0$. Thus, (a) implies (b), (c), (d), and (f).

Now note that $\{0\} \times \mathbf{M}$, $\mathbf{K} \times \{1\}$, and $(\{0\} \times \mathbf{M}) \cup (\mathbf{K} \times \{1\})$ all contain $\langle 0, 1\rangle$, so (b), (c), and (d) all imply (a). As well, $\langle 1^{\Downarrow}, 1\rangle = \langle 0, 1\rangle$ is in $\mathbf{L}$ so (f) implies (a). Finally, $\{0\} \times \mathbf{M} \subseteq \mathbf{L}$ implies that $\langle 0, y\rangle$ is in $\mathbf{L}$ for all y in $\mathbf{M}$. And so, $y^{\Downarrow} = 0$ for all y in $\mathbf{M}$. Moreover, $1^{\Downarrow} = 0$ and $\langle 0, 1\rangle$ is in $\mathbf{L}$. ■

Lemma 2.6 states that $a \wedge b = b$ implies $a^{\Uparrow} \wedge b^{\Uparrow} = b^{\Uparrow}$ for all a and b in $\mathbf{K}$. This fact, combined with Lemma 2.7, yield the following corollary.

Corollary 2.8. *Let* $\mathbf{L}$ *be a subdirect product of finite* $\mathbf{K}\times\mathbf{M}$ *containing* $(\{0\}\times\mathbf{M})\cup(\mathbf{K}\times\{1\})$. *Then*

(a) $\mathbf{L} = \bigcup_{y\in\mathbf{M}}\{\langle x,y\rangle \in \mathbf{K}\times\mathbf{M} : 0 \le x \le y^{\Uparrow}\} = \bigcup_{y\in\mathbf{M}}\{\langle x,y\rangle \in \mathbf{K}\times\mathbf{M} : x \le y^{\Uparrow}\}$, *and*

(b) $y \le z$ *implies* $y^{\Uparrow} \le z^{\Uparrow}$ *for all* y *and* z *in* $\mathbf{M}$.

Before starting the main part of the proof, the following necessary lemma shows that certain subsets of elements of $\mathbf{K}\times\mathbf{M}$ are sublattices of $\mathbf{K}\times\mathbf{M}$.

Lemma 2.9. *If* L *is a subset of* $\mathbf{K}\times\mathbf{M}$ *such that*

$$L = \bigcup_{y\in\mathbf{M}}\{\langle x,y\rangle \in \mathbf{K}\times\mathbf{M} : 0 \le x \le f(y)\},$$

where f *is a meet-homomorphism such that* $f : \mathbf{M}\to\mathbf{K}$ *and* $y \le z$ *implies* $f(y) \le f(z)$ *for all* y *and* z *in* $\mathbf{M}$, *then* $\mathbf{L} \le \mathbf{K}\times\mathbf{M}$. *Similarly, if*

$$L = \bigcup_{x\in\mathbf{K}}\{\langle x,y\rangle \in \mathbf{K}\times\mathbf{M} : g(x) \le y \le 1\},$$

where g *is a join-homomorphism such that* $g : \mathbf{K}\to\mathbf{M}$ *and* $x \le z$ *implies* $g(x) \le g(z)$ *for all* x *and* z *in* $\mathbf{K}$, *then* $\mathbf{L} \le \mathbf{K}\times\mathbf{M}$.

Proof. First, consider $L = \bigcup_{y\in\mathbf{M}}\{\langle x,y\rangle \in \mathbf{K}\times\mathbf{M} : 0 \le x \le f(y)\}$. Let $\langle s,t\rangle$ and $\langle u,v\rangle$ be in L. Then $\langle s,t\rangle \wedge \langle u,v\rangle = \langle s\wedge u, t\wedge v\rangle$ where $t\wedge v$ is in $\mathbf{M}$. Also, since $s \le f(t)$ and $u \le f(v)$, the element $s\wedge u \le f(t)\wedge f(v) \le f(t\wedge v)$. Hence $\langle s\wedge u, t\wedge v\rangle = \langle s,t\rangle \wedge \langle u,v\rangle$ is in L.

Now, $\langle s,t\rangle \vee \langle u,v\rangle = \langle s\vee u, t\vee v\rangle$ where $t\vee v$ is in $\mathbf{M}$. Also, since $s \le f(t)$ and $u \le f(v)$, the element $s\vee u \le f(t)\vee f(v) \le f(t\vee v)\vee f(t\vee v) = f(t\vee v)$. Thus $\langle s\vee u, t\vee v\rangle = \langle s,t\rangle \vee \langle u,v\rangle$ is in L. Therefore, since L is closed under meet and join, the lattice $\mathbf{L}$ is a sublattice of $\mathbf{K}\times\mathbf{M}$.

If $L = \bigcup_{x \in \mathbf{K}} \{\langle x, y\rangle \in \mathbf{K} \times \mathbf{M} : g(x) \leq y \leq 1\}$, then by a dual argument, $\mathbf{L} \leq \mathbf{K} \times \mathbf{M}$. ■

Corollary 2.8 (a) implies that a sublattice of $\mathbf{S}_7 \times \mathbf{S}_7$ containing $(\{0\} \times \mathbf{S}_7) \cup (\mathbf{S}_7 \times \{1\})$ is uniquely determined by the values of the $^{\Uparrow}$ function. The following series of lemmas show that all but five subdirect products of $\mathbf{S}_7 \times \mathbf{S}_7$ containing $(\{0\} \times \mathbf{S}_7) \cup (\mathbf{S}_7 \times \{1\})$ are always congruence lattices of algebras on the underlying set of $\mathbf{A} \times \mathbf{A}$. The remaining five sublattices will be congruence lattices on the underlying set of $\mathbf{A} \times \mathbf{A}$ if and only if a particular one of the five is. As it turns out, the following proofs are dependent almost entirely on $y^{\Uparrow}$ where y is any of the three co-atoms a, b, or c of $\mathbf{S}_7$.

I prove the various subdirect products containing $(\{0\} \times \mathbf{S}_7) \cup (\mathbf{S}_7 \times \{1\})$ are congruence lattices by considering first in Lemma 2.10 the ones that satisfy $a^{\Uparrow} \leq c^{\Uparrow}$ or $b^{\Uparrow} \leq c^{\Uparrow}$. I next show in Lemma 2.11 that if $\mathbf{L}$ satisfies $a^{\Uparrow} > c^{\Uparrow}$ and $b^{\Uparrow} > c^{\Uparrow}$, then $\mathbf{L}$ is a congruence lattice. After this, I show in Lemma 2.12 that two of the the four subdirect products satisfying $a^{\Uparrow} > c^{\Uparrow}$ and $b^{\Uparrow} \parallel c^{\Uparrow}$ are congruence lattices of algebras on the underlying set of $\mathbf{A} \times \mathbf{A}$. Similarly, by a symmetric argument to Lemma 2.12, I show that two of the the four subdirect products satisfying $b^{\Uparrow} > c^{\Uparrow}$ and $a^{\Uparrow} \parallel c^{\Uparrow}$ are congruence lattices. Then, in Lemma 2.13, I show that one of the two subdirect products satisfying $a^{\Uparrow} \parallel c^{\Uparrow}$ and $b^{\Uparrow} \parallel c^{\Uparrow}$ is a congruence lattice. Finally, in Theorem 2.14, I deal with the five remaining subdirect products. Throughout the proofs, the elements of $\mathbf{S}_7$ will be referred to according to the labelling given in Figure 2.1.

Lemma 2.10. *Suppose* $\mathbf{L}$ *is a subdirect product of* $\mathbf{S}_7 \times \mathbf{S}_7$ *containing the elements* $(\{0\} \times \mathbf{S}_7) \cup (\mathbf{S}_7 \times \{1\})$. *If* $a^{\Uparrow} \leq c^{\Uparrow}$ *or* $b^{\Uparrow} \leq c^{\Uparrow}$, *then* $\mathbf{L}$ *is the congruence lattice of an algebra on the underlying set of* $\mathbf{A} \times \mathbf{A}$.

Proof. First, suppose $a^{\Uparrow} \leq c^{\Uparrow}$. I show in the following that $L = U \cap V \cap W$ where

$$\begin{aligned} U &= \{\langle x, y\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : \langle x, y\rangle \geq \langle 0, b\rangle \text{ or } \langle x, y\rangle \leq \langle c^{\Uparrow}, 1\rangle\}, \\ V &= \{\langle x, y\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : \langle x, y\rangle \geq \langle 0, d\rangle \text{ or } \langle x, y\rangle \leq \langle b^{\Uparrow}, 1\rangle\}, \text{ and} \\ W &= \{\langle x, y\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : \langle x, y\rangle \geq \langle 0, e\rangle \text{ or } \langle x, y\rangle \leq \langle a^{\Uparrow}, a\rangle\}. \end{aligned}$$

Then, by Lemma 1.5, since $\mathbf{S}_7 \times \mathbf{S}_7$ is a congruence lattice, so are $\mathbf{U}, \mathbf{V}$, and $\mathbf{W}$. Therefore, since $\mathbf{L} = \mathbf{U} \cap \mathbf{V} \cap \mathbf{W}$ as shown below, Lemma 2.4 states that $\mathbf{L}$ is also a congruence lattice.

First, I show $L \subseteq U$. Note that for all $\langle x, y\rangle$ in L such that $y^{\Uparrow} \leq c^{\Uparrow}$, the pair $\langle x, y\rangle \leq \langle c^{\Uparrow}, 1\rangle$ by Lemma 2.6 (a). By assumption $a^{\Uparrow} \leq c^{\Uparrow}$, so it follows from Corollary 2.8 that $d^{\Uparrow}, e^{\Uparrow}$, and $0^{\Uparrow}$ are all less than $c^{\Uparrow}$. For every $\langle x, y\rangle$ in L such that $y = b$ or $y = 1$ the pair $\langle x, y\rangle$ is greater than or equal to $\langle 0, b\rangle$. Therefore, $L \subseteq U$.

Next, I show $L \subseteq V$. For all $\langle x, y\rangle$ in L such that $y \geq d$ it follows directly that $\langle x, y\rangle \geq \langle 0, d\rangle$. This leaves for consideration the set $\{\langle x, y\rangle \in L : y \leq b\}$. Since $e^{\Uparrow} \leq b^{\Uparrow}$ and $0^{\Uparrow} \leq b^{\Uparrow}$, for all $\langle x, y\rangle$ in L such that $y \leq b$ it follows that $\langle x, y\rangle \leq \langle b^{\Uparrow}, 1\rangle$. Therefore, $L \subseteq V$.

Now to show $L \subseteq W$, I first note that for all $\langle x, y\rangle$ in L where $y \geq e$ the pair $\langle x, y\rangle \geq \langle 0, e\rangle$. This leaves the set $\{\langle x, y\rangle \in L : y \leq a\}$ for consideration. Since $d^{\Uparrow} \leq a^{\Uparrow}$ and $0^{\Uparrow} \leq a^{\Uparrow}$ by Corollary 2.8, for all $\langle x, y\rangle$ in L such that $y \leq a$ it follows that $\langle x, y\rangle \leq \langle a^{\Uparrow}, a\rangle$. Hence $L \subseteq W$. Thus, because L is a subset of U, V, and W, I have $L \subseteq U \cap V \cap W$.

I next show that $U \cap V \cap W \subseteq L$. Suppose $\langle x, y\rangle$ is an element of $U \cap V \cap W$. Then $\langle x, y\rangle$ is either greater than or less than a defining element in each of the three sets. Hence there are eight cases. Only six cases must be considered, though, as there are no elements of $\mathbf{S}_7 \times \mathbf{S}_7$ that satisfy $\langle x, y\rangle \geq \langle 0, b\rangle$ and $\langle x, y\rangle \leq \langle a^{\Uparrow}, a\rangle$.

Fix $\langle x, y\rangle$ in $U \cap V \cap W$. First, assume $\langle x, y\rangle \geq \langle 0, b\rangle$, $\langle x, y\rangle \geq \langle 0, d\rangle$, and $\langle x, y\rangle \geq$

$\langle 0, e\rangle$. Thus $\langle x, y\rangle \geq \langle 0, b\rangle \vee \langle 0, d\rangle = \langle 0, 1\rangle$. Therefore, $\langle x, y\rangle$ is in $(\mathbf{S}_7 \times \{1\}) \subseteq L$.

Second, assume $\langle x, y\rangle \geq \langle 0, b\rangle$, $\langle x, y\rangle \leq \langle b^{\Uparrow}, 1\rangle$, and $\langle x, y\rangle \geq \langle 0, e\rangle$. Since $\langle 0, b\rangle > \langle 0, e\rangle$, the pair $\langle x, y\rangle$ is an element in $\mathbf{S}_7 \times \mathbf{S}_7$ such that $0 \leq x \leq b^{\Uparrow}$ and $b \leq y \leq 1$. But this just means $\langle x, y\rangle$ is in one of the sets $\{\langle x, b\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq x \leq b^{\Uparrow}\}$ or $\{\langle x, 1\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq x \leq b^{\Uparrow}\} \subseteq \{\langle x, 1\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq x \leq 1^{\Uparrow}\}$ and both are contained in L by Corollary 2.8.

Third, assume $\langle x, y\rangle \leq \langle c^{\Uparrow}, 1\rangle$, $\langle x, y\rangle \geq \langle 0, d\rangle$, and $\langle x, y\rangle \geq \langle 0, e\rangle$. Then $\langle x, y\rangle \geq \langle 0, d\rangle \vee \langle 0, e\rangle = \langle 0, c\rangle$. Hence $\langle x, y\rangle$ is an element such that $0 \leq x \leq c^{\Uparrow}$ and $c \leq y \leq 1$. Thus $\langle x, y\rangle$ is in one of the sets $\{\langle x, c\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq x \leq c^{\Uparrow}\}$ or $\{\langle x, 1\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq x \leq c^{\Uparrow}\} \subseteq \{\langle x, 1\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq x \leq 1^{\Uparrow}\}$, both contained in L by Corollary 2.8.

Fourth, assume $\langle x, y\rangle \leq \langle c^{\Uparrow}, 1\rangle$, $\langle x, y\rangle \geq \langle 0, d\rangle$, and $\langle x, y\rangle \leq \langle a^{\Uparrow}, a\rangle$. Since $a^{\Uparrow} \leq c^{\Uparrow}$ it follows that $\langle a^{\Uparrow}, a\rangle \wedge \langle c^{\Uparrow}, 1\rangle = \langle a^{\Uparrow}, a\rangle$. Then, $\langle x, y\rangle$ satisfies $0 \leq x \leq a^{\Uparrow}$ and $d \leq y \leq a$. Now, from Lemma 2.6 (c), $a \wedge c = d$ implies that $a^{\Uparrow} \wedge c^{\Uparrow} = d^{\Uparrow}$ which in turn implies that $a^{\Uparrow} = d^{\Uparrow}$. Therefore, $\langle x, y\rangle$ is either in $\{\langle x, a\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq x \leq a^{\Uparrow}\}$ or $\{\langle x, d\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq x \leq a^{\Uparrow}\} = \{\langle x, d\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq x \leq d^{\Uparrow}\}$, both of which are contained in L by Corollary 2.8.

Fifth, assume $\langle x, y\rangle \leq \langle c^{\Uparrow}, 1\rangle$, $\langle x, y\rangle \leq \langle b^{\Uparrow}, 1\rangle$, and $\langle x, y\rangle \geq \langle 0, e\rangle$. Then $\langle x, y\rangle \leq \langle c^{\Uparrow}, 1\rangle \wedge \langle b^{\Uparrow}, 1\rangle = \langle e^{\Uparrow}, 1\rangle$ by Lemma 2.6. Hence $\langle x, y\rangle$ satisfies $0 \leq x \leq e^{\Uparrow}$ and $e \leq y \leq 1$ which means $\langle x, y\rangle$ is in $\{\langle x, e\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq x \leq e^{\Uparrow}\}$, $\{\langle x, b\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq x \leq e^{\Uparrow}\} \subseteq \{\langle x, b\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq x \leq b^{\Uparrow}\}$, $\{\langle x, c\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq x \leq e^{\Uparrow}\} \subseteq \{\langle x, c\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq x \leq c^{\Uparrow}\}$, or $\{\langle x, 1\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq x \leq e^{\Uparrow}\} \subseteq \{\langle x, 1\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq x \leq 1^{\Uparrow}\}$ which are all contained in L by Corollary 2.8.

Finally, assume $\langle x, y\rangle \leq \langle c^{\Uparrow}, 1\rangle$, $\langle x, y\rangle \leq \langle b^{\Uparrow}, 1\rangle$, and $\langle x, y\rangle \leq \langle a^{\Uparrow}, a\rangle$. The pair $\langle x, y\rangle$ must be less than or equal to $\langle c^{\Uparrow}, 1\rangle \wedge \langle b^{\Uparrow}, 1\rangle \wedge \langle a^{\Uparrow}, a\rangle = \langle 0^{\Uparrow}, a\rangle$. Thus $\langle x, y\rangle$ is in one of the sets $\{\langle x, a\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq x \leq 0^{\Uparrow}\} \subseteq \{\langle x, a\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq x \leq a^{\Uparrow}\}$, $\{\langle x, d\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq x \leq 0^{\Uparrow}\} \subseteq \{\langle x, d\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq x \leq d^{\Uparrow}\}$, or $\{\langle x, 0\rangle \in$

$\mathbf{S}_7 \times \mathbf{S}_7 : 0 \le x \le 0^{\Uparrow}\}$, all contained in L by Corollary 2.8.

Therefore, $U \cap V \cap W \subseteq L$ and I conclude that $\mathbf{L} = \mathbf{U} \cap \mathbf{V} \cap \mathbf{W}$. Therefore, by Lemma 2.4, $\mathbf{L}$ is the congruence lattice of an algebra on the universe of $\mathbf{A} \times \mathbf{A}$.

If $b^{\Uparrow} \le c^{\Uparrow}$, then the argument is completely symmetric and may be done by interchanging a and b and interchanging d and e in the previous paragraphs. ■

The previous lemma covers the cases where either $a^{\Uparrow} \le c^{\Uparrow}$ or $b^{\Uparrow} \le c^{\Uparrow}$ and I now consider the cases where $a^{\Uparrow} \not\le c^{\Uparrow}$ and $b^{\Uparrow} \not\le c^{\Uparrow}$.

Lemma 2.11. *Suppose* $\mathbf{L}$ *is a subdirect product of* $\mathbf{S}_7 \times \mathbf{S}_7$ *containing the elements* $(\{0\} \times \mathbf{S}_7) \cup (\mathbf{S}_7 \times \{1\})$. *If* $b^{\Uparrow} > c^{\Uparrow}$ *and* $a^{\Uparrow} > c^{\Uparrow}$, *then* $\mathbf{L}$ *is the congruence lattice of an algebra on the underlying set of* $\mathbf{A} \times \mathbf{A}$.

Proof. Similar to the proof of Lemma 2.10, I show in the following that $L = U \cap V$ where

$$\begin{aligned} U &= \{\langle x, y\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : \langle x, y\rangle \ge \langle 0, b\rangle \text{ or } \langle x, y\rangle \le \langle a^{\Uparrow}, 1\rangle\}, \text{ and} \\ V &= \{\langle x, y\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : \langle x, y\rangle \ge \langle 0, a\rangle \text{ or } \langle x, y\rangle \le \langle b^{\Uparrow}, 1\rangle\}. \end{aligned}$$

Then, by Lemma 1.5, since $\mathbf{S}_7 \times \mathbf{S}_7$ is a congruence lattice, so are $\mathbf{U}$ and $\mathbf{V}$ and it follows, by Lemma 2.4, that $\mathbf{L}$ is also a congruence lattice.

First, I show that $L \subseteq U \cap V$. Since $c^{\Uparrow} < a^{\Uparrow}$, it follows that $d^{\Uparrow}$,$e^{\Uparrow}$, and $0^{\Uparrow}$ are all less than $a^{\Uparrow}$. Then $\langle x, y\rangle \not\le \langle a^{\Uparrow}, 1\rangle$ only if $y = b$ or $y = 1$. But this means $\langle x, y\rangle \ge \langle 0, b\rangle$. Thus, $L \subseteq U$ and by a similar argument $L \subseteq V$. Therefore, $L \subseteq U \cap V$.

Now, to show $U \cap V \subseteq L$, pick $\langle x, y\rangle$ in $U \cap V$. Then $\langle x, y\rangle$ is either greater than or less than an element in both of the sets. First, assume $\langle x, y\rangle \ge \langle 0, b\rangle$ and $\langle x, y\rangle \ge \langle 0, a\rangle$. Then $\langle x, y\rangle$ is greater than or equal to $\langle 0, b\rangle \vee \langle 0, a\rangle = \langle 0, 1\rangle$ and, since $(\mathbf{S}_7 \times \{1\})$ is contained in L, the pair $\langle x, y\rangle$ must be in L.

Next, assume $\langle x, y\rangle \geq \langle 0, b\rangle$ and $\langle x, y\rangle \leq \langle b^{\Uparrow}, 1\rangle$. If $\langle 0, b\rangle \leq \langle x, y\rangle \leq \langle b^{\Uparrow}, 1\rangle$, then $\langle x, y\rangle$ is in $\{\langle x, b\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq x \leq b^{\Uparrow}\}$ or $\{\langle x, 1\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq x \leq b^{\Uparrow}\}$, both of which are contained in L as $b^{\Uparrow} \leq 1^{\Uparrow}$. The case when $\langle x, y\rangle \leq \langle a^{\Uparrow}, 1\rangle$ and $\langle x, y\rangle \geq \langle 0, a\rangle$ follows a symmetric argument.

Finally, assume $\langle x, y\rangle \leq \langle a^{\Uparrow}, 1\rangle$ and $\langle x, y\rangle \leq \langle b^{\Uparrow}, 1\rangle$. Since $\langle a^{\Uparrow}, 1\rangle \wedge \langle b^{\Uparrow}, 1\rangle = \langle 0^{\Uparrow}, 1\rangle$, the pair $\langle x, y\rangle$ is in $\{\langle x, y\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq x \leq 0^{\Uparrow}\}$ which is contained in $\{\langle x, y\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq x \leq y^{\Uparrow}\}$, a subset of L by Corollary 2.8.

Thus, $U \cap V \subseteq L$ and it follows that $\mathbf{L} = \mathbf{U} \cap \mathbf{V}$. Therefore, $\mathbf{L}$ is the congruence lattice of an algebra on the universe of $\mathbf{A} \times \mathbf{A}$. ■

I now reduce the number of remaining cases to be considered. Assume for now that $\mathbf{L}$ is a subdirect product of $\mathbf{S}_7 \times \mathbf{S}_7$ containing the elements $(\{0\} \times \mathbf{S}_7) \cup (\mathbf{S}_7 \times \{1\})$ where $a^{\Uparrow} > c^{\Uparrow}$ and $b^{\Uparrow}$ is incomparable with $c^{\Uparrow}$.

Suppose $c^{\Uparrow}$ is in the set $\{a, b, c\}$. Then $a^{\Uparrow} = 1$ which implies $\langle 1, a\rangle \wedge \langle b^{\Uparrow}, b\rangle = \langle b^{\Uparrow}, 0\rangle$ is in $\mathbf{L}$. Hence $b^{\Uparrow} \leq 0^{\Uparrow}$ but $0^{\Uparrow} \leq c^{\Uparrow}$ so $b^{\Uparrow} \leq c^{\Uparrow}$ which contradicts our assumption. Thus, $c^{\Uparrow} = d$ or $c^{\Uparrow} = e$.

Now, suppose $c^{\Uparrow} = d$. Then $a^{\Uparrow}$ is in $\{a, c\}$ as $a^{\Uparrow} = 1$ would lead to a contradiction similar to the one in the previous paragraph. Also, $b^{\Uparrow}$ is in $\{b, e\}$ which implies $\langle e, b\rangle$ is in $\mathbf{L}$. If $a^{\Uparrow} = c$, then $\langle c, a\rangle$ is in $\mathbf{L}$ which implies $\langle c, a\rangle \wedge \langle e, b\rangle = \langle e, 0\rangle$ is also in $\mathbf{L}$. Thus $\langle e, 0\rangle \vee \langle 0, c\rangle = \langle e, c\rangle$ is in $\mathbf{L}$, as well. But this leads to a contradiction as $\langle e, c\rangle \vee \langle d, c\rangle = \langle c, c\rangle$ will be an element of $\mathbf{L}$ yielding $a^{\Uparrow} \leq c^{\Uparrow}$.

Therefore, if $c^{\Uparrow} = d$, then $a^{\Uparrow} = a$ and $b^{\Uparrow}$ is in $\{b, e\}$. By a parallel argument, if $c^{\Uparrow} = e$, then $a^{\Uparrow} = b$ and $b^{\Uparrow}$ is in $\{a, d\}$. Consequently, there are only four subdirect products of $\mathbf{S}_7 \times \mathbf{S}_7$ containing $(\{0\} \times \mathbf{S}_7) \cup (\mathbf{S}_7 \times \{1\})$ that occur when $a^{\Uparrow} > c^{\Uparrow}$ and $b^{\Uparrow} \parallel c^{\Uparrow}$. I show in the next lemma that the two subdirect products satisfying $c^{\Uparrow} = d$, $a^{\Uparrow} = a$, and $b^{\Uparrow}$ in $\{b, e\}$ are congruence lattices whereas the two subdirect products resulting when $c^{\Uparrow} = e$, $a^{\Uparrow} = b$, and $b^{\Uparrow}$ is in $\{a, d\}$ will be dealt with in the

Theorem 2.14.

Lemma 2.12. *Suppose* $\mathbf{L}$ *is a subdirect product of* $\mathbf{S}_7 \times \mathbf{S}_7$ *containing the elements* $(\{0\} \times \mathbf{S}_7) \cup (\mathbf{S}_7 \times \{1\})$. *If* $c^{\Uparrow} = d$, $a^{\Uparrow} = a$, *and* $b^{\Uparrow}$ *is in* $\{b, e\}$, *then* $\mathbf{L}$ *is the congruence lattice of an algebra on the underlying set of* $\mathbf{A} \times \mathbf{A}$.

Proof. Let h be in $\{b, e\}$ and let $f : \mathbf{S}_7 \to \mathbf{S}_7$ be the meet homomorphism $f(0) = f(e) = 0$, $f(d) = f(c) = f(a) = a$, $f(1) = 1$, and $f(b) = h$. I show in the following that $L = U \cap V$ where

$$\begin{aligned} U &= \{\langle x, y\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : x \leq y\} \text{ and} \\ V &= \{\langle x, y\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : x \leq f(y)\}. \end{aligned}$$

Recall

$$y_V^{\Uparrow} = \bigvee \{x \in \mathbf{S}_7 : \langle x, y\rangle \in V\}.$$

Then Lemma 2.9 states that $\mathbf{V}$ is a lattice and note that $\mathbf{V}$ contains $(\{0\} \times \mathbf{S}_7) \cup (\mathbf{S}_7 \times \{1\})$. Hence, since $a_V^{\Uparrow} \leq c_V^{\Uparrow}$, from Lemma 2.10 it follows that $\mathbf{V}$ is a congruence lattice on the universe of $\mathbf{A} \times \mathbf{A}$. Since the set V satisfies the hypothesis of Lemma 2.9, the lattice $\mathbf{V}$ is a sublattice $\mathbf{S}_7 \times \mathbf{S}_7$. Thus, $\mathbf{U}$ is a congruence lattice on the universe of $\mathbf{A} \times \mathbf{A}$ by Lemma 2.3 and $\mathbf{V}$ is a congruence lattice on the universe of $\mathbf{A} \times \mathbf{A}$ by Lemma 2.10.

The sublattice of $\mathbf{S}_7 \times \mathbf{S}_7$ corresponding to $\mathbf{U} \cap \mathbf{V}$ is a subdirect product of $\mathbf{S}_7 \times \mathbf{S}_7$ containing $(\{0\} \times \mathbf{S}_7) \cup (\mathbf{S}_7 \times \{1\})$ such that $a^{\Uparrow} = a$, $c^{\Uparrow} = c \wedge a = d$, $b^{\Uparrow} = b \wedge h = h$, $d^{\Uparrow} = d \wedge a = d$, $e^{\Uparrow} = e \wedge 0 = 0$, and $0^{\Uparrow} = 0$. This is precisely $\mathbf{L}$, hence $\mathbf{L} = \mathbf{U} \cap \mathbf{V}$. Therefore, $\mathbf{L}$ is a congruence lattice on the universe of $\mathbf{A} \times \mathbf{A}$ by Lemma 2.4. ∎

Now suppose $\mathbf{L}$ is a subdirect product of $\mathbf{S}_7 \times \mathbf{S}_7$ containing the elements $(\{0\} \times \mathbf{S}_7) \cup (\mathbf{S}_7 \times \{1\})$ where $b^{\Uparrow} > c^{\Uparrow}$ and $a^{\Uparrow} \parallel c^{\Uparrow}$. By an argument similar to the one preceding Lemma 2.12, it follows that either $c^{\Uparrow} = e$, $b^{\Uparrow} = b$, and $a^{\Uparrow}$ is in $\{a, d\}$, or $c^{\Uparrow} = d$, $b^{\Uparrow} = a$, and $a^{\Uparrow}$ is in $\{b, e\}$. A symmetric proof to that of Lemma 2.12 shows that the two subdirect products satisfying $c^{\Uparrow} = e$, $b^{\Uparrow} = b$, and $a^{\Uparrow}$ in $\{a, d\}$ are congruence lattices of algebras on the universe of $\mathbf{A} \times \mathbf{A}$. The two subdirect products resulting when $c^{\Uparrow} = d$, $b^{\Uparrow} = a$, and $a^{\Uparrow}$ is in $\{b, e\}$ are dealt with in Theorem 2.14.

The last case to consider is $a^{\Uparrow} \parallel c^{\Uparrow}$ and $b^{\Uparrow} \parallel c^{\Uparrow}$. First, suppose $a^{\Uparrow}$ is comparable to $b^{\Uparrow}$. Without loss of generality, suppose $a^{\Uparrow} \leq b^{\Uparrow}$. Then $\langle a^{\Uparrow}, a\rangle \wedge \langle b^{\Uparrow}, b\rangle = \langle a^{\Uparrow}, 0\rangle$ is an element of $\mathbf{L}$. But this means $a^{\Uparrow} \leq 0^{\Uparrow} \leq c^{\Uparrow}$ which contradicts our assumption. Hence $a^{\Uparrow} \parallel b^{\Uparrow}$ as well.

Now, the only possible three element antichain in $\mathbf{S}_7$ is the set $\{a, b, c\}$ which means $\{a^{\Uparrow}, b^{\Uparrow}, c^{\Uparrow}\} = \{a, b, c\}$.

Suppose $c^{\Uparrow} = a$. That means $b^{\Uparrow} = c$ or $b^{\Uparrow} = b$. If $b^{\Uparrow} = c$, then $a^{\Uparrow} = b$ so $\langle c, b\rangle \wedge \langle b, a\rangle = \langle e, 0\rangle$ is in $\mathbf{L}$. Thus $\langle 0, c\rangle \vee \langle e, 0\rangle = \langle e, c\rangle$ is also in $\mathbf{L}$. But then $\langle e, c\rangle \vee \langle a, c\rangle = \langle 1, c\rangle$ is in $\mathbf{L}$ which means $b^{\Uparrow} \leq c^{\Uparrow}$, a contradiction. Similarly, if $b^{\Uparrow} = b$ when $c^{\Uparrow} = a$, then $\langle c, a\rangle \wedge \langle b, b\rangle = \langle e, 0\rangle$ is in $\mathbf{L}$ so by the same reasoning $a^{\Uparrow} \leq c^{\Uparrow}$, another contradiction. Supposing $c^{\Uparrow} = b$ leads to the the same contradictions.

Therefore, in the case when $a^{\Uparrow} \parallel c^{\Uparrow}$ and $b^{\Uparrow} \parallel c^{\Uparrow}$, the subdirect product must satisfy $c^{\Uparrow} = c$ and either $a^{\Uparrow} = a$ and $b^{\Uparrow} = b$ or $a^{\Uparrow} = b$ and $b^{\Uparrow} = a$. The first of these is dealt with in the following lemma, the second in Theorem 2.14.

Lemma 2.13. *Suppose* $\mathbf{L}$ *is a subdirect product of* $\mathbf{S}_7 \times \mathbf{S}_7$ *containing the elements* $(\{0\} \times \mathbf{S}_7) \cup (\mathbf{S}_7 \times \{1\})$. *If* $\mathbf{L}$ *satisfies* $c^{\Uparrow} = c$, $a^{\Uparrow} = a$, *and* $b^{\Uparrow} = b$, *then* $\mathbf{L}$ *is a congruence lattice of an algebra on the underlying set of* $\mathbf{A} \times \mathbf{A}$.

Proof. If $c^{\Uparrow} = c$, $a^{\Uparrow} = a$, and $b^{\Uparrow} = b$, then, by Lemma 2.6 (c), $d^{\Uparrow} = d$, $e^{\Uparrow} = e$, and $0^{\Uparrow} = 0$. Thus $\mathbf{L} = \{\langle x, y\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : x \leq y\}$ and is a congruence lattice by Lemma 2.3. ■

Now, define $\mathbf{L}_0$ to be the subdirect product of $\mathbf{S}_7 \times \mathbf{S}_7$ containing $(\{0\} \times \mathbf{S}_7) \cup (\mathbf{S}_7 \times \{1\})$ that satisfies $c^{\Uparrow} = c$, $a^{\Uparrow} = b$, and $b^{\Uparrow} = a$ (see Figure 2.2). Then, by

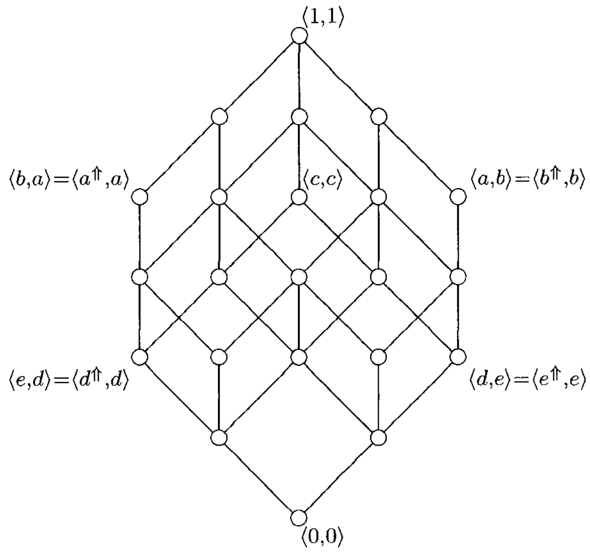

Figure 2.2: The lattice $\mathbf{L}_0$ in Theorem 2.14 with some elements labelled

Lemma 2.6 (c), $d^{\Uparrow} = e$, $e^{\Uparrow} = d$, and $0^{\Uparrow} = 0$ and by Corollary 2.8 the lattice $\mathbf{L}_0$ can be written as

$$\begin{aligned}\mathbf{L}_0 = \{\langle 0,0\rangle\} \\ &\cup \{\langle x,d\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq x \leq e\} \\ &\cup \{\langle x,e\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq x \leq d\} \\ &\cup \{\langle x,a\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq x \leq b\} \\ &\cup \{\langle x,c\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq x \leq c\} \\ &\cup \{\langle x,b\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq x \leq a\} \\ &\cup \{\langle x,1\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq x \leq 1\}.\end{aligned}$$

It turns out that the remaining five subdirect products will be congruence lattices of algebras on the universe of $\mathbf{A} \times \mathbf{A}$ if and only if the lattice $\mathbf{L}_0$ is a congruence lattice of an algebra on the universe of $\mathbf{A} \times \mathbf{A}$. Thus, $\mathbf{S}_7$ is power-hereditary if $\mathbf{L}_0$ is a congruence lattice — the main result of this section.

Theorem 2.14. $\mathbf{Con}(\mathbf{A})$ *is a power-hereditary congruence lattice (and a power-hereditary representation of* $\mathbf{S}_7$*) if and only if* $\mathbf{L}_0$ *is the congruence lattice of an algebra on the underlying set of* $\mathbf{A} \times \mathbf{A}$.

Proof. Suppose $\mathbf{Con}(\mathbf{A})$ is a power-hereditary congruence lattice. Then, by definition, every 0-1 sublattice of $\mathbf{S}_7 \times \mathbf{S}_7$ is the congruence lattice of an algebra on the universe of $\mathbf{A} \times \mathbf{A}$. Therefore, $\mathbf{L}_0$ is also the congruence lattice of an algebra on the universe of $\mathbf{A} \times \mathbf{A}$.

For the other direction, suppose $\mathbf{L}_0$ is the congruence lattice of an algebra on $\mathbf{A} \times \mathbf{A}$. This leaves four subdirect products mentioned previously: the lattices that occur when $c^{\Uparrow} = e$, $a^{\Uparrow} = b$, and $b^{\Uparrow} = k$, where $k = a$ or d, and the lattices that occur when $c^{\Uparrow} = d$, $b^{\Uparrow} = a$, and $a^{\Uparrow} = h$, where $h = b$ or e.

Let $f : \mathbf{S}_7 \to \mathbf{S}_7$ be the meet homomorphism $f(0) = f(e) = 0$, $f(d) = f(c) = f(a) = b$, $f(1) = 1$, and $f(b) = k$ where k is in $\{a, d\}$. If $\mathbf{L}$ is a subdirect product of $\mathbf{S}_7 \times \mathbf{S}_7$ containing the elements $(\{0\} \times \mathbf{S}_7) \cup (\mathbf{S}_7 \times \{1\})$ that satisfies $c^{\Uparrow} = e$, $a^{\Uparrow} = b$, and $b^{\Uparrow} = k$, then $L = L_0 \cap V$ where

$$V = \{\langle x, y\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : x \leq f(y)\}.$$

Recall

$$y_V^{\Uparrow} = \bigvee \{x \in \mathbf{S}_7 : \langle x, y\rangle \in V\}.$$

Then Lemma 2.9 states that $\mathbf{V}$ is a lattice and note that $\mathbf{V}$ contains $(\{0\} \times \mathbf{S}_7) \cup (\mathbf{S}_7 \times \{1\})$. Hence, since $a_V^{\Uparrow} \leq c_V^{\Uparrow}$, from Lemma 2.10 it follows that $\mathbf{V}$ is a congruence lattice on the universe of $\mathbf{A} \times \mathbf{A}$. Thus, since $\mathbf{L}_0$ is a congruence lattice by assumption, $\mathbf{L}$ is also the congruence lattice of an algebra on the universe of $\mathbf{A} \times \mathbf{A}$. The rest of the argument is similar to that of Lemma 2.12.

Now, let $g : \mathbf{S}_7 \to \mathbf{S}_7$ be the meet homomorphism $g(0) = g(d) = 0$, $g(e) = g(c) =$

$g(b) = a$, $g(1) = 1$, and $g(a) = h$ where h is in $\{b, e\}$. If $\mathbf{L}$ is a subdirect product of $\mathbf{S}_7 \times \mathbf{S}_7$ containing the elements $(\{0\} \times \mathbf{S}_7) \cup (\mathbf{S}_7 \times \{1\})$ that satisfies $c^{\Uparrow} = d$, $b^{\Uparrow} = a$, and $a^{\Uparrow} = h$ in $\{b, e\}$, then $L = L_0 \cap W$ where

$$W \;=\; \{\langle x, y\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : x \leq g(y)\}.$$

Recall

$$y^{\Uparrow}_W = \bigvee\{x \in \mathbf{S}_7 : \langle x, y\rangle \in W\}.$$

Then Lemma 2.9 states that $\mathbf{W}$ is a lattice and note that $\mathbf{W}$ contains $(\{0\} \times \mathbf{S}_7) \cup (\mathbf{S}_7 \times \{1\})$. Hence, since $b^{\Uparrow}_W \leq c^{\Uparrow}_W$, from Lemma 2.10 it follows that $\mathbf{W}$ is a congruence lattice on the universe of $\mathbf{A} \times \mathbf{A}$. And so, since $\mathbf{L}_0$ is a congruence lattice by assumption, $\mathbf{L}$ is also the congruence lattice of an algebra on the universe of $\mathbf{A} \times \mathbf{A}$. The proof of this is again similar to that of Lemma 2.12.

Therefore, since every subdirect product of $\mathbf{S}_7 \times \mathbf{S}_7$ containing $(\{0\} \times \mathbf{S}_7) \cup (\mathbf{S}_7 \times \{1\})$ is the congruence lattice of an algebra on $\mathbf{A} \times \mathbf{A}$, it follows from Theorem 1.10 that $\mathbf{Con}(\mathbf{A})$ is a power-hereditary congruence lattice. ■

2.3 A necessary and sufficient condition for a representation of $\mathbf{S}_7{}^*$ to be power-hereditary

Theorem 2.14 gives a necessary and sufficient condition for a representation of $\mathbf{S}_7$ to be power-hereditary. In this section, I prove a similar condition for $\mathbf{S}_7{}^*$. Note that $\mathbf{S}_7{}^*$ is isomorphic to the dual of $\mathbf{S}_7$ under the mapping ψ where $\psi(1) = 1$, $\psi(0) = 0$, $\psi(s) = a$, $\psi(t) = b$, $\psi(r) = c$, $\psi(p) = d$ and $\psi(q) = e$ (see Figure 2.1).

For the sake of brevity, though, I note two important details that allow me to refer one to the proofs in the previous section. All of the proofs for the following lemmas and theorem are the dual arguments of Section 2.2, applied to the opposite coordinate of $\mathbf{S_7}^* \times \mathbf{S_7}^*$. Hence $\langle d, b\rangle \wedge \langle e, c\rangle = \langle 0, e\rangle$ in $\mathbf{S_7} \times \mathbf{S_7}$ is the dual statement of $\langle q, s\rangle \vee \langle r, t\rangle = \langle q, 1\rangle$ in $\mathbf{S_7}^* \times \mathbf{S_7}^*$ in opposite coordinates. Now, let $\mathbf{L}$ be a subdirect product of $\mathbf{K} \times \mathbf{M}$ for finite lattices $\mathbf{K}$ and $\mathbf{M}$. For all x in $\mathbf{K}$, define

$$x^{\uparrow} = \bigvee\{y \in \mathbf{M} : \langle x, y\rangle \in \mathbf{L}\} \text{ and}$$
$$x^{\downarrow} = \bigwedge\{y \in \mathbf{M} : \langle x, y\rangle \in \mathbf{L}\}.$$

Note that Lemma 2.6, Corollary 2.8, and Lemma 2.9 all hold in symmetric forms for $^{\uparrow}$ and $^{\downarrow}$.

For this section, let $\mathbf{A}$ be a finite algebra such that $\mathbf{Con}(\mathbf{A})$ is isomorphic to $\mathbf{S_7}^*$. Notationally then, when $\mathbf{S_7}^*$ is referred to it is as the lattice of congruences of $\mathbf{A}$. It is not known if such an $\mathbf{A}$ exists.

The main part of the proof begins with the following lemma.

Lemma 2.15. *Suppose* $\mathbf{L}$ *is a subdirect product of* $\mathbf{S_7}^* \times \mathbf{S_7}^*$ *containing the elements* $(\{0\} \times \mathbf{S_7}^*) \cup (\mathbf{S_7}^* \times \{1\})$*. If* $s^{\downarrow} \geq r^{\downarrow}$ *or* $t^{\downarrow} \geq r^{\downarrow}$*, then* $\mathbf{L}$ *is the congruence lattice of an algebra on the underlying set of* $\mathbf{A} \times \mathbf{A}$*.*

Proof. If $s^{\downarrow} \geq r^{\downarrow}$, then as in Lemma 2.10 $L = U \cap V \cap W$ where

$$\begin{aligned} U &= \{\langle x, y\rangle \in \mathbf{S_7}^* \times \mathbf{S_7}^* : \langle x, y\rangle \geq \langle 0, r^{\downarrow}\rangle \text{ or } \langle x, y\rangle \leq \langle t, 1\rangle\}, \\ V &= \{\langle x, y\rangle \in \mathbf{S_7}^* \times \mathbf{S_7}^* : \langle x, y\rangle \geq \langle 0, t^{\downarrow}\rangle \text{ or } \langle x, y\rangle \leq \langle p, 1\rangle\}, \text{ and} \\ W &= \{\langle x, y\rangle \in \mathbf{S_7}^* \times \mathbf{S_7}^* : \langle x, y\rangle \geq \langle s, s^{\downarrow}\rangle \text{ or } \langle x, y\rangle \leq \langle q, 1\rangle\}. \end{aligned}$$

Then, by Lemma 1.5, since $\mathbf{S_7}^* \times \mathbf{S_7}^*$ is a congruence lattice, so are $\mathbf{U}, \mathbf{V}$, and $\mathbf{W}$. Therefore, since $\mathbf{L} = \mathbf{U} \cap \mathbf{V} \cap \mathbf{W}$, by Lemma 2.4, $\mathbf{L}$ is also the congruence lattice of an algebra on $\mathbf{A} \times \mathbf{A}$.

If $t^{\downarrow} \geq r^{\downarrow}$, then the argument is symmetric to the previous paragraphs. ■

In the next lemma I consider the case where $s^{\downarrow} < r^{\downarrow}$ and $t^{\downarrow} < r^{\downarrow}$.

Lemma 2.16. *Suppose* $\mathbf{L}$ *is a subdirect product of* $\mathbf{S_7}^* \times \mathbf{S_7}^*$ *containing the elements* $(\{0\} \times \mathbf{S_7}^*) \cup (\mathbf{S_7}^* \times \{1\})$. *If* $s^{\downarrow} < r^{\downarrow}$ *and* $t^{\downarrow} < r^{\downarrow}$, *then* $\mathbf{L}$ *is the congruence lattice of an algebra on the underlying set of* $\mathbf{A} \times \mathbf{A}$.

Proof. If $s^{\downarrow} < r^{\downarrow}$ and $t^{\downarrow} < r^{\downarrow}$, then as in Lemma 2.11 $L = U \cap V$ where

$$\begin{aligned} U &= \{\langle x, y\rangle \in \mathbf{S_7}^* \times \mathbf{S_7}^* : \langle x, y\rangle \geq \langle 0, s^{\downarrow}\rangle \text{ or } \langle x, y\rangle \leq \langle t, 1\rangle\}, \text{ and} \\ V &= \{\langle x, y\rangle \in \mathbf{S_7}^* \times \mathbf{S_7}^* : \langle x, y\rangle \geq \langle 0, t^{\downarrow}\rangle \text{ or } \langle x, y\rangle \leq \langle s, 1\rangle\}. \end{aligned}$$

Then, by Lemma 1.5, since $\mathbf{S_7} \times \mathbf{S_7}$ is a congruence lattice, so are $\mathbf{U}$ and $\mathbf{V}$ and it follows, by Lemma 2.4, that $\mathbf{L}$ is also a congruence lattice. ■

In the case where $s^{\downarrow} < r^{\downarrow}$ and $t^{\downarrow} \parallel r^{\downarrow}$ there are four subdirect products of $\mathbf{S_7}^* \times \mathbf{S_7}^*$ that need to be considered — the lattices satisfying $r^{\downarrow} = p$, $s^{\downarrow} = s$, and $t^{\downarrow}$ is in $\{q, t\}$, and the lattices satisfying $r^{\downarrow} = q$, $s^{\downarrow} = t$, and $t^{\downarrow}$ is in $\{p, s\}$. I deal with the first two lattices in the following lemma.

Lemma 2.17. *Suppose* $\mathbf{L}$ *is a subdirect product of* $\mathbf{S_7}^* \times \mathbf{S_7}^*$ *containing the elements* $(\{0\} \times \mathbf{S_7}^*) \cup (\mathbf{S_7}^* \times \{1\})$. *If* $r^{\downarrow} = p$, $s^{\downarrow} = s$, *and* $t^{\downarrow}$ *is in* $\{q, t\}$, *then* $\mathbf{L}$ *is the congruence lattice of an algebra on the underlying set of* $\mathbf{A} \times \mathbf{A}$.

Proof. Let h be in $\{q, t\}$ and $f : \mathbf{S_7} \to \mathbf{S_7}$ be the join homomorphism such that $f(0) = f(q) = 1$, $f(s) = f(p) = f(r) = s$, $f(t) = h$, and $f(0) = 0$. If $\mathbf{L}$ satisfies $r^{\downarrow} = p$, $s^{\downarrow} = s$, and $t^{\downarrow} = h$, where $h = q$ or t, then, as in Lemma 2.12, $L = U \cap V$ where

$$\begin{aligned} U &= \{\langle x, y\rangle \in \mathbf{S_7}^* \times \mathbf{S_7}^* : x \leq y\} \text{ and} \\ V &= \{\langle x, y\rangle \in \mathbf{S_7}^* \times \mathbf{S_7}^* : f(x) \leq y\}. \end{aligned}$$

Recall

$$x_V^{\downarrow} = \bigwedge\{y \in \mathbf{S_7}^* : \langle x, y\rangle \in V\}.$$

Then a symmetric form of Lemma 2.9 states that $\mathbf{V}$ is a lattice and note that $\mathbf{V}$ contains $(\{0\} \times \mathbf{S_7}^*) \cup (\mathbf{S_7}^* \times \{1\})$. Hence, since $s_V^{\downarrow} \geq r_V^{\downarrow}$, from Lemma 2.15 it follows that $\mathbf{V}$ is a congruence lattice on the universe of $\mathbf{A} \times \mathbf{A}$. By Lemma 2.3, $\mathbf{U}$ is a congruence lattice and so $\mathbf{U} \cap \mathbf{V}$ is also a congruence lattice by Lemma 2.4. Therefore, $\mathbf{L}$ will also be a congruence lattice. ■

The subdirect products satisfying $r^{\downarrow} = q$, $s^{\downarrow} = t$, and $t^{\downarrow}$ is in $\{p, s\}$, are dealt with in Theorem 2.18.

In the case where $t^{\downarrow} < r^{\downarrow}$ and $s^{\downarrow} \parallel r^{\downarrow}$, there are again four subdirect products of $\mathbf{S_7}^* \times \mathbf{S_7}^*$ containing $(\{0\} \times \mathbf{S_7}^*) \cup (\mathbf{S_7}^* \times \{1\})$ — the lattices satisfying $r^{\downarrow} = q$, $t^{\downarrow} = t$, and $s^{\downarrow}$ is in $\{p, s\}$, and the lattices satisfying $r^{\downarrow} = p$, $t^{\downarrow} = s$, and $s^{\downarrow}$ is in $\{q, t\}$. The first two are congruence lattices on the universe of $\mathbf{A} \times \mathbf{A}$ by an argument symmetric to that of Lemma 2.17 and the last two are dealt with later in Theorem 2.18.

There are two subdirect products of $\mathbf{S_7}^* \times \mathbf{S_7}^*$ containing $(\{0\} \times \mathbf{S_7}^*) \cup (\mathbf{S_7}^* \times \{1\})$

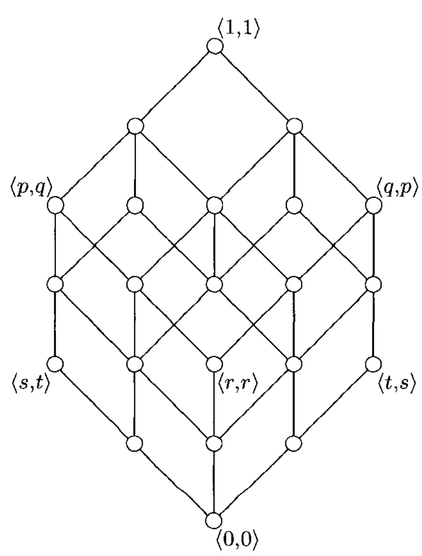

Figure 2.3: The lattice $\mathbf{L_0}^*$ in Theorem 2.18 with some elements labelled

left to deal with. The first subdirect product is the lattice $\mathbf{L} = \{\langle x, y\rangle \in \mathbf{S_7}^* \times \mathbf{S_7}^* : x \leq y\}$ and is a congruence lattice by Lemma 2.3. The second, $\mathbf{L_0}^*$, can be seen in Figure 2.3. Not surprisingly, the lattice $\mathbf{L_0}^*$ is isomorphic to the dual of $\mathbf{L_0}$ and, as in Section 2.2, provides the key to the main theorem of this section.

Theorem 2.18. **Con**($\mathbf{A}$) *is a power-hereditary congruence lattice (and a power-hereditary representation of* $\mathbf{S_7}^*$*) if and only if* $\mathbf{L_0}^*$ *is the congruence lattice of an algebra on the underlying set of* $\mathbf{A} \times \mathbf{A}$.

Proof. The proof is the dual argument of the proof of Theorem 2.14 with the first and second coordinates interchanged. ■

Chapter 3

The Four Element Boolean Meet-semilattice

In Chapter 2, I provided a necessary and sufficient condition for a congruence lattice representation of $\mathbf{S}_7$ to be power-hereditary. In this chapter I discuss an algebra $(\mathbf{2}_\wedge)^2$ whose congruence lattice is isomorphic to $\mathbf{S}_7$. First, I give the isomorphism between the congruence lattice of $(\mathbf{2}_\wedge)^2$ and $\mathbf{S}_7$ in Section 3.1. Secondly, I prove in Section 3.2 that $\mathbf{Con}((\mathbf{2}_\wedge)^2)$ is a power-hereditary congruence lattice by showing it satisfies Theorem 2.14. After that, Section 3.3 contains an alternate proof suggested to me by John Snow. Finally, in Section 3.4, I discuss a result about the finite lattices in the variety of $\mathbf{S}_7$ that follows from Theorem 3.7.

3.1 The meet-semilattice on $\mathbf{2}^2$

Recall from Section 1.2 that a meet-semilattice is a partially ordered set of elements that respects the meet operation as defined on a lattice but does not necessarily satisfy the join operation. The meet-semilattice that is discussed in this chapter has the underlying set $\{\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,0\rangle, \langle 1,1\rangle\}$ and the meet operation defined

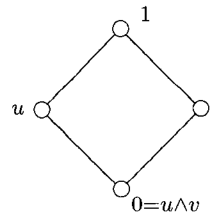

Figure 3.1: The meet-semilattice $(\mathbf{2}_\wedge)^2$

coordinatewise. For ease of notation, though, in this chapter let $0 = \langle 0,0\rangle$, $u = \langle 0,1\rangle$, $v = \langle 1,0\rangle$, and $1 = \langle 1,1\rangle$. A diagram of this Boolean meet-semilattice, which is referred to as $(\mathbf{2}_\wedge)^2$, can be seen in Figure 3.1.

The algebra $(\mathbf{2}_\wedge)^2$ is quite important to the research in this thesis as the congruence lattice of $(\mathbf{2}_\wedge)^2$ is isomorphic to $\mathbf{S}_7$. This is stated in the following lemma and the proof is provided for completeness.

Lemma 3.1. [24] $\mathbf{Con}((\mathbf{2}_\wedge)^2)$ *is isomorphic to* $\mathbf{S}_7$. *Thus, the lattice* $\mathbf{S}_7$ *is representable by* $(\mathbf{2}_\wedge)^2$.

Proof. I proceed by finding all the congruences of $(\mathbf{2}_\wedge)^2$ and then constructing $\mathbf{Con}((\mathbf{2}_\wedge)^2)$. The isomorphism with $\mathbf{S}_7$ is then obvious.

First, consider $Cg(0,0)$, the congruence generated by placing 0 in the same congruence class as itself. Since $0 \equiv 0$ this does not cause any other elements of $(\mathbf{2}_\wedge)^2$ to be congruent. Thus, $Cg(0,0)$ yields the classes $\{\{0\},\{u\},\{v\},\{1\}\}$ and is equal to the trivial congruence on $(\mathbf{2}_\wedge)^2$. Similarly $Cg(u,u)$, $Cg(v,v)$, and $Cg(1,1)$ all yield the trivial congruence. Next, consider the other principal congruences of $(\mathbf{2}_\wedge)^2$.

The congruence $Cg(0,u)$ has classes $\{\{0,u\},\{v\},\{1\}\}$ as $0 = 0\wedge v \equiv u \wedge v = 0$ and $0 = 0 \wedge 1 \equiv u \wedge 1 = u$. Using a similar argument, it follows that $Cg(0,v)$ has classes $\{\{0,v\},\{u\},\{1\}\}$.

The principal congruence $Cg(u,1)$ has $1 \wedge v \equiv u \wedge v$ which implies $v \equiv 0$. Thus, v and 0 are in the same congruence class. Then note $v = 1 \wedge v \equiv 1 \wedge 0 = 0$ and

$\bigvee$	$Cg(0,0)$	$Cg(0,u)$	$Cg(0,v)$	$Cg(v,1)$	$Cg(u,1)$	$Cg(u,v)$	$Cg(0,1)$
$Cg(0,0)$	$Cg(0,0)$	$Cg(0,u)$	$Cg(0,v)$	$Cg(v,1)$	$Cg(u,1)$	$Cg(u,v)$	$Cg(0,1)$
$Cg(0,u)$	$Cg(0,u)$	$Cg(0,u)$	$Cg(u,v)$	$Cg(v,1)$	$Cg(0,1)$	$Cg(u,v)$	$Cg(0,1)$
$Cg(0,v)$	$Cg(0,v)$	$Cg(u,v)$	$Cg(0,v)$	$Cg(0,1)$	$Cg(u,1)$	$Cg(u,v)$	$Cg(0,1)$
$Cg(v,1)$	$Cg(v,1)$	$Cg(v,1)$	$Cg(0,1)$	$Cg(v,1)$	$Cg(0,1)$	$Cg(0,1)$	$Cg(0,1)$
$Cg(u,1)$	$Cg(u,1)$	$Cg(0,1)$	$Cg(u,1)$	$Cg(0,1)$	$Cg(u,1)$	$Cg(0,1)$	$Cg(0,1)$
$Cg(u,v)$	$Cg(u,v)$	$Cg(u,v)$	$Cg(u,v)$	$Cg(0,1)$	$Cg(0,1)$	$Cg(u,v)$	$Cg(0,1)$
$Cg(0,1)$	$Cg(0,1)$	$Cg(0,1)$	$Cg(0,1)$	$Cg(0,1)$	$Cg(0,1)$	$Cg(0,1)$	$Cg(0,1)$

Table 3.1: Joins of congruences of $(\mathbf{2}_\wedge)^2$

$0 = u \wedge v \equiv u \wedge 0 = 0 \wedge 0$ so there are no other congruences forced. Therefore, $Cg(u,1)$ has classes $\{\{0,v\},\{u,1\}\}$. Similarly, $Cg(v,1)$ has classes $\{\{0,u\},\{v,1\}\}$.

In $Cg(u,v)$, $u \equiv v$ so $u \wedge v \equiv u \wedge u = u$. But $u \wedge v = 0$ so 0 is in the same congruence class as u and v. Now notice that the only way to generate 1 with the meet operation is as $1 \wedge 1 = 1$ so 1 is still in its own class. Therefore, $Cg(u,v)$ separates $(\mathbf{2}_\wedge)^2$ into classes $\{\{0,u,v\},\{1\}\}$.

Finally, if 0 and 1 are congruent, then $1 \wedge u \equiv 0 \wedge u$ and $1 \wedge v \equiv 0 \wedge v$. This causes u and v to be in the same congruence class as 0. Thus, $Cg(0,1) = Cg(0,u,v,1) = Cg((\mathbf{2}_\wedge)^2)$ and, by Lemma 2.2(a), no other principal congruences need to be checked.

Now, the set of all the possible joins of the non-trivial principal congruences can be seen in Table 3.1. Since the set of all $Cg(a,b)$ in $\mathbf{Con}((\mathbf{2}_\wedge)^2)$ is closed under join and, by Lemma 2.2(d), every $\theta \in \mathbf{Con}((\mathbf{2}_\wedge)^2)$ is the join of principal congruences, then every non-trivial θ is principal. Therefore, there are seven unique congruences on $(\mathbf{2}_\wedge)^2$ – $Cg(0,0)$, $Cg(0,u)$, $Cg(0,v)$, $Cg(u,1)$, $Cg(v,1)$, $Cg(u,v)$, and $Cg(0,1)$ – which lead to a partial ordering of the congruences:

$$Cg(0,0) \subset Cg(0,u) \subset Cg(v,1) \subset Cg(0,1),\ Cg(0,u) \subset Cg(u,v) \subset Cg(0,1),$$
$$Cg(0,0) \subset Cg(0,v) \subset Cg(u,1) \subset Cg(0,1), \text{ and } Cg(0,v) \subset Cg(u,v) \subset Cg(0,1).$$

The corresponding lattice of congruences $\mathbf{Con}((\mathbf{2}_\wedge)^2)$ is isomorphic to $\mathbf{S}_7$ by the

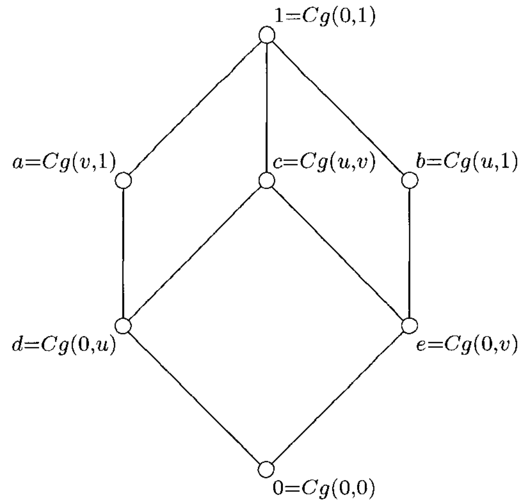

Figure 3.2: The congruence lattice of $(\mathbf{2}_\wedge)^2$

obvious isomorphism (see Figure 3.2). Hence, by definition, $\mathbf{S}_7$ is representable by $(\mathbf{2}_\wedge)^2$. ■

The natural question to ask is whether $\mathbf{Con}((\mathbf{2}_\wedge)^2)$ is hereditary, power-hereditary, or neither. It turns out it is fairly easy to show that $\mathbf{Con}((\mathbf{2}_\wedge)^2)$ is hereditary. However, I show in the next section that, in fact, the congruence lattice of $(\mathbf{2}_\wedge)^2$ is power-hereditary — a much stronger result.

3.2 $\mathbf{Con}((\mathbf{2}_\wedge)^2)$ is power-hereditary

In the following series of arguments I show that $\mathbf{Con}((\mathbf{2}_\wedge)^2)$ is a power-hereditary representation of the lattice $\mathbf{S}_7$. For the rest of this chapter, the elements of the congruence lattice $\mathbf{S}_7$ in Figure 2.1 are referred to specifically as the congruences of $(\mathbf{2}_\wedge)^2$ as seen in Figure 3.2. By considering the algebra on the underlying set of $(\mathbf{2}_\wedge)^2 \times (\mathbf{2}_\wedge)^2$ whose congruence lattice is

$$\mathbf{L}_1 = \{\langle \alpha, \beta \rangle \in \mathbf{Con}((\mathbf{2}_\wedge)^2) \times \mathbf{Con}((\mathbf{2}_\wedge)^2) : \alpha \leq \beta\},$$

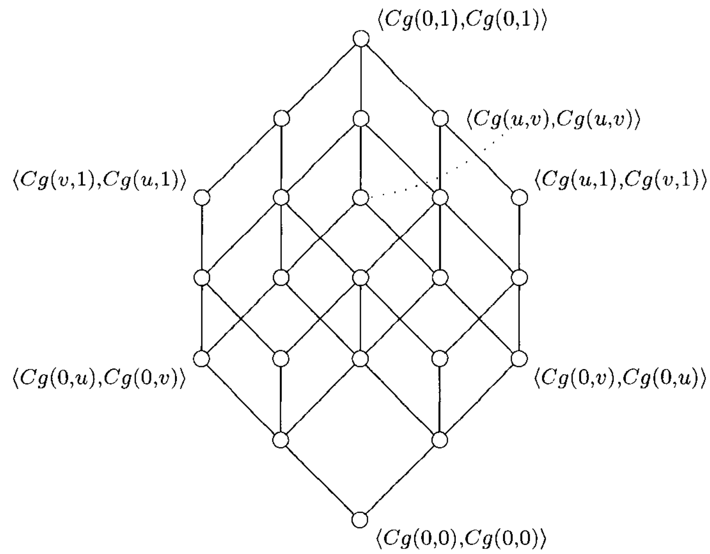

Figure 3.3: $\mathbf{L}_0$ as a 0-1 sublattice of $\mathbf{Con}((\mathbf{2}_\wedge)^2) \times \mathbf{Con}((\mathbf{2}_\wedge)^2)$ with some of the elements labelled

another algebra can be described on the same universe whose congruence lattice is $\mathbf{L}_0$ (see Figure 3.3). Thus, by Theorem 2.14, $\mathbf{Con}((\mathbf{2}_\wedge)^2)$ is a power-hereditary congruence lattice.

Before proceeding with the main part of the proof, though, some general results about congruence lattices are required. This first theorem describes the join of elements in a congruence lattice.

Theorem 3.2 ([1, 4.6]). *If α and β are congruences on an algebra $\mathbf{A}$ then $\langle x, y\rangle$ is in $\alpha \vee \beta$ if and only if there exists a sequence of elements $z_1, \ldots, z_n$ in $\mathbf{A}$ such that*

$$\langle z_i, z_{i+1}\rangle \in \alpha \qquad \textit{or} \qquad \langle z_i, z_{i+1}\rangle \in \beta$$

for $i = 1, \ldots, n-1$, and $x = z_1$ and $y = z_n$.

Recall from Section 1.2 that an automorphism Φ of the congruence lattice of an algebra $\mathbf{A}$ is carried by a function ϕ if

$$\Phi(\theta) = \{\langle \phi(x), \phi(y)\rangle \in \mathbf{A} \times \mathbf{A} \mid x\; \theta\; y\}$$

for all θ in $\mathbf{Con}(\mathbf{A})$. The next lemma shows that every automorphism of an algebra carries an automorphism of the congruence lattice of that algebra.

Lemma 3.3. *Let* $\mathbf{A} = \langle A, F \rangle$ *be an algebra. Every automorphism* ϕ *in* $\mathbf{Aut}(\mathbf{A})$ *carries an automorphism* Γ *in* $\mathbf{Aut}(\mathbf{Con}(\mathbf{A}))$ *defined such that*

$$\Gamma(\theta) = \{\langle \phi(x), \phi(y) \rangle \in \mathbf{A} \times \mathbf{A} \mid x \; \theta \; y\}$$

for all θ *in* $\mathbf{Con}(\mathbf{A})$.

Proof. Fix ϕ in $\mathbf{Aut}(\mathbf{A})$ and θ in $\mathbf{Con}(\mathbf{A})$. First, I claim that

$$\begin{aligned}\Gamma(\theta) &= \{\langle \phi(x), \phi(y) \rangle \in \mathbf{A} \times \mathbf{A} \mid x \; \theta \; y\} \\ &= \left\{\langle x, y \rangle \in \mathbf{A} \times \mathbf{A} \mid \phi^{-1}(x) \; \theta \; \phi^{-1}(y)\right\}\end{aligned}$$

is a congruence on $\mathbf{A}$. Let x, y, w, and z be elements of $\mathbf{A}$. For all x in $\mathbf{A}$, the pair $\langle x, x \rangle$ is in θ as it is reflexive. Then, since ϕ^{-1} is a bijection, for all x the pair $\phi^{-1}(x) \; \theta \; \phi^{-1}(x)$ is also in θ. Hence $\langle \phi(\phi^{-1}(x)), \phi(\phi^{-1}(x)) \rangle = \langle x, x \rangle$ is in $\Gamma(\theta)$ and $\Gamma(\theta)$ is reflexive. Next, suppose $\langle x, y \rangle$ is in $\Gamma(\theta)$. Then $\langle \phi^{-1}(x), \phi^{-1}(y) \rangle$ is in θ and, since θ is symmetric, $\langle \phi^{-1}(y), \phi^{-1}(x) \rangle$ is also in θ. Hence $\langle y, x \rangle$ is in $\Gamma(\theta)$ and $\Gamma(\theta)$ is symmetric. Now let $\langle x, y \rangle$ and $\langle y, w \rangle$ be in $\Gamma(\theta)$. This means $\langle \phi^{-1}(x), \phi^{-1}(y) \rangle$ and $\langle \phi^{-1}(y), \phi^{-1}(w) \rangle$ are in θ. Since θ is transitive, $\langle \phi^{-1}(x), \phi^{-1}(w) \rangle$ is in θ, and subsequently, $\langle x, w \rangle$ is in $\Gamma(\theta)$. Thus, $\Gamma(\theta)$ is transitive and an equivalence relation.

Now, for any n-ary operation f in F and $\langle x_i, y_i \rangle$ in $\Gamma(\theta)$, where $1 \leq i \leq n$, I have

$$\begin{aligned}x_i \; \Gamma(\theta) \; y_i \text{ for all } i &\Leftrightarrow \phi^{-1}(x_i) \; \theta \; \phi^{-1}(y_i) \text{ for all } i \\ &\Leftrightarrow f(\phi^{-1}(x_1), \ldots, \phi^{-1}(x_n)) \; \theta \; f(\phi^{-1}(y_1), \ldots, \phi^{-1}(y_n)) \\ &\Leftrightarrow \phi^{-1}(f(x_1, \ldots, x_n)) \; \theta \; \phi^{-1}(f(y_1, \ldots, y_n))\end{aligned}$$

since ϕ^{-1} is also an automorphism. Hence

$$f(x_1, \ldots, x_n) \, \Gamma(\theta) \, f(y_1, \ldots, y_n)$$

so $\Gamma(\theta)$ preserves the operations of $\mathbf{A}$. Therefore, $\Gamma(\theta)$ is in $\mathbf{Con}(\mathbf{A})$.

I now show that Γ is an automorphism of $\mathbf{Con}(\mathbf{A})$. By the previous paragraph, $\Gamma(\theta)$ is a congruence of $\mathbf{A}$, so Γ is well-defined. Suppose, for some θ_1 and θ_2 in $\mathbf{Con}(\mathbf{A})$, that $\Gamma(\theta_1) = \Gamma(\theta_2)$. In other words,

$$\{\langle \phi(x), \phi(y) \rangle \in \mathbf{A} \times \mathbf{A} \mid x \; \theta_1 \; y\} = \{\langle \phi(x), \phi(y) \rangle \in \mathbf{A} \times \mathbf{A} \mid x \; \theta_2 \; y\}.$$

It follows that

$$\begin{aligned} &\left\{\langle \phi^{-1}(\phi(x)), \phi^{-1}(\phi(y)) \rangle \in \mathbf{A} \times \mathbf{A} \mid x \; \theta_1 \; y\right\} \\ &\qquad = \left\{\langle \phi^{-1}(\phi(x)), \phi^{-1}(\phi(y)) \rangle \in \mathbf{A} \times \mathbf{A} \mid x \; \theta_2 \; y\right\} \end{aligned}$$

hence

$$\{\langle x, y \rangle \in \mathbf{A} \times \mathbf{A} \mid x \; \theta_1 \; y\} = \{\langle x, y \rangle \in \mathbf{A} \times \mathbf{A} \mid x \; \theta_2 \; y\}$$

so $\theta_1 = \theta_2$. Thus, Γ is one-to-one.

Now, let θ be in $\mathbf{Con}(\mathbf{A})$. Since ϕ^{-1} is an automorphism,

$$\theta_{\phi^{-1}} = \left\{\langle \phi^{-1}(x), \phi^{-1}(y) \rangle \in \mathbf{A} \times \mathbf{A} \mid x \; \theta \; y\right\}$$

is also in $\mathbf{Con}(\mathbf{A})$. Then

$$\begin{aligned} \Gamma(\theta_{\phi^{-1}}) &= \left\{\langle \phi(\phi^{-1}(x)), \phi(\phi^{-1}(y)) \rangle \in \mathbf{A} \times \mathbf{A} \mid x \; \theta \; y\right\} \\ &= \{\langle x, y \rangle \in \mathbf{A} \times \mathbf{A} \mid x \; \theta \; y\} \\ &= \theta \end{aligned}$$

and Γ is onto.

Finally, I show that Γ preserves meet and join. Let θ_1 and θ_2 be in $\mathbf{Con}(\mathbf{A})$. I have

$$\begin{aligned}\Gamma(\theta_1 \wedge \theta_2) &= \{\langle \phi(x), \phi(y)\rangle \in \mathbf{A} \times \mathbf{A} \mid x\ \theta_1 \wedge \theta_2\ y\} \\ &= \{\langle \phi(x), \phi(y)\rangle \in \mathbf{A} \times \mathbf{A} \mid x\ \theta_1\ y\} \\ &\qquad \wedge \{\langle \phi(x), \phi(y)\rangle \in \mathbf{A} \times \mathbf{A} \mid x\ \theta_2\ y\} \\ &= \Gamma(\theta_1) \wedge \Gamma(\theta_2).\end{aligned}$$

Thus Γ preserves meet.

Now, suppose $\langle x, y\rangle$ is in $\Gamma(\theta_1 \vee \theta_2)$. This means $\langle \phi^{-1}(x), \phi^{-1}(y)\rangle$ is in $\theta_1 \vee \theta_2$ and, by Theorem 3.2, this is equivalent to having a sequence of elements $z_1, \ldots, z_n$ in $\mathbf{A}$ such that $\langle z_i, z_{i+1}\rangle$ in θ_1 or $\langle z_i, z_{i+1}\rangle$ in θ_2 for $i = 1, \ldots, n-1$ and $\phi^{-1}(x) = z_1$ and $\phi^{-1}(y) = z_n$. Then

$$\begin{aligned}\langle x, y\rangle \in \Gamma(\theta_1 \vee \theta_2) &\Leftrightarrow \langle \phi^{-1}(x), \phi^{-1}(y)\rangle \in \theta_1 \vee \theta_2 \\ &\Leftrightarrow \phi^{-1}(x)\ \theta_j\ z_2\ \theta_j \cdots \theta_j\ z_{n-1}\ \theta_j\ \phi^{-1}(y) \text{ for } j \in \{1, 2\} \\ &\Leftrightarrow x\ \Gamma(\theta_j)\ \phi(z_2)\ \Gamma(\theta_j) \cdots \Gamma(\theta_j)\ \phi(z_{n-1})\ \Gamma(\theta_j)\ y \text{ for } j \in \{1, 2\} \\ &\Leftrightarrow \langle x, y\rangle \in \Gamma(\theta_1) \vee \Gamma(\theta_2).\end{aligned}$$

Thus Γ preserves join.

Therefore, Γ is an automorphism and, consequently, ϕ carries an automorphism of $\mathbf{Con}(\mathbf{A})$. ■

Now, let $\phi_1 : (\mathbf{2}_\wedge)^2 \to (\mathbf{2}_\wedge)^2$ be the automorphism of $(\mathbf{2}_\wedge)^2$ where $\phi_1(0) = 0$, $\phi_1(1) = 1$, $\phi_1(u) = v$, and $\phi_1(v) = u$. For any θ in $\mathbf{Con}((\mathbf{2}_\wedge)^2)$, define $\Phi_1(\theta)$ to be the congruence in $\mathbf{Con}((\mathbf{2}_\wedge)^2)$ such that

$$\Phi_1(\theta) = \{\langle \phi_1(x), \phi_1(y) \rangle \in (\mathbf{2}_\wedge)^2 \times (\mathbf{2}_\wedge)^2 : x\ \theta\ y\}.$$

Under this mapping, $\Phi_1(Cg(0,0))$ has congruence classes

$$\{\{\phi_1(0)\}, \{\phi_1(u)\}, \{\phi_1(v)\}, \{\phi_1(1)\}\} = \{\{0\}, \{v\}, \{u\}, \{1\}\}$$

so $\Phi_1(Cg(0,0)) = Cg(0,0)$. Similarly,

$$\Phi_1(Cg(0,u)) = Cg(0,v),\ \ \Phi_1(Cg(0,v)) = Cg(0,u),\ \ \Phi_1(Cg(u,1)) = Cg(v,1),$$

$$\Phi_1(Cg(v,1)) = Cg(u,1),\ \ \Phi_1(Cg(u,v)) = Cg(u,v),\ \text{ and } \Phi_1(Cg(0,1)) = Cg(0,1).$$

Note that, under this definition, $\Phi_1(Cg(s,t)) = Cg(\phi_1(s), \phi_1(t))$ for all s and t in $(\mathbf{2}_\wedge)^2$. The following corollary to Lemma 3.3 concerns the mapping Φ_1.

Corollary 3.4. *The mapping* $\Phi_1 : \mathbf{Con}((\mathbf{2}_\wedge)^2) \to \mathbf{Con}((\mathbf{2}_\wedge)^2)$ *is an automorphism on* $\mathbf{Con}((\mathbf{2}_\wedge)^2)$ *(see Figure 3.4). Moreover,* $\Phi_1(\Phi_1(\theta)) = \theta$ *for all* θ *in* $\mathbf{Con}((\mathbf{2}_\wedge)^2)$.

Proof. This follows immediately from the definition of Φ_1 and Lemma 3.3. ∎

These two automorphisms provide the key to proving $\mathbf{L}_0$ is a congruence lattice on the underlying set of $(\mathbf{2}_\wedge)^2 \times (\mathbf{2}_\wedge)^2$ by allowing one to exploit the symmetry of $(\mathbf{2}_\wedge)^2$ and $\mathbf{Con}((\mathbf{2}_\wedge)^2)$.

Now, for any $\langle x, y \rangle$ in $(\mathbf{2}_\wedge)^2 \times (\mathbf{2}_\wedge)^2$ and $\langle \alpha, \beta \rangle$ in $\mathbf{Con}((\mathbf{2}_\wedge)^2) \times \mathbf{Con}((\mathbf{2}_\wedge)^2)$, define the *companion* of $\langle x, y \rangle$ as $\overline{\langle x, y \rangle} = \langle \phi_1(x), y \rangle$ and the companion of $\langle \alpha, \beta \rangle$ as $\overline{\langle \alpha, \beta \rangle} = \langle \Phi_1(\alpha), \beta \rangle$. Note that, since $\phi_1(\phi_1(x)) = x$ for all x in $(\mathbf{2}_\wedge)^2$, it follows that $\overline{\overline{\langle x, y \rangle}} = \langle x, y \rangle$ as well. The next lemma provides the necessary connection between $\mathbf{L}_0$ and the lattice $\mathbf{L}_1$.

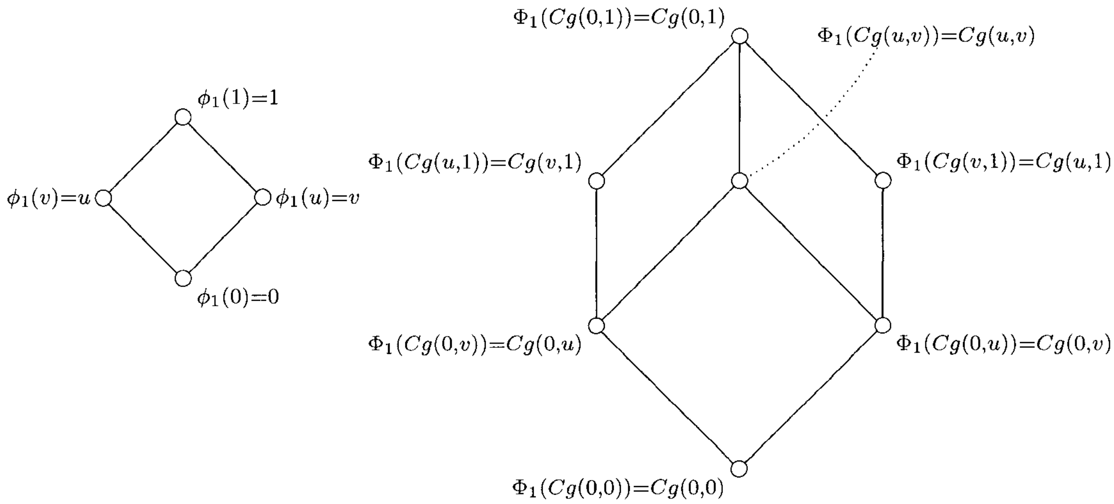

Figure 3.4: The automorphisms ϕ_1 on $(\mathbf{2}_\wedge)^2$ and Φ_1 on $\mathbf{Con}((\mathbf{2}_\wedge)^2)$

Lemma 3.5. *The lattice* $\mathbf{L}_0 \leq \mathbf{Eq}(2^2 \times 2^2)$ *is isomorphic to* $\mathbf{L}_1$ *under the companion mapping. Consequently, for all* $\langle x_1, y_1\rangle$ *and* $\langle x_2, y_2\rangle$ *in* $(\mathbf{2}_\wedge)^2 \times (\mathbf{2}_\wedge)^2$ *and* $\langle \alpha, \beta\rangle \in \mathbf{L}_1$,

$$\langle x_1, y_1\rangle \; \langle \alpha, \beta\rangle \; \langle x_2, y_2\rangle \text{ if and only if } \overline{\langle x_1, y_1\rangle} \; \overline{\langle \alpha, \beta\rangle} \; \overline{\langle x_2, y_2\rangle}.$$

Proof. First, I rewrite $\mathbf{L}_0$ in a form more similar to that of $\mathbf{L}_1$. Note that, as a 0-1 sublattice of $\mathbf{Con}((\mathbf{2}_\wedge)^2) \times \mathbf{Con}((\mathbf{2}_\wedge)^2)$, the definition of $\mathbf{L}_0$ on page 31 is equivalent to

$$\begin{aligned}
\mathbf{L}_0 = \{&\langle Cg(0,0), Cg(0,0)\rangle\} \\
&\cup \{\langle \alpha, Cg(0,u)\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq \alpha \leq Cg(0,v)\} \\
&\cup \{\langle \alpha, Cg(0,v)\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq \alpha \leq Cg(0,u)\} \\
&\cup \{\langle \alpha, Cg(v,1)\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq \alpha \leq Cg(u,1)\} \\
&\cup \{\langle \alpha, Cg(u,v)\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq \alpha \leq Cg(u,v)\} \\
&\cup \{\langle \alpha, Cg(u,1)\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq \alpha \leq Cg(v,1)\} \\
&\cup \{\langle \alpha, Cg(0,1)\rangle \in \mathbf{S}_7 \times \mathbf{S}_7 : 0 \leq \alpha \leq Cg(0,1)\}.
\end{aligned}$$

Also note that, for the lattice $\mathbf{L}_0$, the automorphism Φ_1 satisfies $\Phi_1(\beta) = \beta^{\Uparrow}$ for all β in $\mathbf{Con}((\mathbf{2}_\wedge)^2)$. Hence, by Corollary 2.8,

$$\mathbf{L}_0 = \{\langle \alpha, \beta \rangle \in \mathbf{Con}((\mathbf{2}_\wedge)^2) \times \mathbf{Con}((\mathbf{2}_\wedge)^2) : \alpha \leq \Phi_1(\beta)\}.$$

However, by Corollary 3.4,

$$\begin{aligned} \alpha \leq \Phi_1(\beta) &\Leftrightarrow \alpha \wedge \Phi_1(\beta) = \alpha \\ &\Leftrightarrow \Phi_1(\alpha) \wedge \Phi_1(\Phi_1(\beta)) = \Phi_1(\alpha) \\ &\Leftrightarrow \Phi_1(\alpha) \wedge \beta = \Phi_1(\alpha) \\ &\Leftrightarrow \Phi_1(\alpha) \leq \beta. \end{aligned}$$

Thus

$$\mathbf{L}_0 = \{\langle \alpha, \beta \rangle \in \mathbf{Con}((\mathbf{2}_\wedge)^2) \times \mathbf{Con}((\mathbf{2}_\wedge)^2) : \Phi_1(\alpha) \leq \beta\}.$$

Now, consider the companion mapping restricted to $\mathbf{L}_1$. Suppose $\langle \alpha_1, \beta_1 \rangle$ is in $\mathbf{L}_1$. Then $\alpha_1 \leq \beta_1$ and $\overline{\langle \alpha_1, \beta_1 \rangle} = \langle \Phi_1(\alpha_1), \beta_1 \rangle$. Since $\Phi_1(\Phi_1(\alpha_1)) = \alpha_1$ by Corollary 3.4, it follows that $\langle \Phi_1(\alpha_1), \beta_1 \rangle$ is in $\mathbf{L}_0$. Thus, the companion mapping sends elements of $\mathbf{L}_1$ to $\mathbf{L}_0$.

Now, suppose $\overline{\langle \alpha_1, \beta_1 \rangle} = \overline{\langle \alpha_2, \beta_2 \rangle}$ for $\overline{\langle \alpha_1, \beta_1 \rangle}$ and $\overline{\langle \alpha_2, \beta_2 \rangle}$ in $\mathbf{L}_0$. Then

$$\begin{aligned} \overline{\langle \alpha_1, \beta_1 \rangle} = \overline{\langle \alpha_2, \beta_2 \rangle} \quad &\Leftrightarrow \quad \langle \Phi_1(\alpha_1), \beta_1 \rangle = \langle \Phi_1(\alpha_2), \beta_2 \rangle \\ &\Leftrightarrow \quad \beta_1 = \beta_2 \text{ and } \Phi_1(\alpha_1) = \Phi_1(\alpha_2) \end{aligned}$$

and since Φ_1 is bijective, it follows that $\langle \alpha_1, \beta_1 \rangle = \langle \alpha_2, \beta_2 \rangle$. Therefore, the companion mapping is one-to-one.

Next, pick $\langle \alpha_1, \beta_1 \rangle$ in $\mathbf{L}_0$. Since $\Phi_1(\alpha_1) \leq \beta_1$ by definition, it follows that $\langle \Phi_1(\alpha_1), \beta_1 \rangle$ is an element of $\mathbf{L}_1$. Therefore, $\overline{\langle \Phi_1(\alpha_1), \beta_1 \rangle} = \langle \Phi_1(\Phi_1(\alpha_2)), \beta_2 \rangle =$

$\langle \alpha_1, \beta_1 \rangle$ and the mapping is onto, as well.

I show in the following that the companion map preserves join. Pick $\langle \alpha_1, \beta_1 \rangle$ and $\langle \alpha_2, \beta_2 \rangle$ in $\mathbf{L}_1$. By Corollary 3.4, $\Phi_1(\alpha_1 \vee \alpha_2) = \Phi_1(\alpha_1) \vee \Phi_1(\alpha_2)$. I then have

$$\begin{aligned}
\overline{\langle \alpha_1, \beta_1 \rangle \vee \langle \alpha_2, \beta_2 \rangle} &= \overline{\langle \alpha_1 \vee \alpha_2, \beta_1 \vee \beta_2 \rangle} \\
&= \langle \Phi_1(\alpha_1 \vee \alpha_2), \beta_1 \vee \beta_2 \rangle \\
&= \langle \Phi_1(\alpha_1) \vee \Phi_1(\alpha_2), \beta_1 \vee \beta_2 \rangle \\
&= \langle \Phi_1(\alpha_1), \beta_1 \rangle \vee \langle \Phi_1(\alpha_2), \beta_2 \rangle \\
&= \overline{\langle \alpha_1, \beta_1 \rangle} \vee \overline{\langle \alpha_2, \beta_2 \rangle}.
\end{aligned}$$

Hence the companion mapping preserves join and, by a dual argument, it also preserves meet. Thus, since the companion mapping is one-to-one, onto, and preserves join and meet, the congruence lattice $\mathbf{L}_0$ is isomorphic to $\mathbf{L}_1$.

Now, pick $\langle x_1, y_1 \rangle$ and $\langle x_2, y_2 \rangle$ in $(\mathbf{2}_\wedge)^2 \times (\mathbf{2}_\wedge)^2$ and $\langle \alpha, \beta \rangle \in \mathbf{L}_1$ such that $\langle x_1, y_1 \rangle \; \langle \alpha, \beta \rangle \; \langle x_2, y_2 \rangle$. I then have

$$\begin{aligned}
\langle x_1, y_1 \rangle \; \langle \alpha, \beta \rangle \; \langle x_2, y_2 \rangle &\Leftrightarrow x_1 \, \alpha \, x_2 \text{ and } y_1 \, \beta \, y_2 \\
&\Leftrightarrow \phi_1(x_1) \, \Phi_1(\alpha) \, \phi_1(x_2) \text{ and } y_1 \, \beta \, y_2 \\
&\Leftrightarrow \langle \phi_1(x_1), y_1 \rangle \; \langle \Phi_1(\alpha), \beta \rangle \; \langle \phi_1(x_2), y_2 \rangle
\end{aligned}$$

Therefore, $\langle x_1, y_1 \rangle \; \langle \alpha, \beta \rangle \; \langle x_1, y_1 \rangle$ if and only if $\overline{\langle x_1, y_1 \rangle} \; \overline{\langle \alpha, \beta \rangle} \; \overline{\langle x_1, y_1 \rangle}$. ■

Now, the next lemma states that there exists an algebra on the universe of $(\mathbf{2}_\wedge)^2 \times (\mathbf{2}_\wedge)^2$ whose congruence lattice is equal to $\mathbf{L}_0$.

Lemma 3.6. *There exists an algebra* $\mathbf{A}_0$ *on the underlying set of* $(\mathbf{2}_\wedge)^2 \times (\mathbf{2}_\wedge)^2$ *such that* $\mathbf{Con}(\mathbf{A}_0) = \mathbf{L}_0$.

Proof. By Lemma 2.3 the lattice $\mathbf{L}_1$ is the congruence lattice of an algebra on the underlying set of $(\mathbf{2}_\wedge)^2 \times (\mathbf{2}_\wedge)^2$ so let $\mathbf{A}_1 = \langle 2^2 \times 2^2; F\rangle$ such that $\mathbf{Con}(\mathbf{A}_1) = \mathbf{L}_1$. Then, for all s_i and t_i in $2^2 \times 2^2$ and $\langle \alpha, \beta\rangle$ in $\mathbf{L}_1$,

$$s_i \, \langle \alpha, \beta \rangle \, t_i$$

for $i = 1, \ldots, n$ implies

$$f(s_1, \ldots, s_n) \, \langle \alpha, \beta \rangle \, f(t_1, \ldots, t_n)$$

for all n-ary operations f in F.

I define in the following an algebra $\mathbf{A}_0$ such that $\mathbf{Con}(\mathbf{A}_0) = \mathbf{L}_0$. For each n-ary operation f in F, define g_f such that for all s_i in $2^2 \times 2^2$,

$$g_f(s_1, \ldots, s_n) = \overline{f(\overline{s_1}, \ldots, \overline{s_n})}.$$

Let $G = \{g_f : f \in F\}$ and define $\mathbf{A}_0 = \langle 2^2 \times 2^2; G\rangle$.

I claim that each θ in $\mathbf{L}_0$ is a congruence of $\mathbf{A}_0$. Since $\mathbf{L}_1$ and $\mathbf{L}_0$ are isomorphic under the companion mapping, there exists $\langle \alpha_1, \beta_1\rangle \in \mathbf{L}_1$ such that $\theta = \overline{\langle \alpha_1, \beta_1\rangle}$. Suppose $s_i \, \theta \, t_i$ for $i = 1, \ldots, n$. Then for all i

$$\begin{aligned} s_i \, \theta \, t_i \quad &\Leftrightarrow \quad s_i \, \overline{\langle \alpha_1, \beta_1 \rangle} \, t_i \\ &\Leftrightarrow \quad \overline{s_i} \, \langle \alpha_1, \beta_1 \rangle \, \overline{t_i}. \end{aligned}$$

Thus, for all n-ary operations f in F, it follows that

$$\begin{aligned} s_i \, \theta \, t_i \text{ for } i = 1 \ldots n &\Leftrightarrow f(\overline{s_1}, \ldots, \overline{s_n}) \, \langle \alpha_1, \beta_1 \rangle \, f(\overline{t_1}, \ldots, \overline{t_n}) \\ &\Leftrightarrow \overline{f(\overline{s_1}, \ldots, \overline{s_n})} \; \overline{\langle \alpha_1, \beta_1 \rangle} \; \overline{f(\overline{t_1}, \ldots, \overline{t_n})} \end{aligned}$$

$$\Leftrightarrow \overline{f(\overline{s_1}, \ldots, \overline{s_n})}\, \theta\, \overline{f(\overline{t_1}, \ldots, \overline{t_n})}$$

$$\Leftrightarrow g_f(s_1, \ldots, s_n)\, \theta\, g_f(t_1, \ldots, t_n)$$

Hence, for all s_i and t_i in $2^2 \times 2^2$ and θ in $\mathbf{L}_0$, when $s_i\, \theta\, t_i$ for $i = 1, \ldots, n$ it follows that

$$g_f(s_1, \ldots, s_n)\, \theta\, g_f(t_1, \ldots, t_n)$$

for all n-ary operations g_f in G. Therefore, every θ in $\mathbf{L}_0$ is a congruence on the algebra $\mathbf{A}_0$.

Now, suppose μ in $\mathbf{Eq}(2^2 \times 2^2)$ is a congruence on $\mathbf{A}_0$, as well. That means $s_i\, \mu\, t_i$ for $i = 1, \ldots, n$ implies

$$g_f(s_1, \ldots, s_n)\, \mu\, g_f(t_1, \ldots, t_n)$$

for all n-ary operations g_f in G. Let γ in $\mathbf{Eq}(2^2 \times 2^2)$ be the equivalence relation such that $s_i\, \gamma\, t_i$ if and only if $\overline{s_i}\, \mu\, \overline{t_i}$. Then, for all s_i and t_i in $2^2 \times 2^2$ and g_f in G,

$$\begin{aligned} s_i\, \gamma\, t_i \text{ for } i = 1, \ldots, n \;&\Leftrightarrow\; \overline{s_i}\, \mu\, \overline{t_i} \text{ for } i = 1, \ldots, n \\ &\Rightarrow\; g_f(\overline{s_1}, \ldots, \overline{s_n})\, \mu\, g_f(\overline{t_1}, \ldots, \overline{t_n}) \\ &\Leftrightarrow\; \overline{f(\overline{\overline{s_1}}, \ldots, \overline{\overline{s_n}})}\, \mu\, \overline{f(\overline{\overline{t_1}}, \ldots, \overline{\overline{t_n}})} \\ &\Leftrightarrow\; \overline{f(s_1, \ldots, s_n)}\, \mu\, \overline{f(t_1, \ldots, t_n)} \\ &\Leftrightarrow\; f(s_1, \ldots, s_n)\, \gamma\, f(t_1, \ldots, t_n) \end{aligned}$$

Therefore, γ preserves f for all f in F which means γ is in $\mathbf{L}_1$. But $\overline{\gamma} = \mu$ and since, by Lemma 3.5, $\mathbf{L}_1$ is isomorphic to $\mathbf{L}_0$ under the companion map, the congruence μ must be in $\mathbf{L}_0$.

Therefore, since $\mathbf{L}_0$ contains exactly the congruences on the algebra $\mathbf{A}_0$, the congruence lattice $\mathbf{Con}(\mathbf{A}_0)$ is $\mathbf{L}_0$. ■

Now Theorem 2.14, combined with the fact that $\mathbf{L}_0$ is the congruence lattice of an algebra on the universe of $(\mathbf{2}_\wedge)^2 \times (\mathbf{2}_\wedge)^2$, yields the main theorem of this section.

Theorem 3.7. $\mathbf{Con}((\mathbf{2}_\wedge)^2)$ *is a power-hereditary representation of the lattice* $\mathbf{S}_7$.

3.3 An alternate proof

An alternate proof of Theorem 3.7 exists that has a distinctly different flavour than that of Section 3.2. This section includes the proof — a sketch of which was provided in [17] by John Snow — not only for interest's sake but also because it makes use of primitive positive formulas. As well, Lemma 3.8 has important implications in Chapter 5.

Recall that the graph of an automorphism Φ of a lattice $\mathbf{L}$ is the subset G of $\mathbf{L} \times \mathbf{L}$ such that $G = \{\langle x, y\rangle \in \mathbf{L} \times \mathbf{L} : \Phi(x) = y\}$. This first lemma describes how the graph of an automorphism of the congruence lattice of an algebra $\mathbf{A}$ can yield another congruence lattice of an algebra on the universe of $\mathbf{A} \times \mathbf{A}$.

Lemma 3.8. [17] *Let* Φ *be an automorphism of* $\mathbf{Con}(\mathbf{A})$ *for some algebra* $\mathbf{A}$ *and define sublattices* $\mathbf{M} \leq \mathbf{Con}(\mathbf{A}) \times \mathbf{Con}(\mathbf{A})$ *and* $\mathbf{N} \leq \mathbf{Con}(\mathbf{A}) \times \mathbf{Con}(\mathbf{A})$ *to have the universes*

$$M = \{\langle \alpha, \beta\rangle \in \mathbf{Con}(\mathbf{A}) \times \mathbf{Con}(\mathbf{A}) : \Phi(\alpha) = \beta\} \text{ and}$$
$$N = \{\langle \alpha, \beta\rangle \in \mathbf{Con}(\mathbf{A}) \times \mathbf{Con}(\mathbf{A}) : \Phi(\alpha) \leq \beta\}.$$

If $\mathbf{M}$ *is the congruence lattice of an algebra on the universe of* $\mathbf{A} \times \mathbf{A}$*, then so is* $\mathbf{N}$.

Proof. Let $m = \binom{n}{2}$. Define $P(\alpha_{1,2}, \alpha_{1,3}, \ldots, \alpha_{n-2,n}, \alpha_{n-1,n})$ as the equivalence relation σ such that

$$\sigma(x_1, x_2) \leftrightarrow \exists x_3, \ldots, x_n \bigwedge_{1 \le i < j \le n} \alpha_{i,j}(x_i, x_j).$$

For example, when $n = 4$, the equivalence relation $P(\alpha_{1,2}, \alpha_{1,3}, \alpha_{1,4}, \alpha_{2,3}, \alpha_{2,4}, \alpha_{3,4})$ is defined such that

$$x_1 \; P(\alpha_{1,2}, \alpha_{1,3}, \alpha_{1,4}, \alpha_{2,3}, \alpha_{2,4}, \alpha_{3,4}) \; x_2$$

if and only if

$$\exists x_3, x_4 \; \alpha_{1,2}(x_1, x_2) \wedge \alpha_{1,3}(x_1, x_3) \wedge \alpha_{1,4}(x_1, x_4) \wedge \alpha_{2,3}(x_2, x_3) \wedge \alpha_{2,4}(x_2, x_4) \wedge \alpha_{3,4}(x_3, x_4).$$

Assume that $\mathbf{M}$ is the congruence lattice of an algebra on the universe of $\mathbf{A} \times \mathbf{A}$. By Lemma 1.4, the congruence lattice $\mathbf{M}$ is closed under primitive positive formulas, hence $P(\langle \alpha_1, \beta_1 \rangle, \ldots, \langle \alpha_m, \beta_m \rangle)$ is in $\mathbf{M}$ for all $\langle \alpha_i, \beta_i \rangle$ in $\mathbf{M}$.

Now, since P applies coordinatewise,

$$P(\langle \alpha_1, \beta_1 \rangle, \ldots, \langle \alpha_m, \beta_m \rangle) = \langle P(\alpha_1, \ldots, \alpha_m), P(\beta_1, \ldots, \beta_m) \rangle.$$

Then, since $\langle \alpha_i, \Phi(\alpha_i) \rangle$ is in $\mathbf{M}$ for all α_i in $\mathbf{Con}(\mathbf{A})$, it follows that

$$P(\langle \alpha_1, \Phi(\alpha_1) \rangle, \ldots, \langle \alpha_m, \Phi(\alpha_m) \rangle) = \langle P(\alpha_1, \ldots, \alpha_m), P(\Phi(\alpha_1), \ldots, \Phi(\alpha_m)) \rangle$$

is also in $\mathbf{M}$. But from the definition of $\mathbf{M}$, this just means

$$\Phi(P(\alpha_1, \ldots, \alpha_m)) = P(\Phi(\alpha_1), \ldots, \Phi(\alpha_m)).$$

Next, I claim that P is order preserving.

Suppose, for

$$\boldsymbol{\alpha} = \langle \alpha_{1,2}, \alpha_{1,3}, \ldots, \alpha_{n-2,n}, \alpha_{n-1,n} \rangle \text{ and } \boldsymbol{\beta} = \langle \beta_{1,2}, \beta_{1,3}, \ldots, \beta_{n-2,n}, \beta_{n-1,n} \rangle$$

in $\mathbf{Con}(\mathbf{A})^m$, that $\boldsymbol{\alpha} \leq \boldsymbol{\beta}$. Let $\langle x_1, x_2 \rangle$ be in $P(\boldsymbol{\alpha})$. Hence

$$\exists x_3, \ldots, x_n \bigwedge_{1 \leq i < j \leq n} \alpha_{i,j}(x_i, x_j).$$

But since $\alpha_{i,j} \subseteq \beta_{i,j}$ for $1 \leq i < j \leq n$,

$$\exists x_3, \ldots, x_n \bigwedge_{1 \leq i < j \leq n} \beta_{i,j}(x_i, x_j).$$

Thus $\langle x_1, x_2 \rangle$ is in $P(\boldsymbol{\beta})$ and $P(\boldsymbol{\alpha}) \leq P(\boldsymbol{\beta})$.

Hence for all α_i and β_i in $\mathbf{Con}(\mathbf{A})$ such that $\langle \alpha_i, \beta_i \rangle$ is in N,

$$P(\Phi(\alpha_1), \ldots, \Phi(\alpha_m)) \leq P(\beta_1, \ldots, \beta_m).$$

But since $\Phi(P(\alpha_1, \ldots, \alpha_m)) = P(\Phi_1(\alpha_1), \ldots, \Phi(\alpha_m))$, it follows that

$$\Phi(P(\alpha_1, \ldots, \alpha_m)) \leq P(\beta_1, \ldots, \beta_m).$$

By Lemma 1.4 again, $\mathbf{Con}(\mathbf{A})$ is closed under primitive positive formulas so $P(\alpha_1, \ldots, \alpha_m)$ and $P(\beta_1, \ldots, \beta_m)$ are elements of $\mathbf{Con}(\mathbf{A})$. Thus, by definition of N, the pair

$$\langle P(\alpha_1, \ldots, \alpha_m), P(\beta_1, \ldots, \beta_m) \rangle = P(\langle \alpha_1, \beta_1 \rangle, \ldots, \langle \alpha_m, \beta_m \rangle)$$

is also in N and N is closed under P. Therefore, by Lemma 1.4, N is a congruence lattice of an algebra on the universe of $\mathbf{A} \times \mathbf{A}$. ■

The next lemma describes an algebra on the universe of $(\mathbf{2}_\wedge)^2 \times (\mathbf{2}_\wedge)^2$ whose congruence lattice is equal to

$$\mathbf{M} = \left\{ \langle \alpha, \beta \rangle \in \mathbf{Con}((\mathbf{2}_\wedge)^2) \times \mathbf{Con}((\mathbf{2}_\wedge)^2) : \Phi_1(\alpha) = \beta \right\}$$

where Φ_1 is the automorphism defined on page 45. It is important to note that in his correspondence, Snow acknowledges Ralph McKenzie for the idea of the construction.

Lemma 3.9. [17] *There exists an algebra* $\mathbf{A}$ *such that*

$$\mathbf{Con}(\mathbf{A}) = \left\{ \langle \theta, \Phi_1(\theta) \rangle \mid \theta \in \mathbf{Con}((\mathbf{2}_\wedge)^2) \right\}.$$

Proof. Consider the algebra $\mathbf{A} = \langle 2^2 \times 2^2; \wedge, D, P \rangle$ where the meet operation is coordinatewise on $2^2 \times 2^2$. In addition, for all $\langle a, b \rangle$ and $\langle c, d \rangle$ in $2^2 \times 2^2$, the binary operation D is defined

$$D(\langle a, b \rangle, \langle c, d \rangle) = \langle a, d \rangle$$

and the unary operation P is defined

$$P(\langle a, b \rangle) = \langle \phi_1(b), \phi_1(a) \rangle$$

where ϕ_1 is the automorphism defined in Section 3.2. Using the Universal Algebra Calculator [5] created by Ralph Freese and Emil Kiss to generate the congruence lattice of $\mathbf{A}$ yields

$$\mathbf{Con}(\mathbf{A}) = \left\{ \langle \theta, \Phi_1(\theta) \rangle \mid \theta \in \mathbf{Con}((\mathbf{2}_\wedge)^2) \right\},$$

displayed in Figure 3.5. ∎

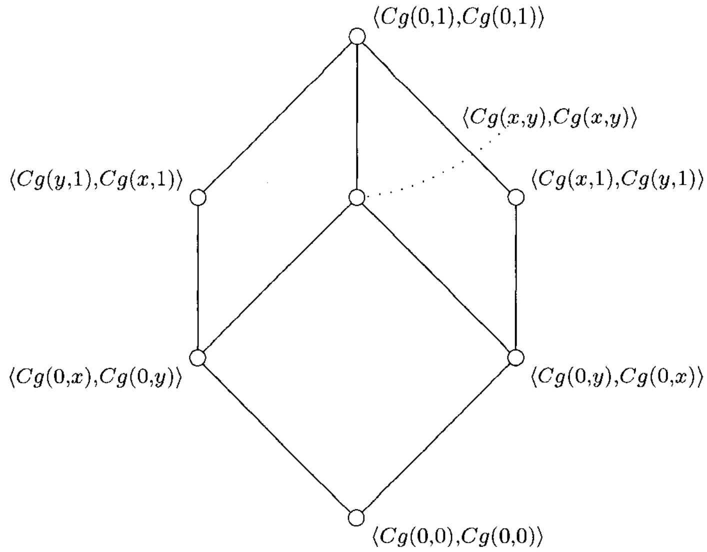

Figure 3.5: The congruence lattice of $\langle 2^2 \times 2^2, \wedge, D, P\rangle$

Now, the lattice $\mathbf{L}_0$ from the previous section can be written as

$$\mathbf{L}_0 = \{\langle \alpha, \beta\rangle \in \mathbf{Con}((\mathbf{2}_\wedge)^2) \times \mathbf{Con}((\mathbf{2}_\wedge)^2) : \Phi_1(\alpha) \leq \beta\}.$$

Hence, Lemma 3.8 and Lemma 3.9 show that $\mathbf{L}_0$ is the congruence lattice of an algebra on the universe of $(\mathbf{2}_\wedge)^2 \times (\mathbf{2}_\wedge)^2$.

3.4 Representations of lattices in $\mathcal{V}(\{\mathbf{S}_7\})$ by algebras with meet operations

Recall that the variety $\mathcal{V}(\{\mathbf{A}\})$ of an algebra $\mathbf{A}$ is the set of all homomorphic images of subalgebras of direct products of $\mathbf{A}$. I prove using Theorem 3.7, that every finite lattice in the variety of $\mathbf{S}_7$ is representable by a finite algebra with a meet operation. In order to prove this in Theorem 3.15, though, a few useful results are needed starting with a form of Birkhoff's Subdirect Representation Theorem adapted from [10].

Theorem 3.10. [10] *If $\mathcal{V}(K)$ is a variety, then every finite member of $\mathcal{V}(K)$ is isomorphic to a subdirect product of finitely many subdirectly irreducible members of $\mathcal{V}(K)$.*

The next two theorems concern congruence-distributive varieties.

Theorem 3.11. [2] *The lattice* $\mathbf{Con}(\mathbf{L})$ *is distributive for any lattice* $\mathbf{L}$.

Theorem 3.12 ([10, 4.99 and 4.104]). *If $\mathcal{V}(K)$ is finitely generated and congruence distributive, then every finite subdirectly irreducible algebra in $\mathcal{V}(K)$ belongs to the set of homomorphic images of subalgebras of elements of K.*

Next, I show in the following two lemmas that every finite lattice in $\mathcal{V}(\{\mathbf{S}_7\})$ is isomorphic to a 0-1 sublattice of ${\mathbf{S}_7}^n$ for some finite n.

Lemma 3.13. *Every non-trivial homomorphic image of a sublattice of* $\mathbf{S}_7$ *can be embedded as a 0-1 sublattice of* $\mathbf{S}_7$.

Proof. There are 45 non-trivial sublattices of $\mathbf{S}_7$. The set of all non-trivial homomorphic images of these sublattices are isomorphic to copies of $\mathbf{2}$, $\mathbf{3}$, $\mathbf{4}$, $\mathbf{2}\times\mathbf{2}$, $\mathbf{1}\oplus(\mathbf{2}\times\mathbf{2})$, $(\mathbf{2}\times\mathbf{2})\oplus\mathbf{1}$, N_5, $\mathbf{2}\times\mathbf{3}$, and $\mathbf{S}_7$. These are all isomorphic to 0-1 sublattices of $\mathbf{S}_7$ (see Figure 3.6). ■

Lemma 3.14. *Every non-trivial finite lattice in $\mathcal{V}(\{\mathbf{S}_7\})$ can be embedded as a 0-1 sublattice of ${\mathbf{S}_7}^n$, for some n.*

Proof. Let $\mathbf{L}$ be a non-trivial finite element of $\mathcal{V}(\{\mathbf{S}_7\})$. By Theorem 3.10, the lattice $\mathbf{L}$ is isomorphic to a subdirect product of finitely many subdirectly irreducible members $\mathbf{L}_i$ of $\mathcal{V}(\{\mathbf{S}_7\})$ and each subdirectly irreducible lattice is finite and non-trivial. Hence, there exists a lattice $\mathbf{L}''$ such that

$$\mathbf{L} \cong \mathbf{L}'' \underset{\mathrm{sd}}{\leq} \prod_{i=1}^{n} \mathbf{L}_i$$

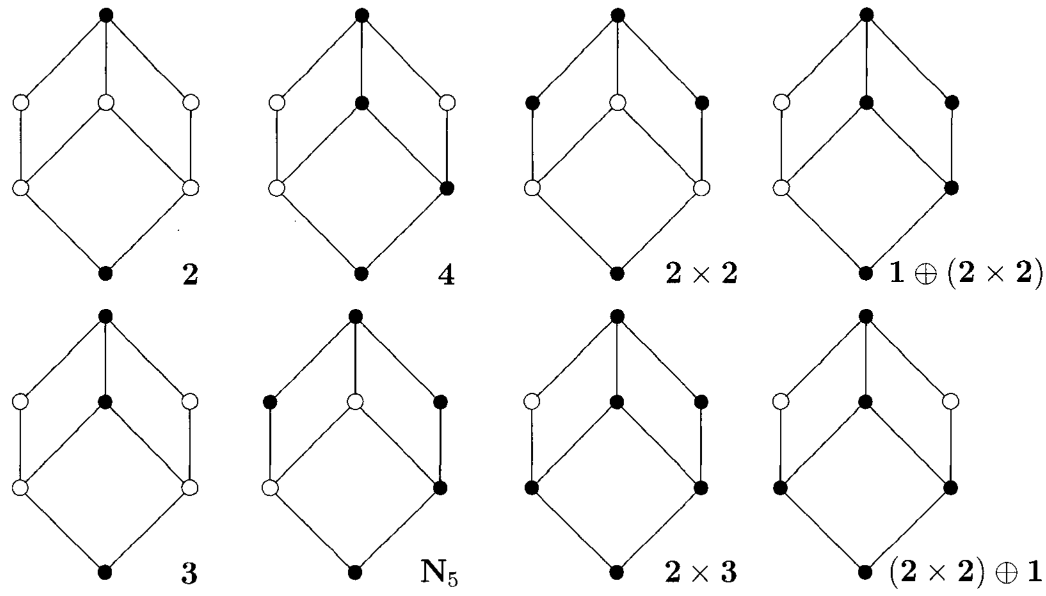

Figure 3.6: The 0-1 sublattices of $\mathbf{S}_7$, up to isomorphism

and each $\mathbf{L}_i$ is finite and subdirectly irreducible. Moreover, by Theorem 3.12, since lattices are congruence distributive and $\mathcal{V}(\{\mathbf{S}_7\})$ is finitely generated, each $\mathbf{L}_i$ is the homomorphic image of a sublattice of $\mathbf{S}_7$.

Now note that, since $\mathbf{L}''$ is a subdirect product, the join of all its elements is $\langle 1, \ldots, 1 \rangle$ and the meet of all of its elements is $\langle 0, \ldots, 0 \rangle$. Thus, $\mathbf{L}''$ is a 0-1 sublattice of $\prod_{i=1}^{n} \mathbf{L}_i$; that is, $\mathbf{L}'' \underset{0\text{-}1}{\leq} \prod_{i=1}^{n} \mathbf{L}_i$. Next, since every $\mathbf{L}_i$ is a homomorphic image of a sublattice of $\mathbf{S}_7$, by Lemma 3.13 they can be embedded as 0-1 sublattices of $\mathbf{S}_7$. Hence

$$\prod_{i=1}^{n} \mathbf{L}_i \cong \prod_{j=1}^{n} \mathbf{L}'_j$$

where $\mathbf{L}'_j \underset{0\text{-}1}{\leq} \mathbf{S}_7$ for all j. Thus,

$$\mathbf{L} \cong \mathbf{L}'' \underset{0\text{-}1}{\leq} \prod_{i=1}^{n} \mathbf{L}_i \cong \prod_{j=1}^{n} \mathbf{L}'_j \underset{0\text{-}1}{\leq} {\mathbf{S}_7}^n.$$

Therefore, $\mathbf{L}$ can be embedded as a 0-1 sublattice of ${\mathbf{S}_7}^n$ for some n. ■

I now prove the main result of this section.

Theorem 3.15. *Every finite lattice in $\mathcal{V}(\{\mathbf{S}_7\})$ is representable by the congruence lattice of a finite algebra with a meet operation.*

Proof. Let $\mathbf{L}$ be a non-trivial finite lattice in $\mathcal{V}(\{\mathbf{S}_7\})$. By Lemma 3.14, $\mathbf{L}$ is isomorphic to $\mathbf{L}'$, a 0-1 sublattice of $\mathbf{Con}((\mathbf{2}_\wedge)^2)^n$ for some $n \geq 1$. Since $\mathbf{Con}((\mathbf{2}_\wedge)^2)$ is power-hereditary by Theorem 3.7, there exists an algebra $\mathbf{A} = \langle (2^2)^n; F\rangle$ such that $\mathbf{Con}(\mathbf{A}) = \mathbf{L}'$.

The meet operation of $((\mathbf{2}_\wedge)^2)^n$ is defined to be the meet operation of $(\mathbf{2}_\wedge)^2$ done coordinate-wise; that is,

$$\langle a_1, \ldots, a_n\rangle \wedge \langle b_1, \ldots, b_n\rangle = \langle a_1 \wedge b_1, \ldots, a_n \wedge b_n\rangle$$

for all $\langle a_1, \ldots, a_n\rangle$ and $\langle b_1, \ldots, b_n\rangle$ in $((\mathbf{2}_\wedge)^2)^n$. I claim that every congruence in $\mathbf{L}'$ preserves the meet operation of $((\mathbf{2}_\wedge)^2)^n$.

Let θ be an equivalence relation in $\mathbf{L}'$. Then $\theta = \langle \alpha_1, \ldots, \alpha_n\rangle$ where each α_i is an element of $\mathbf{Con}((\mathbf{2}_\wedge)^2)$. Let $\langle a_1, \ldots, a_n\rangle$, $\langle b_1, \ldots, b_n\rangle$, $\langle c_1, \ldots, c_n\rangle$, and $\langle d_1, \ldots, d_n\rangle$ in $((\mathbf{2}_\wedge)^2)^n$ be elements such that

$$\langle a_1, \ldots, a_n\rangle \; \langle \alpha_1, \ldots, \alpha_n\rangle \; \langle b_1, \ldots, b_n\rangle$$

and

$$\langle c_1, \ldots, c_n\rangle \; \langle \alpha_1, \ldots, \alpha_n\rangle \; \langle d_1, \ldots, d_n\rangle.$$

This is equivalent to $a_i \; \alpha_i \; b_i$ and $c_i \; \alpha_i \; d_i$ for $1 \leq i \leq n$. Hence $a_i \wedge c_i \; \alpha_i \; b_i \wedge d_i$ for $1 \leq i \leq n$ as every α_i preserves the meet operation on $(\mathbf{2}_\wedge)^2$.

Therefore,

$$\langle a_1 \wedge c_1, \ldots, a_n \wedge c_n \rangle \; \langle \alpha_1, \ldots, \alpha_n \rangle \; \langle b_1 \wedge d_1, \ldots, b_n \wedge d_n \rangle$$

which is equivalent to

$$\langle a_1, \ldots, a_n \rangle \wedge \langle c_1, \ldots, c_n \rangle \; \langle \alpha_1, \ldots, \alpha_n \rangle \; \langle b_1, \ldots, b_n \rangle \wedge \langle d_1, \ldots, d_n \rangle.$$

Thus, every congruence in $\mathbf{L}'$ preserves meet done coordinatewise on $((\mathbf{2}_\wedge)^2)^n$.

Now consider the algebra $\mathbf{A}' = \langle (2^2)^n; F, \wedge \rangle$ which is the original algebra $\mathbf{A}$ with the meet operation added. Since $\mathbf{L}'$ preserves F by definition and preserves meet by the previous paragraph, the congruence lattice of $\mathbf{A}'$ is also $\mathbf{L}'$. Hence, every 0-1 sublattice of $\mathbf{Con}((\mathbf{2}_\wedge)^2)^n$ is the congruence lattice of an algebra with a meet operation.

Therefore, as a result of Lemma 3.14, every finite lattice $\mathbf{L}$ in $\mathcal{V}(\{\mathbf{S}_7\})$ is isomorphic to the congruence lattice of a finite algebra with a meet operation. ■

Chapter 4

The Automorphisms of $(\mathbf{2}_\wedge)^n$ and $\mathbf{Con}((\mathbf{2}_\wedge)^n)$

In this chapter, I discuss the general Boolean meet-semilattice $(\mathbf{2}_\wedge)^n$ and its automorphisms as well as the automorphisms of its congruence lattice. Section 4.1 motivates why the study of these automorphisms is important and revisits the automorphisms of $(\mathbf{2}_\wedge)^2$. Next, in Section 4.2, another example is given which provides more detail to illustrate the upcoming proof. Following that, Section 4.3 contains some useful facts about groups of permutations and automorphisms. Finally, every automorphism of $\mathbf{Con}((\mathbf{2}_\wedge)^n)$ is shown to be carried by an automorphism of $(\mathbf{2}_\wedge)^n$ in Section 4.4.

4.1 Generalizing to the meet-semilattice $(\mathbf{2}_\wedge)^n$

Many natural questions arise from Theorem 3.7 in Chapter 3 and one of the most interesting is "Is $\mathbf{Con}((\mathbf{2}_\wedge)^n)$ power-hereditary for all n?" As of this moment, the question remains unanswered.

It was suggested to me by John Snow that if I could generalize the ideas of the proofs in Section 3.2 it would be hard evidence that the answer is probably "yes". Recall from Section 1.2 that an automorphism Φ of a congruence lattice $\mathbf{Con}(\mathbf{A})$ is carried by a function ϕ if

$$\Phi(\theta) = \{\langle \phi(u), \phi(v) \rangle \in \mathbf{A} \times \mathbf{A} \mid u \,\theta\, v\}$$

for all θ in $\mathbf{Con}(\mathbf{A})$. The idea behind his suggestion is that if every automorphism of $\mathbf{Con}((\mathbf{2}_\wedge)^n)$ is carried by a unique automorphism of $(\mathbf{2}_\wedge)^n$, then the graphs of these automorphisms will be congruence lattices on $(\mathbf{2}_\wedge)^n \times (\mathbf{2}_\wedge)^n$.

The first part of this automorphism conjecture is in fact true for all n, as is shown in Section 4.4, and the second part follows in Chapter 5. I first consider as motivation, though, the case where $n = 2$ from Chapter 3. To discuss all the automorphisms of $(\mathbf{2}_\wedge)^2$ and $\mathbf{Con}((\mathbf{2}_\wedge)^2)$, the following lemma is needed.

Lemma 4.1. *Any meet-semilattice (lattice) automorphism ψ of a finite meet-semilattice (lattice) with 1 fixes 0 and 1. That is, $\psi(0) = 0$ and $\psi(1) = 1$. Moreover, ψ maps every coatom to a coatom and every join-irreducible element to a join-irreducible element.*

Proof. Let ψ be an automorphism of a semilattice $\mathbf{S}$. For any u in $\mathbf{S}$,

$$\psi(u) = \psi(u \wedge 1) = \psi(u) \wedge \psi(1)$$

so $\psi(u) \leq \psi(1)$. Since this is true for every element in $\mathbf{S}$, if $\psi(u) = 1$ then $\psi(1) = 1$. As ψ is one-to-one, $u = 1$. Similarly,

$$\psi(0) = \psi(0 \wedge u) = \psi(0) \wedge \psi(u)$$

so $\psi(0) \leq \psi(u)$ for all u in S. Hence $\psi(0) = 0$.

Now, I claim that ψ maps any coatom of $\mathbf{S}$ to another coatom. For a contradiction, assume there exists a u in $\mathbf{S}$ such that $u \prec 1$ but $\psi(u) \not\prec 1$. Then there exists an u' in $\mathbf{S}$ such that $\psi(u) < u' < 1$. Because ψ is an automorphism, there exists some $v \neq 1$ such that $u' = \psi(v)$. Hence

$$\psi(u) = \psi(u) \wedge u' = \psi(u) \wedge \psi(v) = \psi(u \wedge v)$$

so $u = u \wedge v$. And, since $u \prec 1$ and $v \neq 1$, it follows that $u = v$. But this is a contradiction as ψ is a bijection. Thus $\psi(u) \prec 1$. Moreover, for u, v, and w in $\mathbf{S}$, I have $u \wedge v = w$ if and only $\psi(u) \wedge \psi(v) = \psi(w)$. If w is join-irreducible, then $u = w$ or $v = w$. Hence, either $\psi(u) = \psi(w)$ or $\psi(v) = \psi(w)$ and $\psi(w)$ is join-irreducible.

Note that, since a lattice is also a meet-semilattice, the results are true for lattices as well. ∎

The next theorem describes the automorphisms of $(\mathbf{2}_\wedge)^2$ and $\mathbf{Con}((\mathbf{2}_\wedge)^2)$.

Theorem 4.2. *There are exactly two automorphisms of* $\mathbf{Con}((\mathbf{2}_\wedge)^2)$ *corresponding to the two automorphisms of* $(\mathbf{2}_\wedge)^2$.

Proof. By Lemma 4.1, any automorphism Γ of $\mathbf{Con}((\mathbf{2}_\wedge)^2)$ must permute the coatoms of $\mathbf{Con}((\mathbf{2}_\wedge)^2)$. Moreover, a join-irreducible coatom must be sent to a join-irreducible coatom. Thus, there are two possibilities for Γ. One is the identity map and the other interchanges the two coatoms that are join-irreducible. This latter map is Φ_1 shown in Figure 3.4. ∎

It is trivial that the identity automorphism of $\mathbf{Con}((\mathbf{2}_\wedge)^2)$ is carried by the identity automorphism of $(\mathbf{2}_\wedge)^2$. As well, from the way Φ_1 is defined in Section 3.2, it follows that Φ_1 is also carried by ϕ_1 (see page 46). So for the case of $n = 2$, it follows that every automorphism of $\mathbf{Con}((\mathbf{2}_\wedge)^n)$ is carried by an automorphism of

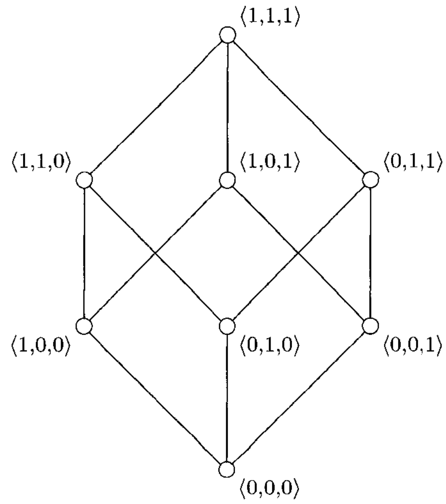

Figure 4.1: The meet-semilattice $(\mathbf{2}_\wedge)^3$

$(\mathbf{2}_\wedge)^n$. In fact, the set of all automorphisms of an algebra form a group as is noted in the following theorem (see Section 1.2 for the definition of a group).

Theorem 4.3. [4] *The set of all automorphisms of an algebra* $\mathbf{A}$ *forms a group,* $\mathbf{Aut}(\mathbf{A}) = \langle Aut(\mathbf{A}); \circ, ^{-1}, id_{\mathbf{A}} \rangle$, *called the automorphism group of* $\mathbf{A}$.

Since $\mathbf{Aut}((\mathbf{2}_\wedge)^2)$ and $\mathbf{Aut}(\mathbf{Con}((\mathbf{2}_\wedge)^2))$ both have only two elements and each element is its own inverse, it is fairly easy to see that the automorphism groups are isomorphic.

4.2 Another example: $\mathbf{Con}((\mathbf{2}_\wedge)^3)$

The first example of $(\mathbf{2}_\wedge)^n$ that has been seen thus far ($n = 2$) provides very little information about the general case. In order to understand the general case better, I provide another example that is much more useful. In this section, the congruence lattice of $(\mathbf{2}_\wedge)^3$ and its corresponding automorphisms are discussed (see Figure 4.1 and 4.2). A sketch is also given of the proof that every automorphism of $\mathbf{Con}((\mathbf{2}_\wedge)^3)$ is carried by an automorphism of $(\mathbf{2}_\wedge)^3$ and refer the reader to Section 4.4 for the rigorous details.

In Chapter 3, finding the congruence lattice of $(\mathbf{2}_\wedge)^2$ was a relatively easy exercise. However, as n grows, the congruence lattice of $(\mathbf{2}_\wedge)^n$ grows large quite quickly. At $n = 3$, the congruence lattice of $(\mathbf{2}_\wedge)^3$ has 61 congruences and is quite hard to find correctly by hand (see Figure 4.2). Luckily, the congruence lattice can be generated using the Universal Algebra Calculator [5] and using this program I am able find all 61 congruences (see Figure 4.2). See Appendix A for a complete description of each congruence in Figure 4.2.

Recall that an element u in a lattice $\mathbf{L}$ is meet-irreducible if $u = v \wedge z$ implies $u = v$ or $u = z$ for all v and z in $\mathbf{L}$. In this section, the only congruences that need discussion are the meet-irreducible congruences of $\mathbf{Con}((\mathbf{2}_\wedge)^3)$. In fact, the meet-irreducible congruences are exactly the coatoms of $\mathbf{Con}((\mathbf{2}_\wedge)^3)$; that is, θ is meet-irreducible in $\mathbf{Con}((\mathbf{2}_\wedge)^3)$ if and only if $\theta \prec \nabla$. These congruences are the 7 congruences of the form

$$\begin{aligned}\Theta_{\langle i_1,i_2,i_3\rangle} = \{&\langle u,v\rangle \in (\mathbf{2}_\wedge)^3 \times (\mathbf{2}_\wedge)^3 : \\ &(u \geq \langle i_1,i_2,i_3\rangle \text{ and } v \geq \langle i_1,i_2,i_3\rangle) \text{ or } (u \ngeq \langle i_1,i_2,i_3\rangle \text{ and } v \ngeq \langle i_1,i_2,i_3\rangle)\}\end{aligned}$$

where $\langle i_1, i_2, i_3\rangle$ in $(\mathbf{2}_\wedge)^3$ is such that $\langle i_1, i_2, i_3\rangle \neq \langle 0,0,0\rangle$. The coatoms and ∇ are illustrated in Figure 4.3.

The meet-irreducible elements are important as they completely determine all the elements of $(\mathbf{2}_\wedge)^3$ and $\mathbf{Con}((\mathbf{2}_\wedge)^3)$. Just as any positive integer can be defined as the product of primes, so too can any element in a meet-semilattice or lattice be written as the meet of meet-irreducible elements. Thus, if it is known where the meet-irreducible elements are mapped by an automorphism, it is also known where every other element is mapped.

Now, any automorphism of $\mathbf{Con}((\mathbf{2}_\wedge)^3)$ will map the coatoms (meet-irreducible congruences) to themselves by a similar argument to the one given in the proof of

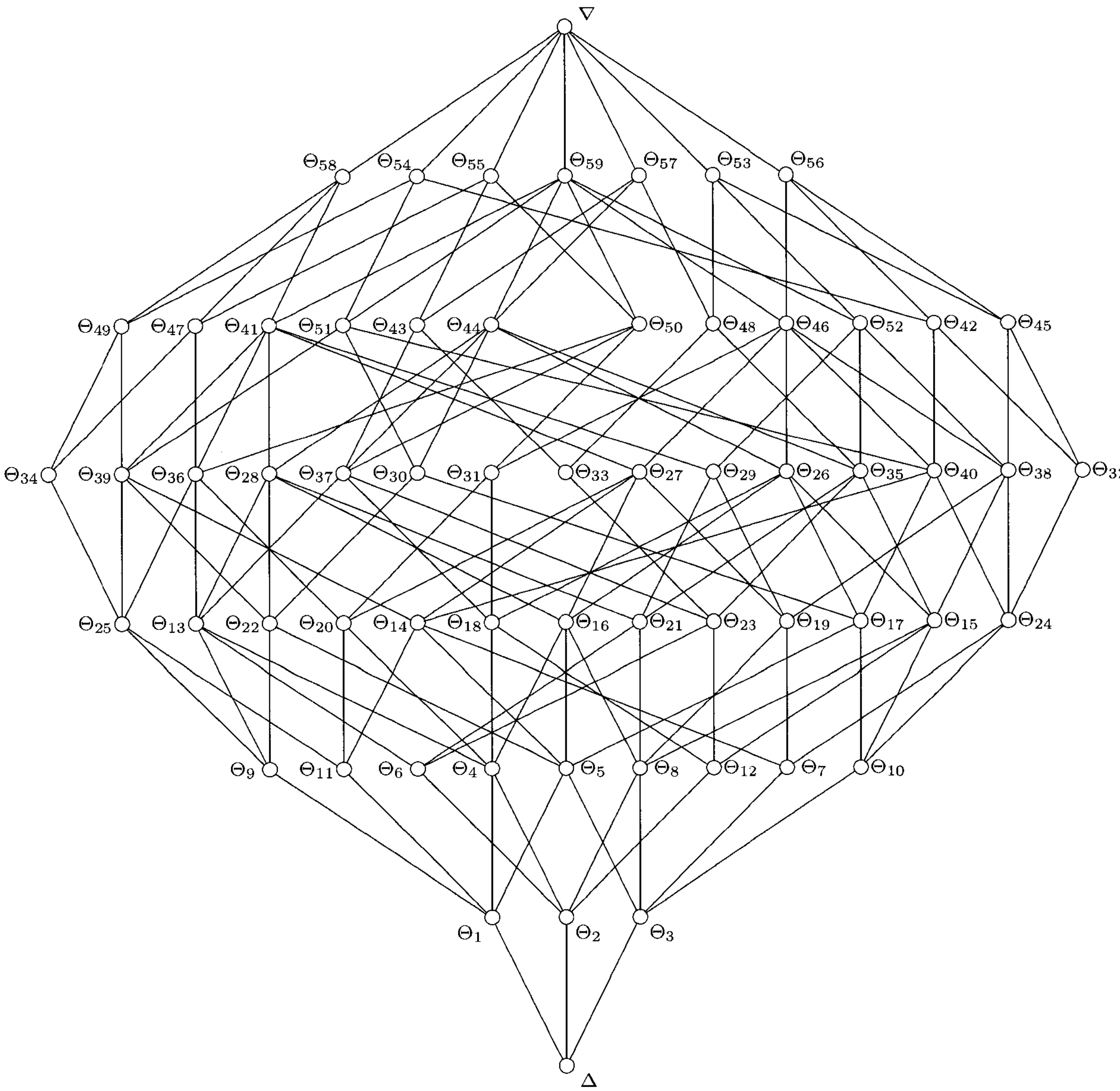

Figure 4.2: The congruence lattice of $(\mathbf{2}_\wedge)^3$

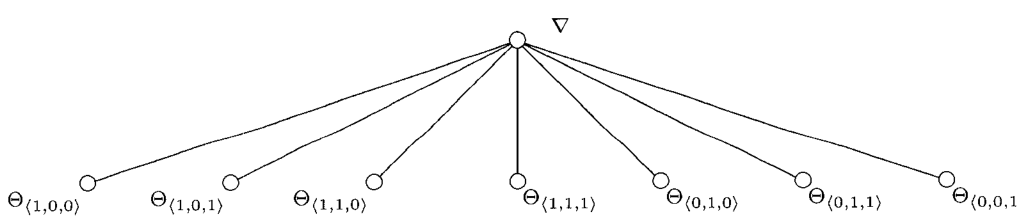

Figure 4.3: The coatoms of $\mathbf{Con}((\mathbf{2}_\wedge)^3)$

Theorem 4.2. What this means is that any automorphism of $\mathbf{Con}((\mathbf{2}_\wedge)^3)$ is completely determined by where it sends the coatoms of $\mathbf{Con}((\mathbf{2}_\wedge)^3)$ as every element of $(\mathbf{2}_\wedge)^3$ is the meet of meet-irreducible elements. Three more useful facts help determine all the automorphisms of $\mathbf{Con}((\mathbf{2}_\wedge)^3)$. First, of all the coatoms in $\mathbf{Con}((\mathbf{2}_\wedge)^3)$, the only three that meet to Δ in $\mathbf{Con}((\mathbf{2}_\wedge)^3)$ are $\Theta_{\langle 1,0,0\rangle}$, $\Theta_{\langle 0,1,0\rangle}$, and $\Theta_{\langle 0,0,1\rangle}$. Any other set of three coatoms will meet to a congruence greater than Δ. Secondly, the meet of any two congruences in $\{\Theta_{\langle 1,0,0\rangle}, \Theta_{\langle 0,1,0\rangle}, \Theta_{\langle 0,0,1\rangle}\}$ is less than a single unique congruence in $\{\Theta_{\langle 1,1,0\rangle}, \Theta_{\langle 1,0,1\rangle}, \Theta_{\langle 0,1,1\rangle}\}$. And thirdly, the congruence $\Theta_{\langle 1,1,1\rangle}$ is the only coatom that covers 6 congruences.

The first fact means that, since every automorphism of $\mathbf{Con}((\mathbf{2}_\wedge)^3)$ maps Δ to itself by Lemma 4.1, the automorphism must also map the set $\{\Theta_{\langle 1,0,0\rangle}, \Theta_{\langle 0,1,0\rangle}, \Theta_{\langle 0,0,1\rangle}\}$ to itself. This, in turn, defines where $\{\Theta_{\langle 1,1,0\rangle}, \Theta_{\langle 1,0,1\rangle}, \Theta_{\langle 0,1,1\rangle}\}$ are sent by the second fact. Finally, the third fact implies that any automorphism must map $\Theta_{\langle 1,1,1\rangle}$ to itself as the automorphism must preserve the number of elements $\Theta_{\langle 1,1,1\rangle}$ covers. Thus every automorphism is completely determined by where it maps the congruences $\Theta_{\langle 1,0,0\rangle}$, $\Theta_{\langle 0,1,0\rangle}$, and $\Theta_{\langle 0,0,1\rangle}$. This, combined with the fact that every permutation of the set $\{\Theta_{\langle 1,0,0\rangle}, \Theta_{\langle 0,1,0\rangle}, \Theta_{\langle 0,0,1\rangle}\}$ generates an automorphism of $\mathbf{Con}((\mathbf{2}_\wedge)^3)$, yields exactly $3! = 6$ automorphisms of $\mathbf{Con}((\mathbf{2}_\wedge)^3)$.

Similarly, for $(\mathbf{2}_\wedge)^3$ there are exactly 6 automorphisms corresponding to permutations of the coatoms. For each coatom c of $(\mathbf{2}_\wedge)^3$ there corresponds a unique atom a such that $a \nleq c$. This atom, in turn, corresponds to a congruence $\Theta_{\langle 1,0,0\rangle}$, $\Theta_{\langle 0,1,0\rangle}$, or $\Theta_{\langle 0,0,1\rangle}$. Thus each automorphism of $\mathbf{Con}((\mathbf{2}_\wedge)^3)$ will correspond to a permutation of the atoms and coatoms of $(\mathbf{2}_\wedge)^3$ and be carried by the corresponding automorphism of $(\mathbf{2}_\wedge)^3$. In addition, the automorphism groups of $(\mathbf{2}_\wedge)^3$ and $\mathbf{Con}((\mathbf{2}_\wedge)^3)$ are in fact isomorphic to each other under the mapping $f : \mathbf{Aut}((\mathbf{2}_\wedge)^\mathbf{3}) \to \mathbf{Aut}(\mathbf{Con}((\mathbf{2}_\wedge)^\mathbf{3}))$ where $f(\phi)$ equals the unique Φ carried by ϕ.

4.3 Permutation and automorphism groups

This section contains two lemmas that are useful for proving the general case in Section 4.4. The first lemma gives a homomorphism between the symmetric group and the automorphism group of the direct product of an algebra. In this lemma, the notation p_σ denotes the automorphism of $\mathbf{A}^I$ corresponding to the permutation σ under the map p.

Lemma 4.4. *Let $\mathbf{\Sigma}_I$ be the symmetric group on the set I and let $\mathbf{A}$ be an algebra. Then the map $p : \mathbf{\Sigma}_I \to \mathbf{Aut}(\mathbf{A}^I)$ defined by $[p_\sigma(x)]_i = x_{\sigma^{-1}(i)}$ (for x in $\mathbf{A}^I$, σ in $\mathbf{\Sigma}_I$, and i in I) is a homomorphism. Moreover, if $|A| \geq 2$ then p is one-to-one.*

Proof. Since $\mathbf{\Sigma}_I$ and $\mathbf{Aut}(\mathbf{A}^I)$ are both groups, I show in the following that p preserves the group operations.

First, consider σ and τ in $\mathbf{\Sigma}_I$ and x in $\mathbf{A}^I$. I have

$$\begin{aligned}
\left[p_{[\sigma\circ\tau]}(x)\right]_i &= x_{[\sigma\circ\tau]^{-1}(i)} \\
&= x_{[\tau^{-1}\circ\sigma^{-1}](i)} \\
&= x_{\tau^{-1}(\sigma^{-1}(i))} \\
&= [p_\tau(x)]_{\sigma^{-1}(i)} \\
&= [p_\sigma(p_\tau(x))]_i \\
&= [[p_\sigma \circ p_\tau](x)]_i \,.
\end{aligned}$$

Since i was arbitrary, it follows that $p_{[\sigma\circ\tau]}(x) = [p_\sigma \circ p_\tau](x)$ and, since x was arbitrary, $p_{[\sigma\circ\tau]} = p_\sigma \circ p_\tau$. Thus, p preserves function composition.

Next, consider $id_{\mathbf{\Sigma}_I}$, the identity element of $\mathbf{\Sigma}_I$. For arbitrary x in $\mathbf{A}^I$ and i in I, it follows that

$$\left[p_{id_{\mathbf{\Sigma}_I}}(x)\right]_i = x_{id^{-1}_{\mathbf{\Sigma}_I}(i)} = x_i = [id_{\mathbf{A}^I}(x)]_i.$$

Thus, $p_{id_{\mathbf{\Sigma}_I}}$ equals $id_{\mathbf{A}^I}$, the identity automorphism of $\mathbf{A}^I$.

Finally, let x be in $\mathbf{A}^I$ and i in I. I have

$$\begin{aligned}\left[\left[p_\sigma \circ p_{\sigma^{-1}}\right](x)\right]_i &= \left[p_{[\sigma\circ\sigma^{-1}]}(x)\right]_i \\ &= \left[p_{id_{\mathbf{\Sigma}_I}}(x)\right]_i \\ &= [id_{\mathbf{A}^I}(x)]_i\end{aligned}$$

and, since i is arbitrary, it follows that $\left[p_\sigma \circ p_{\sigma^{-1}}\right](x) = id_{\mathbf{A}^I}(x)$. Since this is true for all x in $\mathbf{A}^I$, I have $p_\sigma \circ p_{\sigma^{-1}} = id_{\mathbf{A}^I}$. Hence $p_{\sigma^{-1}} = (p_\sigma)^{-1}$ and p preserves inverses. Therefore, p is a homomorphism.

Now, let $|A| \geq 2$ and σ and τ be in $\mathbf{\Sigma}_I$ such that $p_\sigma = p_\tau$. This means that, for all x in $\mathbf{A}^I$,

$$p_\sigma(x) = p_\tau(x).$$

Moreover, for all i in I,

$$[p_\sigma(x)]_i = [p_\tau(x)]_i$$

thus

$$x_{\sigma^{-1}(i)} = x_{\tau^{-1}(i)}.$$

As this holds for all x, consider x with a in the $\sigma^{-1}(j)$-th coordinate and b elsewhere where $a \neq b$. Then $x_{\sigma^{-1}(j)} = a = x_{\tau^{-1}(j)}$, which implies $\sigma^{-1}(j) = \tau^{-1}(j)$. Since j is arbitrary, it follows that $\sigma^{-1}(i) = \tau^{-1}(i)$ for all i in I, hence $\sigma^{-1} = \tau^{-1}$. Therefore, $\sigma = \tau$ and p is one-to-one. ■

The next lemma gives a homomorphism between the automorphism group of an algebra and the automorphism group of the congruence lattice of the same algebra.

Lemma 4.5. *Let* $\mathbf{A}$ *be an algebra. The map* $f : \mathbf{Aut}(\mathbf{A}) \to \mathbf{Aut}(\mathbf{Con}(\mathbf{A}))$ *defined by*

$$f(\phi) = \Phi \text{ where } \phi \text{ carries } \Phi$$

is a homomorphism.

Proof. Recall that Φ is carried by ϕ if

$$\Phi(\theta) = \{\langle \phi(u), \phi(v) \rangle \in \mathbf{A} \times \mathbf{A} \mid u \,\theta\, v\}$$

for all θ in $\mathbf{Con}(\mathbf{A})$. By Lemma 3.3, every ϕ in $\mathbf{Aut}(\mathbf{A})$ carries a unique Φ in $\mathbf{Aut}(\mathbf{Con}(\mathbf{A}))$ so f is well-defined. Hence let Φ and Γ be automorphisms of $\mathbf{Con}(\mathbf{A})$ and ϕ and γ automorphisms of $\mathbf{A}$ such that $f(\phi) = \Phi$ and $f(\gamma) = \Gamma$. Then, for any θ in $\mathbf{Con}(\mathbf{A})$,

$$\begin{aligned}
[f(\phi \circ \gamma)](\theta) &= \{\langle [\phi \circ \gamma]\,(u), [\phi \circ \gamma]\,(v) \rangle \in \mathbf{A} \times \mathbf{A} : u \,\theta\, v\} \\
&= \{\langle \phi(\gamma(u)), \phi(\gamma(v)) \rangle \in \mathbf{A} \times \mathbf{A} : u \,\theta\, v\} \\
&= \{\langle \phi(u), \phi(v) \rangle \in \mathbf{A} \times \mathbf{A} : u \,\Gamma(\theta)\, v\} \\
&= \Phi(\Gamma(\theta)) \\
&= [\Phi \circ \Gamma]\,(\theta).
\end{aligned}$$

Hence $f(\phi \circ \gamma) = \Phi \circ \Gamma = f(\phi) \circ f(\gamma)$.

Now, I claim that $f(\phi^{-1}) = f(\phi)^{-1}$. For any θ' such that $\Phi(\theta) = \theta'$,

$$\begin{aligned}
f(\phi^{-1})(\theta') &= \left\{\langle \phi^{-1}(u), \phi^{-1}(v) \rangle \in \mathbf{A} \times \mathbf{A} : u \,\theta'\, v\right\} \\
&= \left\{\langle \phi^{-1}(u), \phi^{-1}(v) \rangle \in \mathbf{A} \times \mathbf{A} : u \,\Phi(\theta)\, v\right\}
\end{aligned}$$

$$= \{\langle \phi^{-1}(\phi(u)), \phi^{-1}(\phi(v))\rangle \in \mathbf{A} \times \mathbf{A} : u\,\theta\,v\}$$
$$= \{\langle u, v\rangle \in \mathbf{A} \times \mathbf{A} : u\,\theta\,v\}$$
$$= \Phi^{-1}(\theta').$$

Thus, $f(\phi^{-1}) = \Phi^{-1} = f(\phi)^{-1}$.

Finally,

$$\{\langle id_{\mathbf{A}}(u), id_{\mathbf{A}}(v)\rangle \in \mathbf{A} \times \mathbf{A} : u\,\theta\,v\} = \{\langle u, v\rangle \in \mathbf{A} \times \mathbf{A} : u\,\theta\,v\}$$
$$= id_{\mathbf{Con}(\mathbf{A})}(\theta)$$

so $f(id_{\mathbf{A}}) = id_{\mathbf{Con}(\mathbf{A})}$. Thus, since f preserves the group operations, f is a homomorphism. ■

4.4 Every automorphism of $\mathbf{Con}((\mathbf{2}_\wedge)^n)$ is carried by an automorphism of $(\mathbf{2}_\wedge)^n$

I prove in this section that every automorphism of $\mathbf{Con}((\mathbf{2}_\wedge)^n)$ is carried by an automorphism of $(\mathbf{2}_\wedge)^n$ for all n. In addition to this, I show that the automorphism group of $(\mathbf{2}_\wedge)^n$ is isomorphic to the automorphism group of $\mathbf{Con}((\mathbf{2}_\wedge)^n)$ and both are isomorphic to the symmetric group $\mathbf{\Sigma}_n$.

First, some notation used throughout the proofs in this section is defined. Recall from Section 1.2, that the algebra $(\mathbf{2}_\wedge)^n = \langle 2^n; \wedge\rangle$ has the underlying set

$$2^n = \{\langle a_1, \ldots, a_n\rangle \mid a_i \in \{0,1\}\}$$

and the meet operation is applied coordinatewise. For any subset J of $I = \{1, \ldots, n\}$, let x^J be the element of $(\mathbf{2}_\wedge)^n$ such that

$$x_i^J = \begin{cases} 1, & \text{if } i \in J, \text{ and} \\ 0, & \text{if } i \notin J. \end{cases}$$

In addition, let $\overline{x}^J$ be the element in $(\mathbf{2}_\wedge)^n$ such that

$$\overline{x}_i^J = \begin{cases} 0, & \text{if } i \in J, \text{ and} \\ 1, & \text{if } i \notin J. \end{cases}$$

Note that, for every element u in $(\mathbf{2}_\wedge)^n$, there is a set $J \subseteq I$ such that $u = x^J = \overline{x}^{I \setminus J}$. In addition, $\overline{x}^\emptyset$ is the element $\langle 1, 1, \ldots, 1, 1\rangle$ in $(\mathbf{2}_\wedge)^n$, $x^{\{i\}}$ is an atom of $(\mathbf{2}_\wedge)^n$ for all i, and $\overline{x}^{\{i\}}$ is a coatom for all i. Finally, consider the relations Θ^J on $(\mathbf{2}_\wedge)^n$ defined as

$$\Theta^J = \left\{\langle u, v\rangle \in (\mathbf{2}_\wedge)^n \times (\mathbf{2}_\wedge)^n \mid (\overline{x}^J \leq u \text{ and } \overline{x}^J \leq v) \text{ or } (\overline{x}^J \not\leq u \text{ and } \overline{x}^J \not\leq v)\right\}.$$

A useful fact to note is that every Θ^J has exactly two congruence blocks when $J \neq I$. In the next lemma and corollary, I show that every Θ^J is, in fact, a congruence on $(\mathbf{2}_\wedge)^n$.

Lemma 4.6. *Let $a \in \mathbf{S}$ be an element in a finite meet-semilattice $\mathbf{S}$. Then*

$$\Theta_a = \{\langle u, v\rangle \in \mathbf{S} \times \mathbf{S} : (a \leq u \textit{ and } a \leq v) \textit{ or } (a \not\leq u \textit{ and } a \not\leq v)\}$$

is a congruence on $\mathbf{S}$.

Proof. Let u be an element of $\mathbf{S}$. Then surely $a \leq u$ or $a \not\leq u$. Hence $\langle u, u\rangle$ is in Θ_a and Θ_a is reflexive. Now, suppose $\langle u, v\rangle$ is in Θ_a. Then either $a \leq u$ and $a \leq v$ or $a \not\leq u$ and $a \not\leq v$. In each case, the pair $\langle v, u\rangle$ is in Θ_a, so Θ_a is symmetric. Finally, let $\langle u, v\rangle$ and $\langle v, z\rangle$ be in Θ_a. Either $a \leq v$ or $a \not\leq v$, and u and z share the same property. Hence, $\langle u, z\rangle$ is in Θ_a and the relation is transitive. Thus Θ_a is an equivalence relation.

Now, let $u\,\Theta_a\,v$ and $w\,\Theta_a\,z$. If u and v are not greater than or equal to a or w and z are not greater than or equal to a, then $a \not\leq u \wedge w$ and $a \not\leq v \wedge z$. If u, v, w, and z are all greater that a, then $u \wedge w \geq a$ and $v \wedge z \geq a$. Hence, $u \wedge w\,\Theta_a\,v \wedge z$ and Θ_a is a congruence on $\mathbf{S}$. ■

Since every Θ^J is of the form described Lemma 4.6, I have the following immediate corollary.

Corollary 4.7. *Θ^J is a congruence on $(\mathbf{2}_\wedge)^n$ for all $J \subseteq \{1,\ldots,n\}$.*

Now, before the main theorem is stated, a series of lemmas is presented describing the structures of both $(\mathbf{2}_\wedge)^n$ and $\mathbf{Con}((\mathbf{2}_\wedge)^n)$. For the following, since $(\mathbf{2}_\wedge)^n$ has a top element, adopt the convention that $\bigwedge \emptyset = 1$. The next lemma allows a description of all of the elements of $(\mathbf{2}_\wedge)^n$ by considering a much smaller subset of them.

Lemma 4.8. *Every element of $(\mathbf{2}_\wedge)^n$ is the meet of a subset of the coatoms of $(\mathbf{2}_\wedge)^n$. Moreover, this subset of the coatoms is unique.*

Proof. First note that the coatoms of $(\mathbf{2}_\wedge)^n$ are the $\overline{x}^{\{i\}}$ as defined above. If x^J is some element of $(\mathbf{2}_\wedge)^n$, then $x^J = \bigwedge\limits_{k\in I\setminus J} \overline{x}^{\{k\}}$.

Now suppose, for some $L \subseteq I$, that $x^J = \bigwedge\limits_{l\in L} \overline{x}^{\{l\}}$. Then $x_i^J = 0$ if and only if i is in L and i is in $I\setminus J$. Hence L must equal $I\setminus J$. ■

The next lemma describes the automorphisms of $(\mathbf{2}_\wedge)^n$.

Lemma 4.9. *The automorphism group of $(\mathbf{2}_\wedge)^n$ is isomorphic to $\mathbf{\Sigma}_n$.*

Proof. Let ψ be an automorphism of $(\mathbf{2}_\wedge)^n$ and $\overline{x}^{\{i\}}$ a coatom of $(\mathbf{2}_\wedge)^n$. By Lemma 4.1, $\psi(\overline{x}^{\{i\}}) = \overline{x}^{\{j\}}$ for some j. Hence, since ψ is a bijection, ψ permutes the coatoms of $(\mathbf{2}_\wedge)^n$.

Now, for all $u < 1$ in $(\mathbf{2}_\wedge)^n$, by Lemma 4.8, $u = \overline{x}^{\{i_1\}} \wedge \cdots \wedge \overline{x}^{\{i_m\}}$ for some $i_1, \ldots, i_m$. Then

$$\psi(u) = \psi(\overline{x}^{\{i_1\}} \wedge \cdots \wedge \overline{x}^{\{i_m\}}) = \psi(\overline{x}^{\{i_1\}}) \wedge \cdots \wedge \psi(\overline{x}^{\{i_m\}}).$$

Thus, $\psi(u)$ is completely determined by where ψ maps each $\overline{x}^{\{i_j\}}$. Therefore, any automorphism of $(\mathbf{2}_\wedge)^n$ is defined by how it permutes the coatoms of $(\mathbf{2}_\wedge)^n$. Hence, there are at most $n!$ automorphisms of $(\mathbf{2}_\wedge)^n$.

However, by Lemma 4.4, there exists a one-to-one homomorphism p from $\mathbf{\Sigma}_n$ to $\mathbf{Aut}((\mathbf{2}_\wedge)^n)$. Since $|\mathbf{\Sigma}_n| = n!$, it follows that $n! \leq |\mathbf{Aut}((\mathbf{2}_\wedge)^n)|$. Thus

$$|\mathbf{\Sigma}_n| = |\mathbf{Aut}((\mathbf{2}_\wedge)^n)|$$

and p is a bijection. Therefore, $\mathbf{Aut}((\mathbf{2}_\wedge)^n)$ is isomorphic to $\mathbf{\Sigma}_n$. ■

For all u in a finite meet-semilattice $\mathbf{S}$ and θ in $\mathbf{Con}(\mathbf{S})$, define $\lfloor u/\theta \rfloor$ such that

$$\lfloor u/\theta \rfloor = \bigwedge \{v \in \mathbf{S} : v \; \theta \; u\}.$$

By Lemma 4.6, $\Theta_{\lfloor a/\theta \rfloor}$ is in $\mathbf{Con}(\mathbf{S})$. The following lemma provides a useful way of describing the congruences of $\mathbf{S}$.

Lemma 4.10. *For any finite meet-semilattice* $\mathbf{S}$ *and* θ *in* $\mathbf{Con}(\mathbf{S})$,

$$\theta = \bigwedge_{a \in \mathbf{S}} \Theta_{\lfloor a/\theta \rfloor}.$$

Proof. First, I show $\theta \subseteq \bigwedge_{a \in \mathbf{S}} \Theta_{\lfloor a/\theta \rfloor}$. Let $\langle u, v \rangle$ be in θ. Suppose, for a contradiction, that there exists an s such that $\langle u, v \rangle$ is not in $\Theta_{\lfloor s/\theta \rfloor}$. Without loss of generality, this means $\lfloor s/\theta \rfloor \leq u$ and $\lfloor s/\theta \rfloor \not\leq v$. Since $\lfloor s/\theta \rfloor \; \theta \; \lfloor s/\theta \rfloor$ and $u \; \theta \; v$, it follows that $\lfloor s/\theta \rfloor = \lfloor s/\theta \rfloor \wedge u \ \theta \ \lfloor s/\theta \rfloor \wedge v$. But $\lfloor s/\theta \rfloor \not\leq v$ so $\lfloor s/\theta \rfloor \wedge v < \lfloor s/\theta \rfloor$. This contradicts the minimality of $\lfloor s/\theta \rfloor$. Hence the pair $\langle u, v \rangle$ is in $\Theta_{\lfloor a/\theta \rfloor}$ for all a in $\mathbf{S}$ and $\theta \subseteq \bigwedge_{a \in \mathbf{S}} \Theta_{\lfloor a/\theta \rfloor}$.

For the other direction, let $\langle u, v \rangle$ be in the congruence $\bigwedge_{a \in \mathbf{S}} \Theta_{\lfloor a/\theta \rfloor}$. There exists an s such that $\lfloor s/\theta \rfloor \ \theta \ u \wedge v$. I claim that, for any t in $\mathbf{S}$, if $\lfloor t/\theta \rfloor \leq u$ and $\lfloor t/\theta \rfloor \leq v$, then $\lfloor t/\theta \rfloor \leq \lfloor s/\theta \rfloor$. I have that $\lfloor t/\theta \rfloor \leq u$ and $\lfloor t/\theta \rfloor \leq v$ implies $\lfloor t/\theta \rfloor \leq u \wedge v$. Then

$$\lfloor s/\theta \rfloor \wedge \lfloor t/\theta \rfloor \ \theta \ u \wedge v \wedge \lfloor t/\theta \rfloor,$$

so $\lfloor s/\theta \rfloor \wedge \lfloor t/\theta \rfloor \ \theta \ \lfloor t/\theta \rfloor$. But $\lfloor s/\theta \rfloor \wedge \lfloor t/\theta \rfloor \leq \lfloor t/\theta \rfloor$ and, by the minimality of $\lfloor t/\theta \rfloor$, it follows that $\lfloor t/\theta \rfloor \leq \lfloor s/\theta \rfloor \wedge \lfloor t/\theta \rfloor$. Hence $\lfloor t/\theta \rfloor = \lfloor s/\theta \rfloor \wedge \lfloor t/\theta \rfloor$ so $\lfloor t/\theta \rfloor \leq \lfloor s/\theta \rfloor$.

However, there exists an r in $\mathbf{S}$ such that $\lfloor r/\theta \rfloor \ \theta \ u$ so $\lfloor r/\theta \rfloor \leq u$. As $\langle u, v \rangle$ is in $\bigwedge_{a \in \mathbf{S}} \Theta_{\lfloor a/\theta \rfloor}$, it follows that $\lfloor r/\theta \rfloor \leq v$. Thus $\lfloor r/\theta \rfloor \leq u \wedge v \leq u$. By the previous paragraph, $\lfloor r/\theta \rfloor \leq \lfloor s/\theta \rfloor$. Since $\lfloor r/\theta \rfloor \ \theta \ u$,

$$\lfloor r/\theta \rfloor \wedge \lfloor s/\theta \rfloor \ \theta \ u \wedge \lfloor s/\theta \rfloor.$$

Hence $\lfloor r/\theta \rfloor \ \theta \ \lfloor s/\theta \rfloor$ which means $\lfloor r/\theta \rfloor = \lfloor s/\theta \rfloor$. And so, $u \wedge v \ \theta \ \lfloor s/\theta \rfloor$ is equivalent to $u \wedge v \ \theta \ \lfloor r/\theta \rfloor$, thus $u \wedge v \ \theta \ u$. Similarly, $u \wedge v \ \theta \ v$ yielding $u \ \theta \ v$. Thus, $\langle u, v \rangle$ must be in θ. Therefore, $\bigwedge_{a \in \mathbf{S}} \Theta_{\lfloor a/\theta \rfloor} \subseteq \theta$ and $\theta = \bigwedge_{a \in \mathbf{S}} \Theta_{\lfloor a/\theta \rfloor}$. ■

The next lemma describes some important properties of the maximal congruences of $\mathbf{Con}((\mathbf{2}_\wedge)^n)$.

Lemma 4.11. *The following are equivalent:*

(a) *θ is a maximal congruence in $\mathbf{Con}((\mathbf{2}_\wedge)^n)$.*

(b) *θ is meet-irreducible.*

(c) *$\theta = \Theta^J$ for some $J \subsetneq I = \{1, \ldots, n\}$.*

Moreover, every element in $\mathbf{Con}((\mathbf{2}_\wedge)^n)$ *is the meet of a subset of the maximal congruences of* $\mathbf{Con}((\mathbf{2}_\wedge)^n)$ *and this subset is unique.*

Proof. If θ is a maximal congruence, then θ is covered only by the universal relation ∇, hence θ must be meet-irreducible.

I show that if θ is not a maximal congruence, then θ is not meet-irreducible. Assume θ is not a maximal congruence. Then, since $\theta \not\prec \nabla$, there must be at least three congruence classes of θ. Let $T = \{\overline{x}^{J_1}, \overline{x}^{J_2}, \ldots, \overline{x}^{J_m}\}$ be the minimum elements of the congruence classes of θ where m is the number of congruence classes. Note that for some i, the element $\overline{x}^{J_i} = 0$. By Lemma 4.10,

$$\theta = \bigwedge_{i=1}^{m} \Theta^{J_i}.$$

Now, since there are at least three congruence classes of θ, there must must be at least three distinct Θ^{J_i} not equal to θ, two of which are not equal to ∇. Hence θ is not meet-irreducible. Therefore, by contraposition, if θ is meet-irreducible, it must be a maximal congruence.

Next, note that any congruence in any congruence lattice with only two classes is a maximal congruence. Moreover, $0 \; \Theta^J \; 1$ if and only if $J = I$. Thus, Θ^J is a maximal congruence in $\mathbf{Con}((\mathbf{2}_\wedge)^n)$ for all $J \subsetneq I$.

Finally, suppose θ is a maximal congruence. Then θ must have only two congruence blocks; otherwise,

$$\theta = \bigwedge_{i=1}^{m} \Theta^{J_i}$$

for some $m \geq 3$ and θ would not be meet-irreducible. In addition, the two congruence blocks must be $0/\theta$ and $1/\theta$ as $0 \; \theta \; 1$ implies that $\theta = \nabla$. Now, consider $v = \bigwedge \{u \in (\mathbf{2}_\wedge)^n \mid u \; \theta \; 1\}$ which satisfies $v \; \theta \; 1$ and $v \leq u$ for all $u \; \theta \; 1$. Thus,

$$v = v \wedge w \; \theta \; 1 \wedge w = w$$

for all $w \geq v$. Hence, $1/\theta$ is the interval $[v, 1]$ in $(\mathbf{2}_\wedge)^n$ for this v and it follows that $0/\theta$ must be all elements not in the interval. Therefore, $\theta = \Theta^J$, where $J \subseteq \{1, \ldots, n\}$ is chosen so that $v = \overline{x}^J$. ■

In order to understand the rest of this section, consider the following useful definition. Recall that by Lemma 4.11 the meet-irreducible elements of $\mathbf{Con}((\mathbf{2}_\wedge)^n)$ are exactly the maximal congruences of the form Θ^J for some $J \subseteq I$. Let the *dimension of a maximal congruence* in $\mathbf{Con}((\mathbf{2}_\wedge)^n)$ be the cardinality of the corresponding index set J. That is, if $|J| = j$, then Θ^J is a j-dimensional maximal congruence.

Before proceeding with the following lemma, note some important facts about the maximal congruences. Firstly, there are $2^n - 1$ maximal congruences of the form Θ^J corresponding to all elements of $(\mathbf{2}_\wedge)^n$ not equal to 0 (in the case of $\overline{x}^I = 0$, by definition, $\Theta^I = \nabla$.) Secondly, any $(n-1)$-dimensional maximal congruence $\Theta^{\{i_1,\ldots,i_{n-1}\}}$, for some $\{i_1, \ldots, i_{n-1}\} \subseteq I$, is of the form

$$\begin{aligned}\Theta^{\{i_1,\ldots,i_{n-1}\}} = \{\langle u, v\rangle \in (\mathbf{2}_\wedge)^n \times (\mathbf{2}_\wedge)^n \mid \\ (\overline{x}^{\{i_1,\ldots,i_{n-1}\}} \leq u \text{ and } \overline{x}^{\{i_1,\ldots,i_{n-1}\}} \leq v) \text{ or} \\ (\overline{x}^{\{i_1,\ldots,i_{n-1}\}} \not\leq u \text{ and } \overline{x}^{\{i_1,\ldots,i_{n-1}\}} \not\leq v)\}\end{aligned}$$

$$= \{\langle u, v\rangle \in (\mathbf{2}_\wedge)^n \times (\mathbf{2}_\wedge)^n \mid (x^{\{i_n\}} \leq u \text{ and } x^{\{i_n\}} \leq v) \text{ or } (x^{\{i_n\}} \not\leq u \text{ and } x^{\{i_n\}} \not\leq v)\},$$

where $x^{\{i_n\}}$ is an atom of $(\mathbf{2}_\wedge)^n$ and $\{i_n\} = I \backslash \{i_1, \ldots, i_{n-1}\}$. Thirdly, for $m < n$ there are $\binom{n}{m}$ maximal congruences of dimension m corresponding to the number of subsets of I of size m. Finally, in any m-dimensional maximal congruence $\Theta^{\{i_1,\ldots,i_m\}}$ where $m < n-1$, every atom $x^{\{i\}}$ satisfies $\overline{x}^{\{i_1,\ldots,i_m\}} \not\leq x^{\{i\}}$. Hence for all i, the atom $x^{\{i\}}$ is in $0/\Theta^{\{i_1,\ldots,i_m\}}$.

Lemma 4.12. *Let $J_1, J_2, \dots, J_k$ be subsets of I and let $J = \bigcap_{l=1}^{k} J_l$. Then*

$$\bigwedge_{l=1}^{k} \Theta^{J_l} \leq \Theta^{J}.$$

Moreover, if $J \nsubseteq K$, then

$$\bigwedge_{l=1}^{k} \Theta^{J_l} \nleq \Theta^{K}.$$

Proof. Suppose that $\langle u, v\rangle$ is in $\bigwedge_{l=1}^{k} \Theta^{J_l}$. Then there exists an l such that $\overline{x}^{J_l} \nleq u$, or for all l I have $\overline{x}^{J_l} \leq u$. Consider the first case. Since $\overline{x}^{J_l} \nleq u$, it follows that $\overline{x}^{J_l} \nleq v$. However, $\overline{x}^{J_l} \leq \overline{x}^{J}$ so $\overline{x}^{J} \nleq u$ and $\overline{x}^{J} \nleq v$ as $\langle u, v\rangle$ is in Θ^{J_l}. Thus $\langle u, v\rangle$ is in Θ^{J}.

If, on the other hand, for all l I have $\overline{x}^{J_l} \leq u$, then for all l I have $\overline{x}^{J_l} \leq v$. This implies that $\overline{x}^{[\bigcap_l J_l]}$, the least upper bound of the set $\{\overline{x}^{J_1}, \dots, \overline{x}^{J_k}\}$, is less than or equal to both u and v. But $\overline{x}^{[\bigcap_l J_l]} = \overline{x}^{J}$. Hence $\langle u, v\rangle$ is again in Θ^{J} so $\bigwedge_{l=1}^{k} \Theta^{J_l} \leq \Theta^{J}$.

Now, suppose $J \nsubseteq K$. This means there exists an i in J such that i is not in K. Hence $\overline{x}^{K} \nleq \overline{x}^{J}$. However, since $\overline{x}^{J_l} \leq \overline{x}^{J}$ for all l, the pair $\langle \overline{x}^{J}, 1\rangle$ is in Θ^{J_l} for all l. Thus $\langle \overline{x}^{J}, 1\rangle$ is in $\bigwedge_{l=1}^{k} \Theta^{J_l}$. But $\overline{x}^{K} \nleq \overline{x}^{J}$ implies that $\langle \overline{x}^{J}, 1\rangle$ is not in Θ^{K}. Hence $\bigwedge_{l=1}^{k} \Theta^{J_l} \nleq \Theta^{K}$. ■

The following corollary to Lemma 4.12 concerns the $(n-1)$-dimensional maximal congruences.

Corollary 4.13. *If $\Theta^{J_1}, \dots, \Theta^{J_{n-m}}$ are distinct $(n-1)$-dimensional maximal congruences and Θ^{K} is m-dimensional with $\bigwedge_{l=1}^{n-m} \Theta^{J_l} \leq \Theta^{K}$, then $K = \bigcap_{l=1}^{n-m} J_l$.*

Proof. By Lemma 4.12, $\bigwedge_{l=1}^{n-m} \Theta^{J_l} \leq \Theta^K$ implies that $\bigcap_{l=1}^{n-m} J_l \subseteq K$. But

$$\left| \bigcap_{l=1}^{n-m} J_l \right| = m = |K|$$

so $K = \bigcap_{l=1}^{n-m} J_l$. ■

The following lemma describes where an automorphism must send the maximal congruences of $\mathbf{Con}((\mathbf{2}_\wedge)^n)$.

Lemma 4.14. *Any automorphism of* $\mathbf{Con}((\mathbf{2}_\wedge)^n)$ *maps an* m*-dimensional maximal congruence to an* m*-dimensional maximal congruence for* $0 \leq m \leq n-1$.

Proof. Let Φ be an automorphism of $\mathbf{Con}((\mathbf{2}_\wedge)^n)$. Note that, by Lemma 4.1, every maximal congruence in $\mathbf{Con}((\mathbf{2}_\wedge)^n)$ must be mapped to a maximal congruence by Φ.

Next, let $\Theta^{J_1}, \ldots, \Theta^{J_n}$ be n distinct maximal congruences and consider the congruence $\Gamma = \bigwedge_{i=1}^{n} \Theta^{J_i}$. The congruence class of 0 in Γ is equal to $\bigcap_{i=1}^{n} 0/\Theta^{J_i}$. That is,

$$0/\Gamma = \left\{ u \in (\mathbf{2}_\wedge)^n : u \ngeq \overline{x}^{J_i}, 1 \leq i \leq n \right\}.$$

If every Θ^{J_i} is $(n-1)$-dimensional, then $\overline{x}^{J_i}$ is an atom for all i. Hence, since there are exactly n atoms of $(\mathbf{2}_\wedge)^n$ and every element not equal to 0 is greater than or equal to an atom, it follows that

$$0/\Gamma = \{0\}.$$

I claim that Γ is the trivial congruence Δ. For suppose there exists a u and v in $(\mathbf{2}_\wedge)^n$ such that $u \neq v$ but $u/\Gamma = v/\Gamma$. If u is not comparable to v and $u \;\Gamma\; v$, then $u \wedge v < v$ so, without loss of generality, suppose $u < v$. Then, if $u = x^J$ and $v = x^K$, there exists an i in $K\backslash J$. But this means that there also exists an atom $x^{\{i\}}$ of $(\mathbf{2}_\wedge)^n$ such that $x^{\{i\}} \leq v$ but $x^{\{i\}} \not\leq u$ so

$$u \wedge x^{\{i\}} \; \Gamma \; v \wedge x^{\{i\}}.$$

Hence $0 \; \Gamma \; x^{\{i\}}$, which is a contradiction. Thus, if every Θ^{J_i} is $(n-1)$-dimensional, then

$$\Gamma = \bigwedge_{i=1}^{n} \Theta^{J_i} = \Delta.$$

Now, suppose there is a k such that Θ^{J_k} is m-dimensional where $m < n-1$. This means the number of $(n-1)$-dimensional maximal congruences in $\{\Theta^{J_i} \mid i = 1, \ldots, n\}$ is at most $n-1$. Note that when Θ^{J_i} is $(n-1)$-dimensional, $\overline{x}^{J_i}$ is an atom and $0/\Theta^{J_i}$ contains every atom other than $\overline{x}^{J_i}$. As well, for any m-dimensional maximal congruence θ with $m < n-1$, every atom is in $0/\theta$. Hence there exists at least one atom $x^{\{j\}}$ such that, for all i, the element $x^{\{j\}}$ is in $0/\Theta^{J_i}$. Thus $x^{\{j\}}$ must be in $0/\Gamma$ which means that Γ must be greater than Δ. Therefore, the only n maximal congruences that meet to Δ are the $(n-1)$-dimensional maximal congruences and, since there are exactly n of these, every automorphism of $\mathbf{Con}((\mathbf{2}_\wedge)^n)$ must map the $(n-1)$-dimensional maximal congruences to each other.

I next show that, since Φ maps the $(n-1)$-dimensional maximal congruences to each other, Φ must also map the m-dimensional maximal congruences to each other for $0 \leq m \leq n-2$. I show this by induction on $k = n - m$. The base case $k = n - (n-1) = 1$ is true from the previous paragraph. Suppose, for all j where $m < j \leq n-1$, that Φ maps the j-dimensional maximal congruences to each other. Let Θ^J be an m-dimensional maximal congruence. There exists $J_1, \ldots, J_{n-m}$ with each Θ^{J_i} an $(n-1)$-dimensional maximal congruences and $J = \bigcap_{i=1}^{n-m} J_i$. For $i = 1, \ldots, n-m$, let Θ^{K_i} be the $(n-1)$-dimensional maximal congruence such that

$\Phi(\Theta^{J_i}) = \Theta^{K_i}$. By Corollary 4.13, the congruence Θ^J is the only m-dimensional maximal congruence such that $\bigwedge_{i=1}^{n-m} \Theta^{J_i} \leq \Theta^J$. As $\Phi(\Theta^J)$ is a maximal congruence there exists $T \subseteq \{1, \ldots, n\}$ with $\Phi(\Theta^J) = \Theta^T$.

Now

$$\bigwedge_{i=1}^{n-m} \Theta^{J_i} = \bigwedge_{i=1}^{n-m} \Theta^{J_i} \wedge \Theta^J$$

is equivalent to

$$\Phi(\bigwedge_{i=1}^{n-m} \Theta^{J_i}) = \Phi(\bigwedge_{i=1}^{n-m} \Theta^{J_i} \wedge \Theta^J),$$

so

$$\bigwedge_{i=1}^{n-m} \Phi(\Theta^{J_i}) = \bigwedge_{i=1}^{n-m} \Phi(\Theta^{J_i}) \wedge \Phi(\Theta^J).$$

Thus

$$\bigwedge_{i=1}^{n-m} \Theta^{K_i} \leq \Phi(\Theta^J) = \Theta^T.$$

By Lemma 4.12, $\bigcap_{i=1}^{n-m} K_i \subseteq T$, so the congruence Θ^T is of dimension at least m. But, since j-dimensional maximal congruences are mapped by Φ to j-dimensional maximal congruences for all $j > m$, the congruence Θ^T must be m-dimensional. Thus, it follows that every m-dimensional maximal congruence is mapped by Φ to an m-dimensional maximal congruence.

Therefore, by induction on $k = n - m$ with $m = n - 1$ as the base case, every automorphism Φ maps an m-dimensional maximal congruence to an m-dimensional maximal congruence for $0 \leq m \leq n-1$. ■

I am now able to state and prove an important lemma that leads to the main result of this section.

Lemma 4.15. *There are at most $n!$ automorphisms of $\mathbf{Con}((\mathbf{2}_\wedge)^n)$ and they correspond to permutations of the $(n-1)$-dimensional maximal congruences.*

Proof. Let Φ be an automorphism of $\mathbf{Con}((\mathbf{2}_\wedge)^n)$. By Lemma 4.14, the automorphism Φ must map m-dimensional maximal congruences to each other for all m. Then, by Corollary 4.13, every m-dimensional maximal congruence is fixed by a set of $(n-1)$-dimensional maximal congruences of size $n-m$. Hence, Φ is completely determined on the maximal congruences by where it sends the $(n-1)$-dimensional maximal congruences.

In addition, since every congruence in $\mathbf{Con}((\mathbf{2}_\wedge)^n)$ is the meet of maximal congruences by Lemma 4.11, any automorphism of $\mathbf{Con}((\mathbf{2}_\wedge)^n)$ is completely determined by where it maps the maximal congruences. Hence the automorphism Φ is defined entirely by where it maps the $(n-1)$-dimensional maximal congruences which is to themselves. Therefore, since there are exactly n maximal congruences of dimension $n-1$, there are at most $n!$ unique automorphisms of $\mathbf{Con}((\mathbf{2}_\wedge)^n)$. ■

Finally, the next theorem states that every automorphism of $\mathbf{Con}((\mathbf{2}_\wedge)^n)$ is carried by an automorphism of $(\mathbf{2}_\wedge)^n$.

Theorem 4.16. *Every automorphism of $\mathbf{Con}((\mathbf{2}_\wedge)^n)$ is carried by an automorphism of $(\mathbf{2}_\wedge)^n$. Moreover, $\mathbf{Aut}((\mathbf{2}_\wedge)^n)$ is isomorphic to $\mathbf{Aut}(\mathbf{Con}((\mathbf{2}_\wedge)^n))$ under f defined such that*

$$f(\phi) = \Phi \text{ where } \phi \text{ carries } \Phi.$$

Proof. By Lemma 3.3, every automorphism ϕ of $(\mathbf{2}_\wedge)^n$ carries some automorphism of $\mathbf{Con}((\mathbf{2}_\wedge)^n)$. Note that f is the same as the map f defined in Lemma 4.4.

I claim that every distinct automorphism of $(\mathbf{2}_\wedge)^n$ carries a distinct automorphism of $\mathbf{Con}((\mathbf{2}_\wedge)^n)$. That is, the map f is one-to-one. For i in $\{1,2\}$, define $\Gamma_i : \mathbf{Con}((\mathbf{2}_\wedge)^n) \to \mathbf{Con}((\mathbf{2}_\wedge)^n)$ by

$$\Gamma_i(\theta) = \{\langle \phi_i(u), \phi_i(v)\rangle \in (\mathbf{2}_\wedge)^n \times (\mathbf{2}_\wedge)^n \mid u \,\theta\, v\}$$

where ϕ_1 and ϕ_2 are distinct automorphisms of $(\mathbf{2}_\wedge)^n$. There exists a j such that $\phi_1(\overline{x}^{\{j\}}) \neq \phi_2(\overline{x}^{\{j\}})$. Now, $\langle \overline{x}^{\{j\}}, 1\rangle$ is in $\Theta^{\{j\}}$, so

$$\langle \phi_i(\overline{x}^{\{j\}}), \phi_i(1)\rangle = \langle \phi_i(\overline{x}^{\{j\}}), 1\rangle$$

is in $\Gamma_i(\Theta^{\{j\}})$ and, by Lemma 4.1, the elements $\phi_1(\overline{x}^{\{j\}})$ and $\phi_2(\overline{x}^{\{j\}})$ are coatoms of $(\mathbf{2}_\wedge)^n$. But $\overline{x}^{\{j\}}$ is the only coatom in $1/\Theta^{\{j\}}$ so

$$\langle \phi_1(\overline{x}^{\{j\}}), 1\rangle \neq \langle \phi_2(\overline{x}^{\{j\}}), 1\rangle,$$

which implies

$$\Gamma_1(\Theta^{\{j\}}) \neq \Gamma_2(\Theta^{\{j\}}).$$

Thus, $\Gamma_1 \neq \Gamma_2$.

Therefore, since there are $n!$ automorphisms of $(\mathbf{2}_\wedge)^n$ by Lemma 4.9 and at most $n!$ automorphisms of $\mathbf{Con}((\mathbf{2}_\wedge)^n)$ by Lemma 4.15, every automorphism of $\mathbf{Con}((\mathbf{2}_\wedge)^n)$ is carried by an automorphism of $(\mathbf{2}_\wedge)^n$. Moreover, $|\mathbf{Aut}(\mathbf{Con}((\mathbf{2}_\wedge)^n))| = |\mathbf{Aut}((\mathbf{2}_\wedge)^n)|$ and, since f is one-to-one, f is a bijection. Therefore, by Lemma 4.5, f is an isomorphism. ■

Chapter 5

Classifying the Congruence Lattice of $(\mathbf{2}_\wedge)^n$ for all n

In this chapter, I discuss possibilities for determining the congruence heredity of $(\mathbf{2}_\wedge)^n$ for all $n > 2$.

5.1 Ideas for solving the general case

In Chapter 2 and 3, I prove that $\mathbf{Con}((\mathbf{2}_\wedge)^2)$ is power-hereditary by showing that every subdirect product of $\mathbf{Con}((\mathbf{2}_\wedge)^2) \times \mathbf{Con}((\mathbf{2}_\wedge)^2)$ containing $(\{0\} \times \mathbf{Con}((\mathbf{2}_\wedge)^2)) \cup (\mathbf{Con}((\mathbf{2}_\wedge)^2) \times \{1\})$ is the congruence lattice of an algebra on the universe of $(\mathbf{2}_\wedge)^2 \times (\mathbf{2}_\wedge)^2$. I believe the key to solving the general case involves the graphs of the automorphisms of $\mathbf{Con}((\mathbf{2}_\wedge)^n)$. This comes from the fact that proving that $\mathbf{L}_0$ — the only subdirect product coming from a non-identity automorphism of $\mathbf{Con}((\mathbf{2}_\wedge)^2)$ — is a congruence lattice requires further consideration of the underlying set and the equivalence relations involved. Whereas, for every other subdirect product containing $(\{0\} \times \mathbf{Con}((\mathbf{2}_\wedge)^2)) \cup (\mathbf{Con}((\mathbf{2}_\wedge)^2) \times \{1\})$, the proof that each is a congruence lattice requires only lattice-theoretic arguments.

The main result of Chapter 4, that every automorphism of $\mathbf{Con}((\mathbf{2}_\wedge)^n)$ is carried by an automorphism of $(\mathbf{2}_\wedge)^n$, actually yields an even more powerful result thanks to the next lemma.

Lemma 5.1. [22] *Suppose that* $\mathbf{A}$ *is a finite algebra and that* $\Phi : \mathbf{Con}(\mathbf{A}) \to \mathbf{Con}(\mathbf{A})$ *is an automorphism. If* Φ *is carried by an automorphism* ϕ *of* $\mathbf{A}$*, then the graph of the automorphism* Φ *(as a sublattice of* $\mathbf{Con}(\mathbf{A}) \times \mathbf{Con}(\mathbf{A})$*) is closed under all primitive positive formulas yielding equivalence relations.*

Lemma 5.1 provides the step needed to produce, from Theorem 4.16 and Lemma 3.8, the following important result.

Theorem 5.2. *Let* Φ *be an automorphism of* $\mathbf{Con}((\mathbf{2}_\wedge)^n)$*. Every subdirect product* $\mathbf{L}$ *of* $\mathbf{Con}((\mathbf{2}_\wedge)^n) \times \mathbf{Con}((\mathbf{2}_\wedge)^n)$ *containing* $(\{0\} \times \mathbf{Con}((\mathbf{2}_\wedge)^n)) \cup (\mathbf{Con}((\mathbf{2}_\wedge)^n) \times \{1\})$ *and of the form*

$$\mathbf{L} = \{\langle \alpha, \beta \rangle \in \mathbf{Con}((\mathbf{2}_\wedge)^n) \times \mathbf{Con}((\mathbf{2}_\wedge)^n) : \Phi(\alpha) \leq \beta\}$$

is the congruence lattice of an algebra on the universe of $(\mathbf{2}_\wedge)^n \times (\mathbf{2}_\wedge)^n$*.*

Proof. By Theorem 4.16, every automorphism Φ of $\mathbf{Con}((\mathbf{2}_\wedge)^n)$ is carried by an automorphism ϕ of $(\mathbf{2}_\wedge)^n$. Then, by Lemma 5.1, the graph $\mathbf{G}$ of the automorphism Φ is closed under primitive formulas and, by Lemma 1.4, is a congruence lattice on the universe of $(\mathbf{2}_\wedge)^n \times (\mathbf{2}_\wedge)^n$. But

$$\mathbf{G} = \{\langle \alpha, \beta \rangle \in \mathbf{Con}((\mathbf{2}_\wedge)^n) \times \mathbf{Con}((\mathbf{2}_\wedge)^n) : \Phi(\alpha) = \beta\}$$

hence, by Lemma 3.8,

$$\mathbf{L} = \{\langle \alpha, \beta \rangle \in \mathbf{Con}((\mathbf{2}_\wedge)^n) \times \mathbf{Con}((\mathbf{2}_\wedge)^n) : \Phi(\alpha) \leq \beta\}$$

is the congruence lattice of an algebra on the universe of $(\mathbf{2}_\wedge)^n \times (\mathbf{2}_\wedge)^n$. ■

I predict that Theorem 5.2 will allow me to write every subdirect product of $\mathbf{Con}((\mathbf{2}_\wedge)^n) \times \mathbf{Con}((\mathbf{2}_\wedge)^n)$ containing $(\{0\} \times \mathbf{Con}((\mathbf{2}_\wedge)^n)) \cup (\mathbf{Con}((\mathbf{2}_\wedge)^n) \times \{1\})$ as the intersection of known congruence lattices as I do for $(\mathbf{2}_\wedge)^2$ in Chapter 2. In addition, I anticipate Corollary 2.8 will prove very useful. Recall that Corollary 2.8 (a) states any subdirect product $\mathbf{L}$ of $\mathbf{Con}((\mathbf{2}_\wedge)^n) \times \mathbf{Con}((\mathbf{2}_\wedge)^n)$ containing $(\{0\} \times \mathbf{Con}((\mathbf{2}_\wedge)^n)) \cup (\mathbf{Con}((\mathbf{2}_\wedge)^n) \times \{1\})$ is equivalent to

$$\mathbf{L} = \left\{\langle \alpha, \beta \rangle \in \mathbf{Con}((\mathbf{2}_\wedge)^n) \times \mathbf{Con}((\mathbf{2}_\wedge)^n) : \alpha \leq \beta^{\Uparrow}\right\}$$

where $\beta^{\Uparrow} = \bigvee\{\alpha \in \mathbf{Con}((\mathbf{2}_\wedge)^n) : \langle \alpha, \beta \rangle \in \mathbf{L}\}$. I conjecture that thinking of the subdirect products in this way should expedite the process of writing them as the intersection of known congruence lattices.

Chapter 6

Conclusion and Future Directions

6.1 Conclusion

The initial question asked in Chapter 1 is, "What are the power-hereditary representations of the lattice $\mathbf{S}_7$ and its dual?" This question is partially answered in the central theorem of Chapter 3:

Theorem 3.7. *The congruence lattice of* $(\mathbf{2}_\wedge)^2$ *is a power-hereditary representation of the lattice* $\mathbf{S}_7$.

Every other result in this thesis either leads to or, in a certain sense, leads from Theorem 3.7.

Chapter 2 consists of a series of lemmas and theorems leading to Theorems 2.14 and 2.18 — necessary and sufficient conditions for congruence lattice representations of $\mathbf{S}_7$ and its dual to be power-hereditary. Besides being a useful tool to study any representation of $\mathbf{S}_7$, Theorem 2.14 provides the key to the proof of Theorem 3.7.

Then, in Chapter 3, I describe an algebra $(\mathbf{2}_\wedge)^2$ whose congruence lattice is isomorphic to $\mathbf{S}_7$. I supply a proof that $\mathbf{Con}((\mathbf{2}_\wedge)^2)$ is power-hereditary in Section 3.2 and then provide an alternate proof in Section 3.3. In addition, I show in Theorem 3.15 that every finite lattice in $\mathcal{V}(\{\mathbf{S}_7\})$ is representable by an algebra with a meet operation, a result stemming directly from Theorem 3.7. And finally, in Chapters 4 and 5, I begin the process of generalizing the proof of Theorem 3.7 with the goal to prove the conjecture that $\mathbf{Con}((\mathbf{2}_\wedge)^n)$ is power-hereditary for all $n > 2$.

6.2 Further questions

Many questions have come up during my research, some of which I had intended to answer when I began and some which arose from my results. This thesis concludes with a few of them.

The major open problem I would like to answer is discussed fully in the previous section.

Problem 1. *Is* $\mathbf{Con}((\mathbf{2}_\wedge)^n)$ *power-hereditary for all* $n > 2$*?*

With respect to the lattice $\mathbf{S}_7$, the following questions remain to be answered.

Problem 2. *What are other congruence lattice representations of* $\mathbf{S}_7$ *and are these representations hereditary or power-hereditary?*

John Snow has suggested that it would probably be possible to come up with a sufficient condition for a representation of $\mathbf{S}_7$ to not be power-hereditary — one that is similar to that of Theorem 1.12 for the lattice $\mathbf{M}_3$. He speculated, though, that finding a representation to satisfy this graph-theoretic condition could prove quite challenging.

In Chapter 2, a necessary and sufficient condition is given for a representation of ${\mathbf{S}_7}^*$, the dual of $\mathbf{S}_7$, to be power-hereditary. I have yet to find a congruence lattice representation of ${\mathbf{S}_7}^*$ but it is of interest to note that ${\mathbf{S}_7}^*$ is isomorphic to the lattice of convex subsets of a three element chain [8]. Still, many similar questions to those already asked in regards to $\mathbf{S}_7$ could also be asked about ${\mathbf{S}_7}^*$.

Problem 3. *What are other congruence lattice representations of* $\mathbf{S_7}^*$ *and are these representations hereditary or power-hereditary?*

Let $\mathcal{H}$ be the class of all finite lattices $\mathbf{L}$ such that if $\mathbf{Con(A)}$ is isomorphic to $\mathbf{L}$ then $\mathbf{Con(A)}$ is hereditary and let $\mathcal{PH}$ be the class of all finite lattices $\mathbf{L}$ such that if $\mathbf{Con(A)}$ is isomorphic to $\mathbf{L}$ then $\mathbf{Con(A)}$ is power-hereditary.

Problem 4. *What are the elements of* $\mathcal{H}$ *and* $\mathcal{PH}$*?*

Snow's results in [20] show that every congruence lattice representation of the non-modular lattice $\mathbf{N}_5$ is power hereditary. Hence $\mathbf{N}_5$ is an element of $\mathcal{PH}$. The modular non-distributive lattice $\mathbf{M}_3$, on the other hand, has both a power-hereditary and non-power-hereditary representation and so is not an element of $\mathcal{PH}$.

An even more interesting question comes next.

Problem 5. *Under what operations are* $\mathcal{H}$ *and* $\mathcal{PH}$ *closed?*

In other words, are $\mathcal{H}$ and $\mathcal{PH}$ closed under subalgebras, products, homomorphic images and various other operations. The discovery of a non-power-hereditary representation of $\mathbf{M}_3$ in [12] showed that $\mathcal{H}$ is not closed under direct products but I believe $\mathcal{H}$ and $\mathcal{PH}$ may be quite different from each other. A major benefit of showing $\mathcal{H}$ and $\mathcal{PH}$ to be closed under the various possible operations is that it would greatly expand the number of known representable lattices.

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Appendix A

The congruence lattice of $(\mathbf{2}_\wedge)^3$

The following are all 61 congruences of $(\mathbf{2}_\wedge)^3$, arranged according to the number of congruence blocks. For the ease of the reader, I have relabelled $(\mathbf{2}_\wedge)^3$ with elements $\{0, a, b, c, d, e, f, 1\}$ as can be seen in Figure A.1.

$\Delta = \{\{0\}, \{d\}, \{e\}, \{f\}, \{a\}, \{b\}, \{c\}, \{1\}\}$,

$\Theta_1 = \{\{0, d\}, \{e\}, \{f\}, \{a\}, \{b\}, \{c\}, \{1\}\}$,

$\Theta_2 = \{\{0, e\}, \{d\}, \{f\}, \{a\}, \{b\}, \{c\}, \{1\}\}$,

$\Theta_3 = \{\{0, f\}, \{d\}, \{e\}, \{a\}, \{b\}, \{c\}, \{1\}\}$,

$\Theta_4 = \{\{0, d, e\}, \{f\}, \{a\}, \{b\}, \{c\}, \{1\}\}$,

$\Theta_5 = \{\{0, d, f\}, \{e\}, \{a\}, \{b\}, \{c\}, \{1\}\}$,

$\Theta_6 = \{\{0, e\}, \{d, a\}, \{f\}, \{b\}, \{c\}, \{1\}\}$,

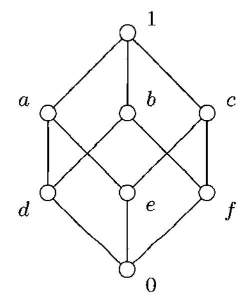

Figure A.1: The meet-semilattice $(\mathbf{2}_\wedge)^3$ relabelled

$\Theta_7 = \{\{0,f\},\{d,b\},\{e\},\{a\},\{c\},\{1\}\}$,

$\Theta_8 = \{\{0,e,f\},\{d\},\{a\},\{b\},\{c\},\{1\}\}$,

$\Theta_9 = \{\{0,d\},\{e,a\},\{f\},\{b\},\{c\},\{1\}\}$,

$\Theta_{10} = \{\{0,f\},\{e,c\},\{d\},\{a\},\{b\},\{1\}\}$,

$\Theta_{11} = \{\{0,d\},\{f,b\},\{e\},\{a\},\{c\},\{1\}\}$,

$\Theta_{12} = \{\{0,e\},\{f,c\},\{d\},\{a\},\{b\},\{1\}\}$,

$\Theta_{13} = \{\{0,d,e,a\},\{f\},\{b\},\{c\},\{1\}\}$,

$\Theta_{14} = \{\{0,d,f,b\},\{e\},\{a\},\{c\},\{1\}\}$,

$\Theta_{15} = \{\{0,e,f,c\},\{d\},\{a\},\{b\},\{1\}\}$,

$\Theta_{16} = \{\{0,d,e,f\},\{a\},\{b\},\{c\},\{1\}\}$,

$\Theta_{17} = \{\{0,d,f\},\{e,c\},\{a\},\{b\},\{1\}\}$,

$\Theta_{18} = \{\{0,d,e\},\{f,c\},\{a\},\{b\},\{1\}\}$,

$\Theta_{19} = \{\{0,e,f\},\{d,b\},\{a\},\{c\},\{1\}\}$,

$\Theta_{20} = \{\{0,d,e\},\{f,b\},\{a\},\{c\},\{1\}\}$,

$\Theta_{21} = \{\{0,e,f\},\{d,a\},\{b\},\{c\},\{1\}\}$,

$\Theta_{22} = \{\{0,d,f\},\{e,a\},\{b\},\{c\},\{1\}\}$,

$\Theta_{23} = \{\{0,e\},\{d,a\},\{f,c\},\{b\},\{1\}\}$,

$\Theta_{24} = \{\{0,f\},\{d,b\},\{e,c\},\{a\},\{1\}\}$,

$\Theta_{25} = \{\{0,d\},\{e,a\},\{f,b\},\{c\},\{1\}\}$,

$\Theta_{26} = \{\{0,d,e,f,c\},\{a\},\{b\},\{1\}\}$,

$\Theta_{27} = \{\{0,d,e,f,b\},\{a\},\{c\},\{1\}\}$,

$\Theta_{28} = \{\{0,d,e,f,a\},\{b\},\{c\},\{1\}\}$,

$\Theta_{29} = \{\{0,e,f\},\{d,a,b\},\{c\},\{1\}\}$,

$\Theta_{30} = \{\{0,d,f\},\{e,a,c\},\{b\},\{1\}\}$,

$\Theta_{31} = \{\{0,d,e\},\{f,b,c\},\{a\},\{1\}\}$,

$\Theta_{32} = \{\{0,f\},\{d,b\},\{e,c\},\{a,1\}\}$,

$\Theta_{33} = \{\{0,e\},\{d,a\},\{f,c\},\{b,1\}\}$,

$\Theta_{34} = \{\{0, d\}, \{e, a\}, \{f, b\}, \{c, 1\}\}$,

$\Theta_{35} = \{\{0, e, f, c\}, \{d, a\}, \{b\}, \{1\}\}$,

$\Theta_{36} = \{\{0, d, e, a\}, \{f, b\}, \{c\}, \{1\}\}$,

$\Theta_{37} = \{\{0, d, e, a\}, \{f, c\}, \{b\}, \{1\}\}$,

$\Theta_{38} = \{\{0, e, f, c\}, \{d, b\}, \{a\}, \{1\}\}$,

$\Theta_{39} = \{\{0, d, f, b\}, \{e, a\}, \{c\}, \{1\}\}$,

$\Theta_{40} = \{\{0, d, f, b\}, \{e, c\}, \{a\}, \{1\}\}$,

$\Theta_{41} = \{\{0, d, e, f, a, b\}, \{c\}, \{1\}\}$,

$\Theta_{42} = \{\{0, d, f, b\}, \{e, c\}, \{a, 1\}\}$,

$\Theta_{43} = \{\{0, d, e, a\}, \{f, c\}, \{b, 1\}\}$,

$\Theta_{44} = \{\{0, d, e, f, a, c\}, \{b\}, \{1\}\}$,

$\Theta_{45} = \{\{0, e, f, c\}, \{d, b\}, \{a, 1\}\}$,

$\Theta_{46} = \{\{0, d, e, f, b, c\}, \{a\}, \{1\}\}$,

$\Theta_{47} = \{\{0, d, e, a\}, \{f, b\}, \{c, 1\}\}$,

$\Theta_{48} = \{\{0, e, f, c\}, \{d, a\}, \{b, 1\}\}$,

$\Theta_{49} = \{\{0, d, f, b\}, \{e, a\}, \{c, 1\}\}$,

$\Theta_{50} = \{\{0, d, e, a\}, \{f, b, c\}, \{1\}\}$,

$\Theta_{51} = \{\{0, d, f, b\}, \{e, a, c\}, \{1\}\}$,

$\Theta_{52} = \{\{0, e, f, c\}, \{d, a, b\}, \{1\}\}$,

$\Theta_{53} = \{\{0, e, f, c\}, \{d, a, b, 1\}\}$,

$\Theta_{54} = \{\{0, d, f, b\}, \{e, a, c, 1\}\}$,

$\Theta_{55} = \{\{0, d, e, a\}, \{f, b, c, 1\}\}$,

$\Theta_{56} = \{\{0, d, e, f, b, c\}, \{a, 1\}\}$,

$\Theta_{57} = \{\{0, d, e, f, a, c\}, \{b, 1\}\}$,

$\Theta_{58} = \{\{0, d, e, f, a, b\}, \{c, 1\}\}$,

$\Theta_{59} = \{\{0, d, e, f, a, b, c\}, \{1\}\}$, and

$\nabla = \{\{0, d, e, f, a, b, c, 1\}\}$.

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